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Modified Jarratt Method Without Memory With Twelfth-Order Convergence

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Abstract. Re-obtaining some old zero-finding iterative methods is not a rarity in numerical analysis. Routinely, most of the improvements of root solvers increase the order of convergence by adding a new evaluation of the function or its derivatives per iteration. In the present article, we give a simple way to develop the local order of convergence by using Jarratt method in the first step of a three-step cycle. The analysis of convergence illustrates that the proposed method without memory is a twelfth-order iterative scheme and its classical efficiency index is 1.644, which is greater than that of Jarratt. Some numerical examples are provided to support and re-verify the novel method.

The discussion of the new iteration in complex plane by presenting basins of attraction will also be given. Although the proposed technique is not optimal due to its local 12th-order convergence with five (functional) evaluations per full iteration, it consists of two evaluations of the first-order derivatives and three evaluations of the function. In fact, there is no optimal method with 5 (functional) evaluations per iteration in the literature with local 16th-order convergence, in which there is two first-order derivative evaluations per cycle.

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1. Introduction

The boundary value problems (BVPs) in Kinetic theory of gases, elasticity and other applied areas are mostly reduced in solving single variable nonlinear equations. Hence, the problem of approximating a solution of the nonlinear equation $f(x) = 0$, is important. The numerical methods for finding the roots of such equations are called iterative methods, [1].

A large number of papers have been written about iterative methods for the solution of nonlinear equations and systems; e.g. [2-5]. However, these iterative schemes can be classified into two main categories; **A**: the derivative-free methods [6]. And, **B**: high order methods that use the derivatives of the function in their structure, see e.g. [7, 8]. Here, we focus on the category B. In this study, we consider iterative methods to find a simple root α of the nonlinear equations, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

There exists an extension of Newton's method, called Potra-Ptak iterative scheme [9], which is of order three and given by

$$x_{n+1} = x_n - \frac{f(x_n) + f(x_n - f(x_n)/f'(x_n))}{f'(x_n)}. \tag{1}$$

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For quite some time, this had been the only reported third-order iterative method with three evaluations per iteration. In 1966, a method of order four, Jarratt method [10], which contains one evaluation of the function and two evaluations of the first derivatives, had been investigated as follows

$$x_{n+1} = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \tag{2}$$

where $k_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$, and

$$J_f(x_n) = \frac{3f'(k_n) + f'(x_n)}{6f'(k_n) - 2f'(x_n)}. \tag{3}$$

Some decades later, in 2009, an improvement of Jarratt method (2) had been proposed in [11], which is a sixth-order method and can be defined as comes next

$$y_n = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \tag{4}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{\psi(y_n)}. \tag{5}$$

Three different ways for calculating $\psi(y_n)$ (note that $f'(y_n) \approx \psi(y_n)$) are provided in [11] as well, such as

$$\psi(y_n) = \frac{2f'(x_n)f'(y_n)}{3f'(x_n) - f'(y_n)}. \tag{6}$$

This iterative scheme consists of two evaluations of the function and two evaluations of the first derivative. Generally speaking, many methods have been developed in the recent years by more calculations of the function or its derivatives per iteration at the new points. In fact, the idea is to compose two or more familiar iterative schemes in order to reach better convergence order. But unfortunately in this way, the method is inefficient, unless the number of (functional) evaluations per full iteration has been decreased by approximating the new-appeared values of the function or its derivatives.

On the other hand, from a practical standpoint, it is fascinating to improve the order of convergence of the known efficient methods. In this work, an accurate twelfth-order iterative scheme is developed by considering a three-step cycle using (2) in its first step.

Note that in 1974, the fundamental work in root finding was published by Kung and Traub that provided methods of order 2^n consisting of $n+1$ evaluation per iteration [12]. As a matter of fact, they conjectured that a multi-point iteration without memory using $n+1$ (functional) evaluation can reach the maximum convergence rate 2^n. Taking into account of this, the optimal efficiency index of an iteration is $2^{\frac{n}{n+1}}$. See for more on this topic or an application of nonlinear equations [13, 14].

2. A new method

Let us consider a three-step cycle in which the Jarratt method is in the first step and we have the Newton's method in the second and third steps in the following way:

$$\begin{cases} y_n = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \end{cases} \tag{7}$$

In order to obtain a novel variant of Jarratt method with better efficiency index, we estimate the two new-appeared first derivatives of the function (at the new points) in the second and third steps, by two different approximations. First, we estimate $f'(y_n)$ by a polynomial of degree three as follows:

$$q(t) = a_1 + a_2(t - x_n) + a_3(t - x_n)^2 + a_4(t - x_n)^3. \tag{8}$$

Note that this estimation function meets the function $f(t)$ in the points x_n, k_n, y_n, i.e., by considering

$$q(x_n) = f(x_n),\ q(y_n) = f(y_n),$$
$$q'(x_n) = f'(x_n),\ q'(k_n) = f'(k_n),$$

we have four linear equations with four unknowns a_1, a_2, a_3 and a_4. By solving this system of linear equations we have $a_1 = f(x_n)$, $a_2 = f'(x_n)$,

$$a_4 = \frac{2f[y_n, x_n](-x_n + k_n) + (x_n - 2k_n + y_n)f'(x_n) + (x_n - y_n)f'(k_n)}{(x_n - k_n)(x_n - y_n)(x_n - 3k_n + 2y_n)}, \tag{9}$$

and

$$a_3 = f[y_n, x_n, x_n] - (y_n - x_n)a_4, \tag{10}$$

where

$$f[y_n, x_n, x_n] = \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n}.$$

Now we have a powerful approximation of $f'(y_n)$ in the following form

$$f'(y_n) \approx q'(y_n) = a_2 + 2a_3(y_n - x_n) + 3a_4(y_n - x_n)^2. \tag{11}$$

Although we have used all of the past four known values in estimating $f'(y_n)$, the order will arrive at 6 (according to Theorem 1) by using (11) at the end of the second step of (7). That is to say, the Kung-Traub hypothesis cannot be achieved. The reason is that, the first two steps in this way consume two evaluations of the function and two evaluations of the first derivatives.

In fact, the maximum order that could be achieved by four evaluations per full cycle in which there are two first-order derivative evaluations is 6. We should remark that there is no optimal three-point without memory method including two derivative evaluations and two function evaluations per full iteration in the literature. Note that we cannot build any weight function at this step to fulfill the conjecture of Kung-Traub. As a matter of fact, any try by the authors for providing weight functions to increase the order from 6 to 8 without more evaluations per step have failed.

Anyhow now, we can double the convergence order by adding only one more evaluation of the function at the new added step (the third step of (7)). For the last new-appeared first derivative of the function in the third step, i.e. $f'(z_n)$, we consider a same interpolating polynomial as in (8) but with different coefficients. Let us consider the interpolating polynomial

$$m(t) = b_0 + b_1(t - x_n) + b_2(t - x_n)^2 + b_4(t - x_n)^3, \tag{12}$$

as an approximation for the function $f(t)$ which meets the function in x_n, y_n, z_n. That is,

$$m(x_n) = f(x_n),\ m'(x_n) = f'(x_n),\ m(y_n) = f(y_n)\ \textit{and}\ m(z_n) = f(z_n).$$

Hence, by solving a new system of linear equations just like the previous case, we obtain the four unknowns (b_1, b_2, b_3 and b_4) and consequently we have an estimation of $f'(z_n)$ as comes next

$$f'(z_n) \approx 2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n)f[y_n, x_n, x_n], \tag{13}$$

wherein $f[x_n, y_n], f[x_n, z_n]$ and $f[y_n, z_n]$ are divided differences. Another interesting point of Jarratt-type methods has appeared now. Although, at the third step we omit using the known value $f'(k_n)$ in approximating $f'(z_n)$, the order has doubled. In fact, the use of $f'(k_n)$ in finding approximates of $f'(z_n)$ just increase the computational load and the order will remain unchanged.

Note that if we consider the interpolation conditions as follows

$$m(x_n) = f(x_n), m'(k_n) = f'(k_n),\ m(y_n) = f(y_n) \text{ and } m(z_n) = f(z_n),$$

then a new approximation for $f'(z_n)$ will be attained.

Now we could write down our novel iterative method without memory as follows:

$$\begin{cases} y_n = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{2f[y_n,x_n]-f'(x_n)+a_4(y_n-x_n)^2}, \\ x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n,z_n]+f[y_n,z_n]-2f[x_n,y_n]+(y_n-z_n)f[y_n,x_n,x_n]}. \end{cases} \tag{14}$$

wherein $J_f(x_n)$ and a_4 are defined by (3) and (9), respectively. The theoretical proof of this without memory Jarratt-type method is given in Theorem 1.

Theorem 1. *Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ in an open interval D, which contains x_0 as an initial approximation of α. Then the method (14) is of order twelve and includes three evaluations of the function and two evaluations of the first derivative per full iteration.*

Proof. We write down the Taylor's series expansion of the function f and its first derivative around the simple zero in the n-th iterate. For simplicity, we assume that

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(\alpha)}{f'(\alpha)},\ k \geq 2. \tag{15}$$

Also let $e_n = x_n - \alpha$. Thus, we have

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + \cdots + O(e_n^{13})), \tag{16}$$

and

$$f'(x_n) = f'(\alpha)(1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+7c_7e_n^6+\cdots+O(e_n^{12})). \tag{17}$$

Dividing the new two expansions (16) and (17) on each other, gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \cdots + O(e_n^{13}). \tag{18}$$

Now we have

$$x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} - \alpha = \frac{e_n}{3} + \frac{2c_2e_n^2}{3} - \frac{4}{3}(c_2^2 - c_3)e_n^3 + \frac{2}{3}(4c_2^3 - 7c_2c_3 + 3c_4)e_n^4$$
$$-\frac{4}{3}(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 + \cdots + O(e_n^{13}). \tag{19}$$

We expand $f'(k_n)$ around the simple root and then we obtain:

$$x_n - \frac{3f'(k_n) + f'(x_n)}{6f'(k_n) - 2f'(x_n)}\frac{f(x_n)}{f'(x_n)} - \alpha = (c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4$$
$$+(-4c_2^4 + 8c_2^2c_3 - 2c_3^2 - \frac{20c_2c_4}{9} + \frac{8c_5}{27})e_n^5 + \frac{2}{27}(135c_2^5 - 405c_2^3c_3$$
$$+165c_2^2c_4 - 99c_3c_4 + 9c_2(27c_3^2 - 5c_5) + 7c_6)e_n^6 + \cdots + O(e_n^{13}). \tag{20}$$

For the second step of (14), we provide the Taylor expansion in the same form for $f(y_n)$ at first, and then for the approximation function $q'(y_n)$. Accordingly, we attain

$$y_n - \frac{f(y_n)}{2f[y_n,x_n] - f'(x_n) + a_4(y_n - x_n)^2} - \alpha = -\frac{(c_4(9c_2^3 - 9c_2c_3 + c_4))}{81c_2}e_n^6$$
$$+\frac{(6c_4(18c_2^5 - 27c_2^3c_3 + 10c_2^2c_4 + c_3c_4) - 8c_2(9c_2^3 - 9c_2c_3 + 2c_4)c_5)}{243c_2^2}e_n^7$$
$$+\frac{1}{729c_2^3}(729c_2^{10} - 1458c_2^8c_3 - 81c_2^7c_4 + \cdots + c_2^2(27c_3c_4^2 - 64c_5^2 - 84c_4c_6))e_n^8 + \cdots + O(e_n^{13}). \tag{21}$$

This shows that our iterative scheme (14) arrives at 6th order of convergence in the end of the second step. For the last step, by writing the Taylor expansion of $f(z_n)$ around the simple root α, we have

$$\begin{aligned}
&f(z_n) = -((c_4(9c_2^3 - 9c_2c_3 + c_4)f(\alpha))e_n^6)/(81c_2) + (2(3c_4(18c_2^5 - 27c_2^3c_3\\
&+10c_2^2c_4 + c_3c_4) - 4c_2(9c_2^3 - 9c_2c_3 + 2c_4)c_5)f(\alpha)e_n^7)/(243c_2^2)\\
&+(1/(729c_2^3))(729c_2^{10} - 1458c_2^8c_3 - 81c_2^7c_4 - 36c_3^2c_4^2 + 27c_2^6(27c_3^2 + 25c_5)\\
&-9c_2^4(77c_4^2 + 123c_3c_5) + c_2(25c_4^3 + 96c_3c_4c_5) + 27c_2^5(45c_3c_4 - 14c_6) + 27c_2^3(-24c_3^2c_4\\
&+27c_4c_5 + 14c_3c_6) + c_2^2(27c_3c_4^2 - 64c_5^2 - 84c_4c_6))f(\alpha)e_n^8 - (1/(2187c_2^4))2((8748c_2^{12}\\
&-26244c_2^{10}c_3 + 8262c_2^9c_4 - 108c_3^3c_4^2 + 162c_2^8(135c_3^2 - 4c_5) + 6c_2c_3c_4(25c_4^2\\
&+48c_3c_5) - 486c_2^7(17c_3c_4 + 3c_6) + 27c_2^5(-81c_3^2c_4 + 124c_4c_5 + 96c_3c_6)\\
&+3c_2^2(27c_3^2c_4^2 - 82c_4^2c_5 - 64c_3c_5^2 - 84c_3c_4c_6) - 9c_2^6(486c_3^3 + 159c_4^2 + 306c_3c_5\\
&-92c_7) + 3c_2^4(999c_3c_4^2 + 675c_3^2c_5 - 332c_5^2 - 564c_4c_6 - 276c_3c_7) + c_2^3(-93c_4^3\\
&+336c_5c_6 + 8c_4(-27c_3c_5 + 23c_7)))f(\alpha))e_n^9 + 1/(6561c_2^5)(236196c_2^{14} - 944784c_2^{12}c_3\\
&+361584c_2^{11}c_4 - 1296c_3^4c_4^2 + 243c_2^{10}(4752c_3^2 - 437c_5) + 108c_2c_3^2c_4(25c_4^2 + 32c_3c_5)\\
&+243c_2^9(-2919c_3c_4 + 50c_6) + c_2^2(972c_3^3c_4^2 - 625c_4^4 - 5904c_3c_4^2c_5 - 144c_3^2(16c_5^2\\
&+21c_4c_6)) + 27c_2^8(-16524c_3^3 + 3003c_4^2 + 5040c_3c_5 + 406c_7) + 9c_2^6(2916c_3^4 + 1764c_3c_4^2\\
&+729c_3^2c_5 - 1514c_5^2 - 2922c_4c_6 - 2322c_3c_7) + 3c_2^3(-864c_3^2c_4c_5 + c_4(1024c_5^2 + 807c_4c_6)\\
&+c_3(-588c_4^3 + 1344c_5c_6 + 736c_4c_7)) + 9c_2^7(28188c_3^2c_4 + 1086c_4c_5 + 1782c_3c_6 - 731c_8)\\
&+c_2^4(-243c_3^2c_4^2 + 3771c_4^2c_5 - 1764c_6^2 + 108c_3(16c_5^2 + 21c_4c_6) - 2944c_5c_7 - 1462c_4c_8)\\
&+3c_2^5(1944c_3^3c_4 - 4755c_4^3 - 5832c_3^2c_6 + 5790c_5c_6 + 4603c_4c_7\\
&+3c_3(-6429c_4c_5 + 731c_8)))f(\alpha)e_n^{10} + O(e_n^{11}).
\end{aligned}$$

By considering the obtained formula and the Taylor expansion of $m'(z_n)$, we have that $2f[x_n,z_n] + f[y_n,z_n] - 2f[x_n,y_n] + (y_n - z_n)f[y_n,x_n,x_n] = f(\alpha) + (7/81)c_4(9c_2^3 - 9c_2c_3 + c_4)f(\alpha)e_n^6 + (2(-3c_4(126c_2^5 - 270c_2^3c_3 + 81c_2c_3^2 + 70c_2^2c_4 - 2c_3c_4) + c_2(171c_2(c_2^2 - c_3) + 47c_4)c_5)f(\alpha)e_n^7)/(243c_2) + 1/(729c_2^2)(1458c_2^{10} - 2916c_2^8c_3 + 7128c_2^7c_4 - 72c_3^2c_4^2 + 54c_2^6(27c_3^2 - 83c_5) + 45c_2^4(169c_4^2 + 210c_3c_5) + c_2(59c_4^3 + 192c_3c_4c_5) + 27c_2^5(-720c_3c_4 + 53c_6) + 27c_2^3(438c_3^2c_4 - 156c_4c_5 - 53c_3c_6) + c_2^2(-5373c_3c_4^2 - 2916c_3^2c_5 + 304c_5^2 + 453c_4c_6))f(\alpha)e_n^8 + O(e_n^9)$. We finally obtain

$$e_{n+1} = -\frac{8(c_4^2(9c_2^3 - 9c_2c_3 + c_4)^2)}{6561c_2}e_n^{12} + O(e_n^{13}), \tag{22}$$

which shows that the convergence order of the proposed three-step algorithm (14) in this contribution is twelve. □

The analysis of error shows that the method can be used as a great tool for solving nonlinear equations. By this new and simple approach, we have developed the order of convergence by one additional calculation of the function in lieu of method (5).

This new method includes three evaluations of the function and two evaluations of the first derivative per iteration.

Clearly, the classical efficiency index [1] of the proposed method (if we suppose that all the evaluations have the same computational cost) is $12^{\frac{1}{5}} \approx 1.644$ which is more than lots of existed methods, such as, $2^{\frac{1}{2}} \approx 1.414$ of Newton's method, $3^{\frac{1}{3}} \approx 1.442$ of method (1), $4^{\frac{1}{3}} \approx 1.587$ of method (2) and $6^{\frac{1}{4}} \approx 1.565$ of method (5).

We note that again the highest possible order up to now in the literature for without memory iterations with two evaluations of the first-order derivatives and three evaluations of the function is 12, and with two evaluations of the first-order derivative and two evaluations of the function is 6. Although such iterations do not satisfy the Kung-Traub conjecture on the optimality, they have some advantages, e.g. the convergence radius of Jarratt-type methods for starting points which are in the vicinity of the root but not so close are greater than that of Newton-type or Steffensen-type methods [15].

3. Examples

In this section, we employ the presented method (PM) (14), to solve some nonlinear single valued equations and compare the results with some famous existing methods. The test functions and their roots are listed as follows.

- $f_1(x) = (\sin x)^2 - x^2 + 1,\quad \alpha \approx 1.4044916482153412260350868177868680771766,$
- $f_2(x) = e^{x^2+7x-30} - 1,\quad \alpha = 3,$
- $f_3(x) = x^3 - 10,\quad \alpha \approx 2.1544346900318837217592935665193504952593,$
- $f_4(x) = x^2 - e^x - 3x + 2,\quad \alpha \approx 0.2575302854398607604553673049372417813845,$
- $f_5(x) = (x-1)^3 - 1,\quad \alpha = 2.$

Note that some optimal three-step eighth-order methods without memory have recently been developed in [16] as comes next

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)}\{\frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{2f[z_n,x_n]-f'(x_n)}\{1 + (\frac{f(z_n)}{f(x_n)})^2 \\ +\frac{f(z_n)}{f(y_n)} + (\frac{f(z_n)}{f(y_n)})^2 - \frac{3}{2}(\frac{f(y_n)}{f(x_n)})^3 - 2(\frac{f(y_n)}{f'(x_n)})^2 - \frac{f(z_n)}{f'(x_n)}\}, \end{cases} \tag{23}$$

and

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)}\{\frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{2f[z_n,x_n]-f'(x_n)}\{1 + (\frac{f(z_n)}{f(x_n)})^2 \\ +\frac{f(z_n)}{f(y_n)} + (\frac{f(z_n)}{f(y_n)})^2 - \frac{3}{2}(\frac{f(y_n)}{f(x_n)})^3 - \frac{31}{4}(\frac{f(y_n)}{f(x_n)})^4 + (\frac{f(y_n)}{f'(x_n)})^3 - 2\frac{f(z_n)}{f'(x_n)}\}, \end{cases} \tag{24}$$

and also

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)}\{\frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{2f[z_n,x_n]-f'(x_n)}\{1 + (\frac{f(z_n)}{f(x_n)})^2 \\ +\frac{f(z_n)}{f(y_n)} + (\frac{f(z_n)}{f(y_n)})^2 - \frac{3}{2}(\frac{f(y_n)}{f(x_n)})^3 - \frac{31}{4}(\frac{f(y_n)}{f(x_n)})^4 - (\frac{f(y_n)}{f'(x_n)})^2 - (\frac{f(z_n)}{f'(x_n)})^2\}. \end{cases} \tag{25}$$

High-order iterative methods are significant because numerical applications use high precision in their computations. By virtue of this, numerical computations have been performed using variable precision arithmetic in MATLAB 7.6 with 1000 significant digits. We use the following stopping criterion in our computations: $|f(x_n)| < \varepsilon$. For numerical illustrations in this section we used the fixed stopping criterion $\varepsilon = 1.10^{-1000}$.

In numerical comparisons, we have used the Newton's method of order two, Jarratt scheme of optimal order four (2), the sixth-order method of Wang et al. by (5), the new and efficient optimal eighth-order method of Wang and Liu by (WLM), relation (16) of [17], the eighth-order methods (23) and (24), alongside with our contributed algorithm (14), shown as PM. We present the numerical test results for the iterative methods in Table 1. In this Table, e.g. 3(154), shows that the absolute value of the test function after three full steps is correct up to 154 decimal places, i.e. it is zero up to 154 decimal places.

To have a fair comparison, we have tried to consider the Total Number of Evaluations (TNE) in different methods at a somewhat similar level. In some cases, we let the iteration to cycle more for reaching the convergence phase.

Table 1.
Comparison of results for presented method (PM) and some well-known methods

Test Functions	x_0	PM	(23)	(24)	WLM	(5)	(2)	NM
f_1	15	3(154)	4(2)	Div.	4(151)	4(81)	5(67)	9(33)
f_1	4.2	3(311)	3(90)	3(106)	4(736)	4(250)	5(180)	9(118)
f_1	0.3	3(4)	Div.	Div.	Div.	5(1)	5(15)	9(10)
f_2	3.2	3(219)	4(227)	4(232)	4(637)	4(247)	5(40)	9(52)
f_2	3.4	3(29)	3(26)	4(227)	4(115)	4(42)	5(31)	9(7)
f_2	2.85	3(128)	Div.	Div.	Div.	4(4)	5(114)	9(8)
f_3	0.5	3(109)	Div.	Div.	Div.	5(1)	5(29)	9(6)
f_3	0.2	3(4)	Div.	Div.	Div.	10(3)	6(5)	12(1)
f_3	9	3(307)	Div.	Div.	4(251)	4(125)	5(81)	9(32)
f_4	-300	3(18)	Div.	Div.	4(13)	4(24)	5(4)	9(1)
f_4	6	3(35)	4(222)	3(19)	4(96)	4(48)	5(53)	9(30)
f_4	-100	3(51)	Div.	Div.	4(47)	4(38)	5(16)	9(8)
f_5	6	3(228)	Div.	Div.	4(182)	4(90)	5(61)	9(24)
f_5	10	3(85)	Div.	Div.	4(64)	4(40)	5(23)	9(8)
f_5	1.3	3(237)	Div.	Div.	7(0)	4(5)	5(64)	9(18)

Note that in Table 1, "Div." represents that the iteration diverges for the considered starting points. The computational order of convergence, namely COC, which is defined as follows

$$COC \approx \frac{\ln |\frac{e_{n+1}}{e_n}|}{\ln |\frac{e_n}{e_{n-1}}|}, \tag{26}$$

where $e_n = x_n - \alpha$, is very close to 12 for PM and 8 for WLM, (23) and (24) in case of converging, which shows that numerical results are in concordance with the theory developed in the previous section. Table 1 also shows that the convergence radius of without memory iterations in which the Jarratt-type methods are employed at the beginning of them is better than that of optimal Newton-type methods. In fact, the results of Table 1 manifest that Jarratt-type methods are better choices when the starting points are in the vicinity of the root but not so close.

In the next section, we will discuss on the basins of attractions for the fourth-order method of Jarratt (2), the sixth-order method has obtained at the end of the second step of (14), i.e.

$$\begin{cases} y_n = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{2f[y_n,x_n]-f'(x_n)+a_4(y_n-x_n)^2}, \end{cases} \quad (27)$$

and also the twelfth-order method (14). According to Table 1, we do not include the comparison with eighth-order methods, because Jarratt-type high order methods have better convergence radius. Note that we need the some basic definitions before going to Section 4. Thus, now we shortly present them. Let $R : \mathbb{C} \to \mathbb{C}$ be a rational map on the Riemann sphere. Then, a periodic point z_0 of period m is such that $R^m(z_0) = z_0$ where m is the smallest such integer. And also point z_0 is called attracting if $|R'(z_0)| < 1$, repelling if $|R'(z_0)| > 1$, and neutral if $|R'(z_0)| = 1$. If the derivative is also zero then the point is called super-attracting.

Note that the Julia set of a nonlinear map $R(z)$, denoted $J(R)$, is the closure of the set of its repelling periodic points. The complement of $J(R)$ is the Fatou set $\mathbb{F}(R)$. A point z_0 belongs to the Julia set if and only if dynamics in a neighborhood of z_0 displays sensitive dependence on the initial conditions, so that nearby initial conditions lead to wildly different behavior after a number of iterations.

4. Basins of Attractions

In this section, we describe the basins of attractions for the higher order Jarratt-type methods for finding the complex roots of the polynomials $x^2 - 1$, $x^3 - 1$ and $x^4 - 1$ for $x \in \mathbb{C}$.

The iteration (14) can be written as Iterative Functions:

$$\begin{cases} y_n = \psi_{4^{th}JM}(x_n), \\ z_n = \psi_{6^{th}JM}(x_n), \\ x_{n+1} = \psi_{12^{th}JM}(x_n). \end{cases} \quad (28)$$

Bahman Kalantari coined the term "polynomiography" to be the art and science of visualization in the approximation of roots of polynomial using Iteration Functions. We describe the methods in (28), (the method at the end of its first step, the method at the end of its second step, the method at the end of its third step) to produce the polynomiographs using MATLAB 7.6.

4.1. Polynomiographs of $x^2 - 1$

We now draw the polynomiographs of $f(x) = x^2 - 1$ with roots $\alpha_1 = -1$ and $\alpha_2 = 1$. Let x_0 be the initial point. A square grid of 80000 points, composed of 400 columns and 200 rows corresponding to the pixels of a computer display would represent a region of the complex plane [18]. We consider the square $[-2, 2] \times [-2, 2]$. Each grid point is used as a starting value x_0 of the sequence $\psi_{IF}(x_n)$ and the number of iterations until convergence is counted for each gridpoint. We assign pale blue color if the iterates x_n of each grid point converge to the root $\alpha_1 = -1$ and green color if they converge to the root $\alpha_2 = 1$ in at most 100 iterations and if $|\alpha_j - x_n| < 1.e - 4$, $j = 1, 2$. In this way, the basin of attraction for each root would be assigned a characteristic color. The common boundaries of these basins of attraction constitute the Julia set of the Iteration Function. If the iterates do not satisfy the above criterion

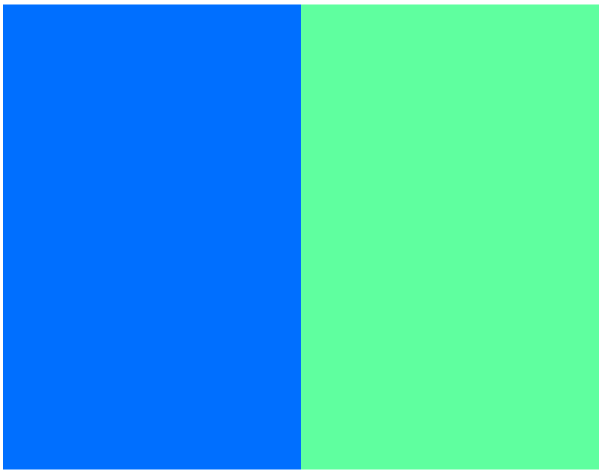

FIGURE 1. Polynomiographs of $4^{th}JM$, $6^{th}JM$ and $12^{th}JM$ methods for $f(x) = x^2 - 1$.

for convergence we assign the dark blue color.

The Iteration Functions of the higher order Jarratt-type methods are given by

$$\psi_{4^{th}JM}(x) = \frac{x^4 + 6x^2 + 1}{x^4 + 6x^2 + 1},$$

$$\psi_{6^{th}JM}(x) = \frac{x^8 + 28x^6 + 70x^4 + 28x^2 + 1}{8x^7 + 56x^5 + 56x^3 + 8x},$$

and

$$\psi_{12^{th}JM}(x)$$

$$= \frac{x^{16} + 120x^{14} + 1820x^{12} + 8008x^{10} + 12870x^8 + 8008x^6 + 1820x^4 + 120x^2 + 1}{16x^{15} + 560x^{13} + 4368x^{11} + 11440x^9 + 11440x^7 + 4368x^5 + 560x^3 + 16x}.$$

We observe as the order increases, the order of the numerator and denominator of the iteration function increases. This shows the complexity of higher order methods.

Fig. (1) shows the polynomiograph of the quadratic polynomial. We observe that it is the same for the 3 methods all starting points converge. The Julia set is the imaginary y-axis.

4.2. Polynomiographs of $x^3 - 1$

The roots are $\alpha_1 = 1$, $\alpha_2 = -0.5000 - 0.8660i$ and $\alpha_3 = -0.5000 + 0.8660i$. Each grid point over the region $[-2, 2] \times [-2, 2]$ is colored accordingly, brownish yellow for convergence to α_1, blue for convergence to α_2 and pale green for convergence to α_3. We use the same conditions for convergence as in the quadratic polynomial.

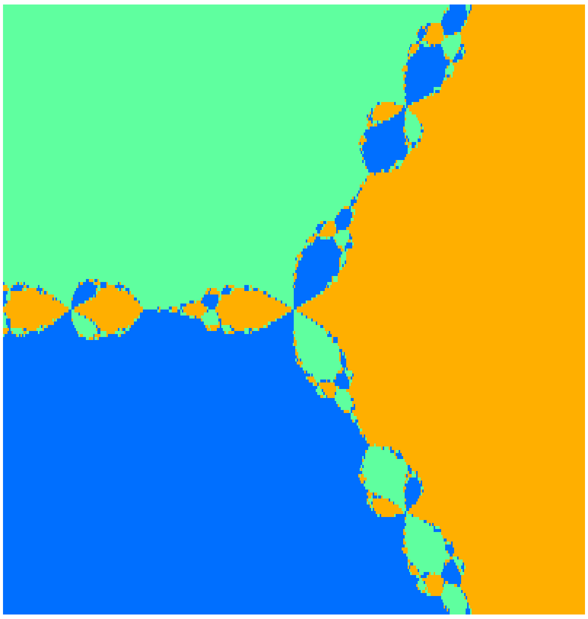

FIGURE 2 (a) Polynomiographs of $4^{th} JM$ method for $f(x) = x^3 - 1$.

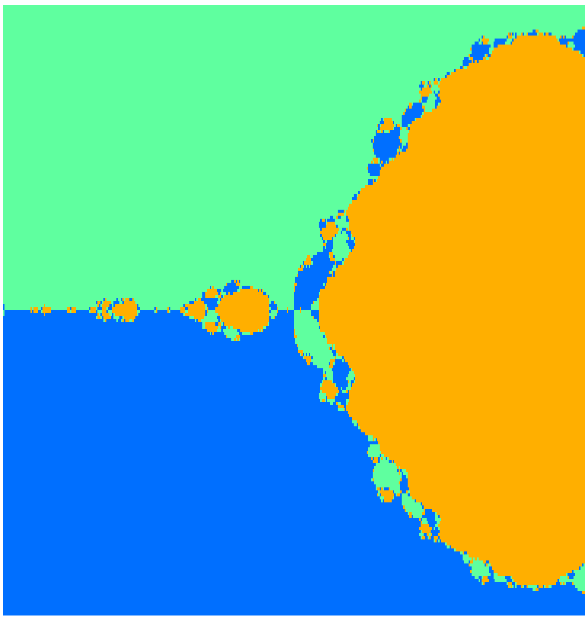

FIGURE 2 (b) Polynomiographs of $6^{th} JM$ method for $f(x) = x^3 - 1$.

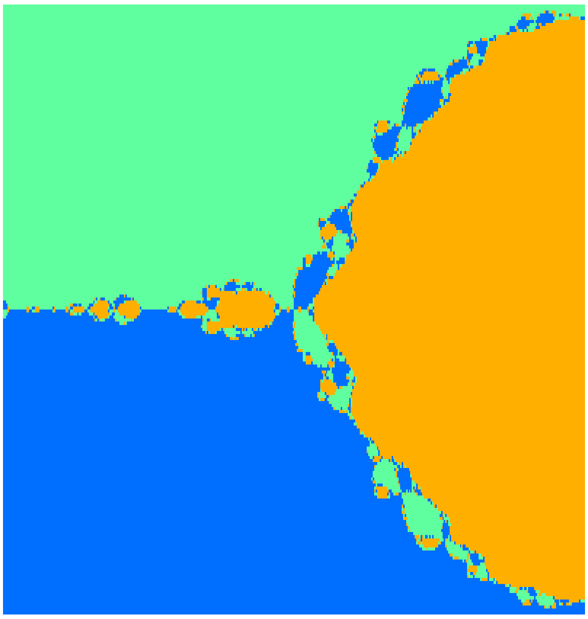

FIGURE 2 (c) Polynomiographs of $12^{th} JM$ method for $f(x) = x^3 - 1$.

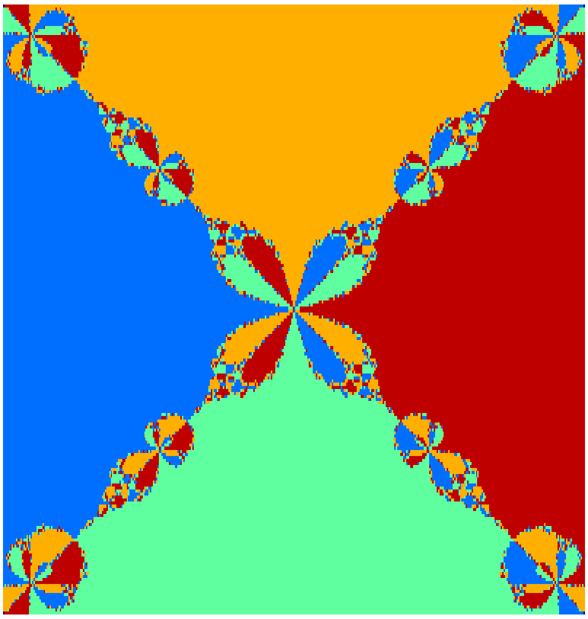

FIGURE 3 (a) Polynomiographs of $4^{th} JM$ method for $f(x) = x^4 - 1$.

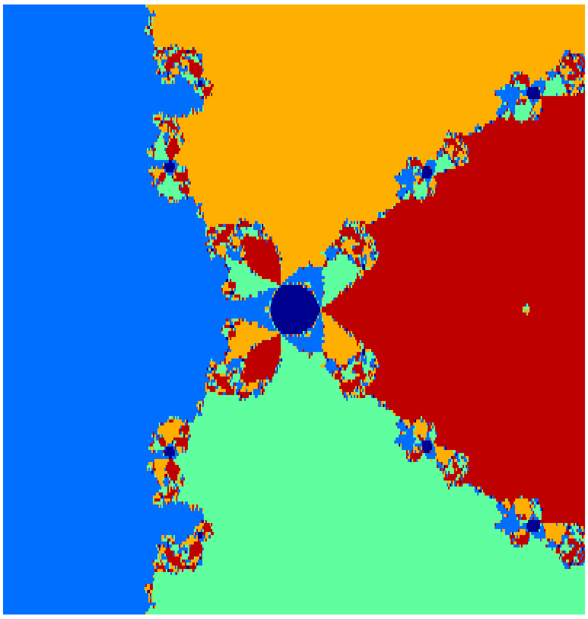

FIGURE 3 (b) Polynomiographs of $6^{th} JM$ method for $f(x) = x^4 - 1$.

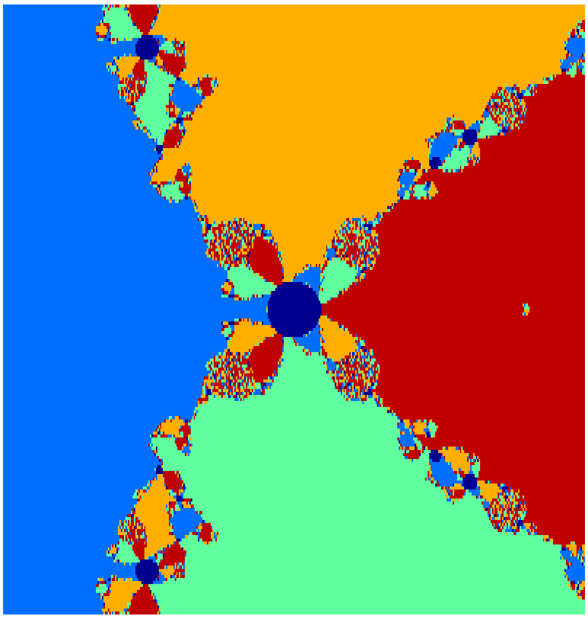

FIGURE 3 (c) Polynomiographs of $12^{th} JM$ method for $f(x) = x^4 - 1$.

The Iteration Functions of the higher order Jarratt methods for cubic polynomial contain higher order polynomials in both the numerator and denominator:

$$\psi_{4^{th}JM}(x) = \frac{O(x^9)}{O(x^8)},\ \psi_{6^{th}JM}(x) = \frac{O(x^{27})}{O(x^{26})},\ \psi_{12^{th}JM}(x) = \frac{O(x^{81})}{O(x^{80})}.$$

Fig. 2 (a)-(c) show the polynomiographs of the methods for the cubic polynomial. All starting points are convergent. Compared to $4^{th}JM$ method, the poynomiographs of the $6^{th}JM$ and $12^{th}JM$ are deformed and the basins of attractions for the roots α_2 and α_3 are bigger. The polynomiograph of $12^{th}JM$ method is almost similar to that of $6^{th}JM$ but the basins of attractions is smaller.

4.3. Polynomiographs of $x^4 - 1$

The roots are $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = i$ and $\alpha_4 = -i$. Note that we used blue for α_2, brown for α_3, red for α_1, green for α_4 in the Polynomiographs in this case. The Iteration Functions of the higher order Jarratt methods for quartic polynomial contain higher order polynomials in both the numerator and denominator:

$$\psi_{4^{th}JM}(x) = \frac{O(x^{16})}{O(x^{15})},\ \psi_{6^{th}JM}(x) = \frac{O(x^{72})}{O(x^{71})},\ \psi_{12^{th}JM}(x) = \frac{O(x^{288})}{O(x^{287})}.$$

Fig. (3) (a)-(c) show the polynomiographs of the methods for the quartic polynomial. There are diverging points for $6^{th}JM$ and $12^{th}JM$ methods. The $12^{th}JM$ method have more diverging points and its fractal shape is bigger. The $6^{th}JM$ and $12^{th}JM$ methods have a larger basin of attraction for the root α_2 but they have smaller basins of attractions for other roots when compared to $4^{th}JM$ method.

From this comparison based on the basins of attractions, we could generally say Jarratt-type methods are reliable in solving nonlinear equations. Anyhow, we should point out that for some roots the higher-order Jarratt-type methods are better while for some other roots of a nonlinear equation in the complex plane the Jarratt method (2) is better.

5. Conclusion

As Table 1 and the error analysis have manifested, our novel second derivative-free iterative scheme, which includes three evaluations of the function and two of its first-order derivative, is efficient and can be used as a great tool in solving single variable nonlinear equations. The efficiency index of the proposed method is $12^{\frac{1}{5}} \approx 1.644$, which is higher than a lot of well-known iterative methods. However, we should mention that the convergence behavior of the considered multi-point methods strongly depends on the structure of tested functions and the accuracy of initial approximations.

We end this paper with an attraction to one of the features of Jarratt-type methods. Although multi-point without memory methods of the same order (e.g. 8) and the same computational cost show a similar convergence behavior and produce results of approximately same accuracy, this equalized behavior is especially valid when such methods are Newton-like methods in essence. As can be seen from Table 1 and the discussion in Section 4, Jarratt-type method have greater convergence radius according to their first step.

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