On a general sequence of Durrmeyer operators

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Abstract. In this paper we establish direct and inverse theorems in simultaneous approximation using weighted Ditzian-Totik modulus of smoothness for a generalized sequence of Bernstein-Durrmeyer polynomials. The particular case are Szász Durrmeyer and Baskakov Durrmeyer operators.

2010 Mathematics Subject Classification. Primary 41A28; Secondary 26A15.
Key words and phrases. Simultaneous approximation, Ditzian-Totik modulus of continuity.

1. Introduction

With the aim of approximating Lebesgue integrable functions on $[0, 1]$, Durrmeyer [5] introduced an integral modification of the well known Bernstein polynomials and were extensively studied by Derrienic [2]. Later in the year 1989 Heilmann [11] considered a general sequence of Durrmeyer operators defined on $[0, \infty)$ for $n > c$ and $x \in [0, \infty)$ as

$$V_{n,r}(f, x) = \int_0^\infty K_{n,r}(x, t)f(t)dt,$$

where the kernel $K_{n,r}$ is given by

$$K_{n,r}(x, t) = \begin{cases} 
(n-c) \sum_{k=0}^\infty p_{n,k}(x)p_{n,k}(t), & r = 0, \\
(n-c)\beta(n,r,c) \sum_{k=0}^\infty p_{n+cr,k}(x)p_{n-cr,k+r}(t), & r > 0.
\end{cases}$$

where $r, n \in \mathbb{R}$, $p_{n,k}(x) = \left(\frac{-x}{k!}\right)^{n} \psi^{(n)}(x)$ and $\beta(n,r,c) = \frac{r^{-1}}{\prod_{l=0}^{r-1} \frac{n+cl}{n-c(l+1)}}$.

The family of operators $V_{n,r}(f, x)$ is linear and positive. The special case $c = 1$, and $r = 0$ was considered very recently by Deo [3] wherein he studied the local asymptotic formula and an error estimation in simultaneous approximation for generalized Durrmeyer operators, which were introduced by [11]. There was several misprints in [3]. The authors in [7] corrected them and obtained local error estimates in simultaneous approximation by the operators $V_{n}(f)(x)$. In this paper we extend the work in [7] and obtain direct and inverse theorems in simultaneous approximation using weighted Ditzian-Totik modulus of smoothness. In the end we mention some of the particular cases of the main theorem.

Received June 02, 2011. Revision received October 25, 2011.
For the functions $W(a)$ and $(b)$ follow from direct calculations and $(c)$ follows in view of the relation

\[ \text{Lemma 3.1.} \]

$W_{m,n}(x)$ given by

\[ W_{m,n}(x) = \sum_{k=0}^{\infty} \left( \frac{k}{n + cr} - x \right) p_{n+cr,k}(x), \]

we have:

(a) $W_{0,n}(x) = 1$, $W_{1,n}(x) = x(n + cr - 1)$;
(b) $(n + cr)W_{m+1,n}(x) = \varphi^2(x) \left\{ W'_{m,n}(x) + m W_{m-1,n}(x) \right\}$, where $m \geq 1$, $x \in [0, \infty)$ and $\varphi^2(x) = x(1 + cr)$;
(c) $W_{2m,n}(x) \leq C_m n^{-m+1} \left( \delta_{n,m}(x)^2 + n^{-1} \right)$, for all $m \in \mathbb{N}$, where $C_m$ is a constant that depends on $m$ and $\delta_n(x) = \varphi(x) + \frac{1}{n}m$.

Proof. (a) and (b) follow from direct calculations and (c) follows in view of the relation $\varphi^2(x)p'_{n+cr,k}(x) = \left( \frac{k}{n + cr} - x \right) p_{n+cr,k}(x)$, the recurrence relation $(b)$ together the
The values of equivalencies:

\[
\delta_n(x) \sim \begin{cases} 
\frac{1}{\sqrt{n}} & \text{for } x \in \left[0, \frac{1}{n}\right] = E_n^L \\
\varphi(x) & \text{for } x \in \left(\frac{1}{n}, \infty\right) = E_n^R
\end{cases}
\]

\[\square\]

Following is a Lorentz type lemma:

**Lemma 3.2.** [10] There exist polynomials \( q_{i,j,r}(x) \) independent of \( n \) and \( k \) such that

\[
\varphi^{2r}(x) \frac{d^r}{dx^r} p_{n+cr,k}(x) = \sum_{2i+j \leq r, i,j \geq 0} (n + cr)^i [k - (n + cr)]^j q_{i,j,r}(x) p_{n+cr,k}(x).
\]

**Lemma 3.3.** [1] Let \( \Omega \) be monotone increasing on \([0, c]\). Then \( \Omega(t) = O(t^\alpha), \ t \to 0^+ \), if for some \( 0 < \alpha < r \) and all \( h, t \in [0, c] \)

\[
\Omega(h) < M \left[ t^\alpha + (h/t)^r \Omega(t) \right].
\]

**Lemma 3.4.** Suppose \( f \) is \( s \) times differentiable on \([0, \infty)\) such that \( f^{(s-1)}(t) = O(t^\alpha), \) for some \( \alpha > 0 \) as \( t \to \infty \). Then for any \( r, s \in \mathbb{R} \) and \( n > \alpha + cs \), we have

\[
D^s V_{n,r}(f, x) = V_{n,r+s}(D^s f, x).
\]

We make use of the Lemma 3.4 to define the operators \( V_{n,r,s}(f, x) \) as follows

\[
V_{n,r,s}(f, x) = \int_0^\infty K_{n,r+s}(t) f(t) \, dt.
\]

Obviously, \( V_{n,r}^{(s)}(f, x) = V_{n,r,s}(f^{(s)}, x) \) and \( V_{n,r,s} \) are linear positive operators.

**Lemma 3.5.** For \( m \in \mathbb{N}^0 \), if we define the \( m \)-th order moment for the operators \( V_{n,r,s} \) by \( T_{n,m}(x) = V_{n,r,s}((t-x)^m, x) \) then

\[
T_{n,0}(x) = \frac{(n-c)(n,r+s,c)}{(n-c)(n+1)}, \quad T_{n,1}(x) = \frac{(n-c)(n,r+s,c)(r+s+1)(1+2cr)}{(n-c)(n+1)(n-c)(n+s+2)}, \text{ and there holds the recurrence relation}
\]

\[
(n - (m + r + s + 2)c)T_{n,m+1}(x) + n(1 - x)T_{n,m}(x)
\]

\[
= (n + r + s + 1)(1 + 2cr)T_{n,m}(x) + 2nc\phi^2(x)T_{n,m-1}(x) + \varphi^2(x)T_{n,m}(x).
\]

**Proof.** The values of \( T_{n,0}(x) \) and \( T_{n,1}(x) \) follow from straight forward calculations. Writing \( \alpha_{n,r+s} = (n-c)(n,r+s,c) \) and using the relation \( \varphi^2(x)\phi_{n+cr,k}(x) = \left( \frac{k}{n+cr} -
\]

From Lemma 3.5, and in view of 

\[ V_{n,r,s}(x) = 0 \]

we obtain

\[
\varphi^2(x) \left( T_{n,m}(x) + mT_{n,m-1}(x) \right)
\]

\[
= \alpha_{n,r+s} \sum_{k=0}^{\infty} \varphi^2(x)p_{n+r,k}(x) \int_{0}^{\infty} p_{n-c(r+s),k+r+s}(t)(t-x)^m, dt
\]

\[
= \alpha_{n,r+s} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{\infty} \varphi^2(t)p_{n-c(r+s),k+r+s}(t)(t-x)^m, dt
\]

\[
+ \left( n - (r+s)c \right) T_{n,m+1}(x) + \left( n - r - s - (n+2c(r+s))x \right) T_{n,m}(x)
\]

\[
= \alpha_{n,r+s} \sum_{k=0}^{\infty} \left\{ \varphi^2 + (1+2cx)(t-x) + c(t-x)^2 \right\} \times
\]

\[
\times \int_{0}^{\infty} p_{n-c(r+s),k+r+s}(t)(t-x)^m, dt
\]

\[
+ \left( n - (r+s)c \right) T_{n,m+1}(x) + \left( n - r - s - (n+2c(r+s))x \right) T_{n,m}(x)
\]

Now, integration by parts and rearrangements of the terms gives the recurrence relation.

\[ \square \]

**Corollary 3.1.** From Lemma 3.5, and in view of \( \alpha_{n,r+s} = O(1) \), it follows that

\[
T_{n,2}(x) = \frac{\alpha_{n,r+s}}{n-c(r+s+1)} \frac{2(n-c)\varphi^2(x) + (r+s+1)(r+s+2)(1+2cx)^2}{\{n-c(r+s+1)\}^2}
\]

This gives \( T_{n,2}(x) \leq C\delta_n^2(x) \), where \( \delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}} \).

Our next result is a Bernstein type lemma which we shall use in inverse theorem.

**Lemma 3.6.** If \( f \in L_B[0,\infty) \), \( f^{(l-1)} \in AC_{loc}(0,\infty) \) and \( l \in \mathbb{N} \) then, there hold the inequality:

\[
\left| V_{n,r,s}^{(l)}(f,x) \right| \leq M \varphi^{-M}(x) \| \varphi^M f^{(l)} \|
\]

where \( M = M(l) \) is a constant that depends on \( r \) but is independent of \( f \) and \( n \).

**Proof.** By the assumption we can write

\[
f(t) = \sum_{\nu=0}^{l-1} \frac{f^{(\nu)}(x)(t-x)^\nu}{\nu!} + R_l(f,t;x), \]

where

\[
R_l(f,t;x) = \frac{1}{(\nu-1)!} \int_{0}^{t} (t-u)^{l-1} f^{(\nu)}(u) du.
\]

Since, from Lemma 3.5 it follows that \( V_{n,r,s}((t-x)^\nu,x) \) are polynomials in \( x \) of degree \( \nu \) so that

\[
V_{n,r,s}^{(r)}((t-x)^\nu,x) = 0
\]

for \( \nu < r \), it is sufficient to consider \( V_{n,r,s}^{(l)}(R_l(f,t;x),x) \).
We write $M = \sup_{\frac{t}{1+c}, \delta \neq 0} \|q_{i,j}(x)\|$ and make use of Hölder’s inequalities for integration and summation, the value $\int_{\mathbb{R}} p_{n-c(r+s), k+r+s}(t) = \frac{1}{n+c(k-1)}$ and Lemma 3.1, Lemma 3.5 to obtain following estimates

\[
I_1 \leq M\frac{\|\varphi^M f(t)\|}{(l-1)!\varphi^{2l+2M}(x)} \sqrt{\alpha_{n,r+s}} \sum_{2+i \leq l, i \geq 0} \left( \sum_{k=0}^{\infty} \left( \frac{k}{n+c} - \frac{k}{n+c} \right)^{2j} p_{n+c(r+s), k}(x) \right)^{\frac{1}{2}} \times
\]

\[
\times \frac{(n+c)^{l+j}}{\sqrt{n+c(k-1)}} \left( \alpha_{n,r+s} \sum_{k=0}^{\infty} \frac{p_{n+c(r+s), k}(x)}{l+1} \int_0^{\infty} (t-x)^{2j} p_{n-c(r+s), k+r+s}(t) dt \right)^{\frac{1}{2}} \leq M \frac{\|\varphi^M f(t)\|}{(l-1)!\varphi^{2l+2M}(x)} \sqrt{n+c(k-1)} \sum_{2+i \leq l, i \geq 0} (n+c)^{l+j} \left( n^{-j+1} \delta_n^j(x) \right)^{\frac{1}{2}} n^{-1/2} \delta_n(x) \leq M \varphi^{-M}(x)\|\varphi^M f(t)\|,
\]

where we have used the equivalence $\delta_n(x) \sim \frac{1}{\sqrt{n}}$ for $x \in \mathbb{E}_n$ and for $x \in \mathbb{E}_n$, $\delta_n(x) \sim \varphi(x)$. Now it follows by direct calculations that $\int_0^{\infty} p_{n-c(r+s), k+r+s}(t)(1+ct)^{-\lambda} dt \leq M(1+ct)^{-1/2}$. Therefore, we get

\[
I_2 \leq \varphi^M \frac{\|\varphi^M f(t)\|}{(l-1)!\varphi^{2l+2M}(x)} \alpha_{n,r+s} \sum_{2+i \leq l, i \geq 0} (n+c)^{l+j} |k - (n+c)x| \frac{|q_{i,j}(x)|}{\varphi^M(x)} \times
\]

\[
\times \frac{(n+c)^{l+j}}{\sqrt{n+c(k-1)}} \left( \alpha_{n,r+s} \sum_{k=0}^{\infty} \frac{p_{n+c(r+s), k}(x)}{l+1} \int_0^{\infty} (t-x)^{2j} p_{n-c(r+s), k+r+s}(t) dt \right)^{\frac{1}{2}} \leq \varphi^{-M}(x)\|\varphi^M f(t)\|.
\]
If we introduce the auxiliary operators approximation by the operators $M$.  

**Main Results**

Then, we have 

$$\left\| \varphi_M f(t) \right\| \leq M \left\| \varphi_M f(t) \right\| \sum_{\alpha, \beta \geq 0} \alpha_{\alpha, \beta} \left( \frac{n}{n + c(r + s)} \right)^{\alpha} \right\| n, r \right\| V \left( \frac{s}{2^{\alpha}} \right) \left( \frac{n}{n + c(r + s)} \right)^{\beta} \left( \frac{k}{n + c(r + s)} \right)^{\beta} \right\| p_{n, c(r + s)}(x) \times$$

$$\left( \int_0^\infty p_{n, c(r + s)}(x)(t - x)^{2j}dt \right)^{\frac{1}{2}} \left( \int_0^\infty p_{n, c(r + s)}(x)(1 + ct)^{-\lambda}dt \right)^{\frac{1}{2}} \leq M \left\| \varphi_M f(t) \right\| \sum_{\alpha, \beta \geq 0} \alpha_{\alpha, \beta} \left( \frac{n}{n + c(r + s)} \right)^{\alpha} \right\| n, r \right\| V \left( \frac{s}{2^{\alpha}} \right) \left( \frac{n}{n + c(r + s)} \right)^{\beta} \left( \frac{k}{n + c(r + s)} \right)^{\beta} \right\| p_{n, c(r + s)}(x) \times$$

$$\left( \int_0^\infty p_{n, c(r + s)}(x)(t - x)^{2j}dt \right)^{\frac{1}{2}} \leq M \left\| \varphi_M f(t) \right\|$$

**Lemma 3.7**. If $f \in L_B[0, \infty)$ and $r \in N$ then, there hold the inequalities:

$$\left\| \varphi^{(r)}(f, x) \right\| \leq M n^{r/2} \delta_n(x) \varphi^{-2r}(x) \| f \|,$$

where $M = M(r)$ is a constant that depends on $r$ but is independent of $f$ and $n$.

The proof of is similar to Lemma 3.6. □

### 4. Main Results

In this section we establish the direct and inverse theorems in simultaneous approximation by the operators $V_{n,r}(f, x)$.

**Theorem 4.1.** If $f \in L_B[0, \infty)$, $f^{(s-1)} \in AC_{loc}(0, \infty)$, $0 \leq \lambda \leq 1$, $0 < \alpha < 2$ and $\varphi(x) = \sqrt{x(1 + cx)}$ then, we have

$$\left\| V_{n,r}(f, x) - f^{(s)}(x) \right\| \leq M \left\| \varphi^{2\alpha} f^{(s)}(x) \right\| \omega_n^{2\alpha} \left( \frac{n - c(r + s + 1)}{n} \right)^{\lambda} \left( \frac{n - c(r + s + 2)}{n} \right)^{\lambda}.$$

**Proof.** Let us take $g_{n,r, \lambda} = g \in W_{2,\lambda}$ such that

$$\left\| f^{(s)} - g \right\| + \left( n - \frac{1}{2} \right) \delta_n^{1-\lambda}(x) \varphi^{2\alpha} g'' \| \leq 2K_{2,\varphi^{2\alpha}} \left( f^{(s)}(x), \left( n - \frac{1}{2} \right) \delta_n^{1-\lambda}(x) \right)^2. \quad (1)$$

We introduce the auxiliary operators $\tilde{V}_{n,r,s}$ defined by

$$\tilde{V}_{n,r,s}(f, x) = \frac{1}{C_{n,r}} \left[ V_{n,r,s}(f, x) - f^{(s)}(x + z) + f^{(s)}(x) \right], \quad (2)$$

where $z = V_{n,r,s}(t - x, x) = \frac{(n-c)(n-r+s+c)(r+s+1)(1+2cx)}{(n-c(r+s+1))(n-c(r+s+2))}$, $C_{n,r} = V_{n,r,s}(1, x) = (n - c)(n-r+s+c)/(n-c(r+s+1))$ and $x \in [0, \infty)$. The operators $\tilde{V}_{n,r,s}$ are linear and preserve the linear functions. Further, $\tilde{V}_{n,r,s}(1, x) = 1$, $\tilde{V}_{n,r,s}(t - x, x) = 0$ and from 2 it follows that $|\tilde{V}_{n,r,s}(f^* - g, x)| \leq M \| f^* - g \|$. Therefore,

$$V_{n,r,s}(f, x) - f^{(s)}(x)$$

$$= C_{n,r} \left[ V_{n,r,s}(f^* - g, x) + \{ g(x) - f^*(x) \} \right]$$

$$\tilde{V}_{n,r,s}(g, x) - g(x) + \{ C_{n,r} - 1 \} f^{(s)}(x) + f^{(s)}(x + z) - f^{(s)}(x)$$
Hence, in view of the limit $C_{n,r} \to 1$ as $n \to \infty$, we get

$$|V_n^{(s)}(f, x) - f(x)| \leq M \left( 4\|f^{(s)}\| + |\hat{V}_{n,r,s}(g, x) - g(x)| + \omega(f^{(s)}, z) \right).$$

Using the smoothness of $g$, and in view of $\hat{V}_{n,r,s}(t - x, x) = 0$, we get

$$|\hat{V}_{n,r,s}(g, x) - g(x)| \leq M \int \left( x + z - u \right) g''(u) du,$$

where $R_2(g, t, x) = \int (t - u) g''(u) du$. Now following holds (see [4] p. 141.)

$$|R_2(g, t, x)| \leq \frac{|t - x|}{x^4} \left( \frac{1}{1 + cx^2} + \frac{1}{1 + ct} \right) \left( \int \frac{\varphi^{2\lambda}(u)}{x^4} |g''(u)| du \right).$$

Also it can be verified (cf. [6]) that $V_{n,r,s+1}((1 + ct)^{-m}, x) \leq C(1 + cx)^{-m}$ and $V_{n,r,s}((t - x)^4, x) \leq C(n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x))^2$. Therefore, we get

$$|V_{n,r,s}(R_2(g, t, x)| \leq \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)} V_{n,r,s}((t - x)^2, x) + \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)} V_{n,r,s}((t - x)^2, x)$$

$$+ \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)} (V_{n,r,s}((t - x)^4, x) + (V_{n,r,s}((1 + ct)^{-2\lambda}, x) \leq \frac{M\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)} \left( n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x) \right)^2.$$

Since, $z \leq C \left( n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x) \right)^2$ for all values of $x$, therefore we obtain

$$\left| \int \frac{x+z}{x} (x + z - u) g''(u) du \right| \leq (n^{-\frac{1}{3}} \delta_n^{1-\lambda}(x))^4 \|g''\|.$$

Collecting these estimates, we get

$$|\hat{V}_{n,r,s}(g, x) - g(x)| \leq M \|\varphi^{2\lambda}g''\| \left( n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x) \right)^2 + (n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x))^4 \|g''\|.$$

Therefore, we have

$$|V_n^{(s)}(f, x) - f^{(s)}(x)| \leq M \left( \|f^{(s)}\| + \|\varphi^{2\lambda}g''\| \left( n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x) \right)^2 + (n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x))^4 \|g''\| \right) + \omega(f^{(s)}, z).$$

This in view of equivalence of $K_{2, \varphi^1}(f, t^2)$ and $\omega_{f^2}(f, t)$ gives

$$|V_n^{(s)}(f, x) - f^{(s)}(x)| \leq M \left( K_{2, \varphi^1}(f, (n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x))^2) + \omega(f^{(s)}, z) \right) \leq M \omega_{f^2}(f, (n^{-\frac{2}{3}} \delta_n^{1-\lambda}(x)) + \omega(f^{(s)}, z).$$

This completes the proof of the theorem.
Corollary 4.1. Now, using Lemma 2.3 [14], it follows that $\omega_{\varphi^2}(f, t) = O(t^\alpha)$, $0 < \alpha$ implies that $\omega(f^{(s)}, t) = O(t^{\alpha(1-\lambda)})$ for $0 < 1 - \lambda < \frac{2}{\alpha}$. Therefore, $\omega_{\varphi^2}(f, t) = O(t^\alpha)$ implies $|V_{n,r}^{(s)}(f, x) - f^{(s)}(x)| = O(t^\alpha)$.

Theorem 4.2 (Inverse). Let $f \in L_B[0, \infty)$, $0 \leq \lambda \leq 1$, $0 < \alpha < 2$ and $\varphi(x) = \sqrt{x(1+cx)}$. Then, there holds the implication:

$$|V_{n,r}^{(s)}(f, x) - f^{(s)}(x)| = O\left(n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)\right) \Rightarrow \omega_{\varphi^2}(f, x) = O(t^\alpha).$$

Proof. We have

$$\left|\nabla_{h\varphi^2(x)}(f^{(s)}(x))\right| \leq \nabla_{h\varphi^2(x)}(f^{(s)}(x) - V_{n,r}^{(s)}(f, x)) + \nabla_{h\varphi^2(x)}V_{n,r}^{(s)}(f^{(s)}, x) \leq M\left(n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)\right) + \int_{-\frac{h\varphi^2(x)}{2}}^{\frac{h\varphi^2(x)}{2}} \int_{-\frac{h\varphi^2(x)}{2}}^{\frac{h\varphi^2(x)}{2}} V_{n,r}^{(s)}(f^{(s)} - g, x + u + v)du \, dv \leq M\left(n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)\right) + \int_{-\frac{h\varphi^2(x)}{2}}^{\frac{h\varphi^2(x)}{2}} \int_{-\frac{h\varphi^2(x)}{2}}^{\frac{h\varphi^2(x)}{2}} V_{n,r}^{(s)}(f^{(s)} - g, x + u + v)du \, dv.$$

Using Lemma 3.6, and Lemma 3.7, we obtain

$$\omega_{\varphi^2}(f, h) \leq M\left(n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)\right)^{\alpha} + \left(h\varphi^2(x)\right)^2 \left(\varphi^{-2\lambda}\left(n^{1/2}\delta_n^{1-\lambda}(x)\right)^{2\alpha} f^{(s)} - g\right) + \left(\varphi^{-2\lambda}\varphi^2 f^{(s)}\right)^{2\alpha} \leq M\left(n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)\right)^{\alpha} \left(\frac{h}{n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)}\right)^{2\alpha} \left(\varphi^{-2\lambda}\varphi^2 f^{(s)}\right)^{2\alpha} \leq M\left(n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)\right)^{\alpha} \left(\frac{h}{n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)}\right)^{2\alpha} \omega_{\varphi^2}(f, n^{-\frac{s}{2}}\delta_n^{1-\lambda}(x)).$$

Using Lemma 3.3 this implies $\omega_{\varphi^2}(f, t) = O(t^\alpha)$.

Remark 4.1. Analogous to Theorem 1, [6] we can obtain the corresponding theorem for the range $0 < \alpha < 1$ while for $s = 0$ from Theorem 4.1 and Theorem 4.2 we obtain following theorem for the range $0 < \alpha < 2$:

Theorem 4.3. Let $f \in L_B[0, \infty)$, $\varphi(x) = \sqrt{x(1+cx)}$, $0 < \lambda \leq 1$ and $0 < \alpha < 2$. Then, there holds the implication (i) $\Leftrightarrow$ (ii) in the following statements:

(i) $|V_{n,r}(f, t) - f(x)| = O\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^{\alpha}$

(ii) $\omega_{\varphi^2}(f, t) = O(t^\alpha)$.

Remark 4.2. We obtain following operators as the special cases of these operators:

For $c = 0, r = 0$ and $\phi_n(x) = e^{-nx}$, we get the Szász-Mirakyan-Durrmeyer operators (see [8], [9], [13]).

For $c = 1, r = 0$ and $\phi_n(x) = e^{-nx}$, we obtain the Baskakov-Durrmeyer operators.
For \( c = 0 \), and \( \phi_n(x) = e^{-nx} \), we get the Szász-Durrmeyer operators (see [13]).

For \( c > 1 \), \( r = 0 \) and \( \phi_n(x) = (1 + cx)^{-n/c} \), we obtain general Baskakov-Durrmeyer operators (see [11]).

For \( c = -1 \), \( r = 0 \) and \( \phi_n(x) = (1 - x)^{-n} \), we obtain Bernstein-Durrmeyer operators (see [5], [12]).

References