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On a general sequence of Durrmeyer operators

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Abstract. In this paper we establish direct and inverse theorems in simultaneous approximation using weighted Ditzian-Totik modulus of smoothness for a generalized sequence of Bernstein-Durrmeyer polynomials. The particular case are Szász Durrmeyer and Baskakov Durrmeyer operators.

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1. Introduction

With the aim of approximating Lebesgue integrable functions on $[0,1]$, Durrmeyer [5] introduced an integral modification of the well known Bernstein polynomials and were extensively studied by Derrienic [2]. Later in the year 1989 Heilmann [11] considered a general sequence of Durrmeyer operators defined on $[0,\infty)$ for $n > c$ and $x \in [0,\infty)$ as

$$V_{n,r}(f,x) = \int_0^\infty K_{n,r}(x,t) f(t) dt,$$

where the kernel $K_{n,r}$ is given by

$$K_{n,r}(x,t) = \begin{cases} (n-c)\sum\limits_{k=0}^{\infty} p_{n,k}(x)\, p_{n,k}(t), & r=0, \\ (n-c)\beta(n,r,c)\sum\limits_{k=0}^{\infty} p_{n+cr,k}(x)\, p_{n-cr,k+r}(t), & r>0. \end{cases}$$

where $r, n \in \mathbb{R}$. $p_{n,k}(x) = \frac{(-x)^k}{k!}\phi_n^{(k)}(x)$ and $\beta(n,r,c) = \prod\limits_{l=0}^{r-1} \frac{n+cl}{n-c(l+1)}$.
The family of operators $V_{n,r}(f,x)$ is linear and positive. The special case $c = 1$, and $r = 0$ was considered very recently by Deo [3] wherein he studied the local asymptotic formula and an error estimation in simultaneous approximation for generalized Durrmeyer operators, which were introduced by [11]. There was several misprints in [3]. The authors in [7] corrected them and obtained local error estimates in simultaneous approximation by the operators $V_n(f)(x)$. In this paper we extend the work in [7] and obtain direct and inverse theorems in simultaneous approximation using weighted Ditzian-Totik modulus of smoothness. In the end we mention some of the particular cases of the main theorem.

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2. Definitions and Notations

The $K-$functional $\overline{K}_{2,\varphi^\lambda}(f^{(s)},t)$ and the corresponding Ditzian-Totik modulus of smoothness $\omega^2_{\varphi^\lambda}(f^{(s)},t)$(cf. [4]) we shall use in our study are defined as:

Let $f^{(s)}\in C_B[0,\infty)$, the class of bounded and continuous functions on $[0,\infty)$, $0\leqslant \lambda\leqslant 1$ $\varphi(x)=\sqrt{x(1+cx)}$, then the Ditzian-Totik weighted modulus of smoothness of second order is given by

$$\omega^2_{\varphi^\lambda}(f^{(s)},t) \;=\; \sup_{0<h\leqslant t}\ \sup_{x+2h\varphi^\lambda(x)\geqslant 0}\big\|\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)}f^{(s)}(t)\big\|,$$

where the second order forward difference of the function $f^{(s)}$ at a point x is given by

$$\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)}f^{(s)}(x)=\begin{cases}\sum\limits_{j=0}^{2}(-1)^{2-j}\binom{2}{j}f^{(s)}\big(x+jh\varphi^\lambda(x)\big)\\ \text{if } x,x+2h\varphi^\lambda(x)\in[0,\infty)\\ \\ 0,\quad \text{otherwise}\end{cases}$$

and

$$\overline{K}_{2,\varphi^\lambda}(f^{(s)},t^2)=\inf_{g\in W_{2,\lambda}}\big\{\|f^{(s)}-g\|+t^2\|\varphi^{2\lambda}g''\|+t^4\|g''\|\big\}$$

where the class $W_{2,\lambda}$ is given by $\big\{g:\|\varphi^{2\lambda}g''\|<\infty, g'\in AC_{loc}(0,\infty)\big\}$ and $\varphi(x)=\sqrt{x(1+cx)}$ is an admissible weight function of Ditzian-Totik modulus of smoothness. It is easy to see that $\varphi^\lambda(x)$ satisfies properties (I)-(III) p.8 [4]. Moreover, the following equivalence is well known (p. 11, [4])

$$\omega^2_{\varphi^\lambda}(f^{(s)},t)\sim\overline{K}_{2,\varphi^\lambda}(f^{(s)},t^2).$$

By $\mathbb{N}^0$ we mean the set of non-negative integers and the constant M is not the same at each occurrence. In the present chapter, we study the rate of convergence in simultaneous approximation for the operators $V_{n,r}(f,x)$ for functions in class $L_B[0,\infty)$.

3. Some Lemmas

The contents of this section are some auxiliary results and lemmas which will be used in our main theorems.

Lemma 3.1. *For the functions $W_{m,n}(x)$ given by*

$$W_{m,n}(x)\equiv\sum_{k=0}^{\infty}\left(\frac{k}{n+cr}-x\right)^m p_{n+cr,k}(x),$$

we have :

(a) $W_{0,n}(x)=1$, $W_{1,n}(x)=x(n+cr-1)$;

(b) $(n+cr)W_{m+1,n}(x)=\varphi^2(x)\big\{W'_{m,n}(x)+\,m\,W_{m-1,n}(x)\big\}$, *where* $m\geqslant 1, x\in[0,\infty)$ *and* $\varphi^2(x)=x(1+cx)$;

(c) $W_{2m,n}(x)\leqslant C_m\,n^{-m+1}\big(\delta_n^{2m}(x)+n^{-1}\big)$, *for all* $m\in\mathbb{N}^0$, *where* C_m *is a constant that depends on* m *and* $\delta_n(x)=\varphi(x)+\frac{1}{\sqrt{n}}$.

Proof. (a) and (b) follow from direct calculations and (c) follows in view of the relation $\varphi^2(x)p'_{n+cr,k}(x)=\big(\frac{k}{n+cr}-x\big)p_{n+cr,k}(x)$, the recurrence relation (b) together the equivalencies:

$$\delta_n(x) \sim \begin{cases} \frac{1}{\sqrt{n}} \text{ for } x \in \left[0, \frac{1}{n}\right] = E_n \\ \varphi(x) \text{ for } x \in \left(\frac{1}{n}, \infty\right) = E_n^c \end{cases}$$

□

Following is a Lorentz type lemma :

Lemma 3.2. [10] *There exist polynomials $q_{i,j,r}(x)$ independent of n and k such that*

$$\varphi^{2r}(x)\frac{d^r}{dx^r}p_{n+cr,k}(x) = \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} (n+cr)^i[k-(n+cr)x]^j q_{i,j,r}(x)\, p_{n+cr,k}(x).$$

Lemma 3.3. [1] *Let Ω be monotone increasing on $[0,c]$. Then $\Omega(t) = O(t^\alpha)$, $t \to 0+$, if for some $0 < \alpha < r$ and all $h, t \in [0,c]$*

$$\Omega(h) < M\left[t^\alpha + (h/t)^r \Omega(t)\right].$$

Lemma 3.4. *Suppose f is s times differentiable on $[0,\infty)$ such that $f^{(s-1)}(t) = O(t^\alpha)$, for some $\alpha > 0$ as $t \to \infty$. Then for any $r, s \in \mathbb{R}$ and $n > \alpha + cs$, we have*

$$D^s V_{n,r}(f,x) = V_{n,r+s}(D^s f, x).$$

We make use of the Lemma 3.4 to define the operators $V_{n,r,s}(f,x)$ as follows

$$V_{n,r,s}(f,x) = V_{n,r+s}(f,x) = \int_0^\infty K_{n,r+s}(t) f(t)\, dt.$$

Obviously, $V_{n;r}^{(s)}(f,x) = V_{n,r,s}(f^{(s)},x)$ and $V_{n,r,s}$ are linear positive operators.

Lemma 3.5. *For $m \in \mathbb{N}^0$, if we define the $m-th$ order moment for the operators $V_{n,r,s}$ by $T_{n,m}(x) = V_{n,r,s}\left((t-x)^m, x\right)$*
then
$T_{n,0}(x) = \frac{(n-c)\beta(n,r+s,c)}{\{n-c(r+s+1)\}}$; $\quad T_{n,1}(x) = \frac{(n-c)\beta(n,r+s,c)(r+s+1)(1+2cx)}{\{n-c(r+s+1)\}\{n-c(r+s+2)\}}$; *and there holds the recurrence relation*

$$\big(n-(m+r+s+2)c\big)T_{n,m+1}(x) + n(1-x)T_{n,m}(x)$$

$$= \big((m+r+s+1)(1+2cx)\big)T_{n,m}(x) + 2m\phi^2(x)T_{n,m-1}(x) + \varphi^2(x)T'_{n,m}(x).$$

Proof. The values of $T_{n,0}(x)$ and $T_{n,1}(x)$ follow from straight forward calculations. Writing $\alpha_{n,r+s} = (n-c)\beta(n,r+s,c)$ and using the relation $\varphi^2(x)p'_{n+cr,k}(x) = \big(\frac{k}{n+cr} -$

$x)p_{n+cr,k}(x)$, we obtain

$$
\begin{aligned}
&\varphi^2(x)\Big(T_{n,m}(x)+mT_{n,m-1}(x)\Big)\\
&=\alpha_{n,r+s}\sum_{k=0}^{\infty}\varphi^2(x)p'_{n+cr,k}(x)\int_0^{\infty}p_{n-c(r+s),k+r+s}(t)(t-x)^m,dt\\
&=\alpha_{n,r+s}\sum_{k=0}^{\infty}p_{n+cr,k}(x)\int_0^{\infty}\varphi^2(t)\,p_{n-c(r+s),k+r+s}(t)(t-x)^m,dt\\
&+\big(n-(r+s)c\big)T_{n,m+1}(x)+\big(n-r-s-(n+2c(r+s))x\big)T_{n,m}(x)\\
&=\alpha_{n,r+s}\sum_{k=0}^{\infty}p_{n+cr,k}(x)\int_0^{\infty}\big\{\varphi^2+(1+2cx)(t-x)+c(t-x)^2\big\}\times\\
&\times p_{n-c(r+s),k+r+s}(t)(t-x)^m,dt\\
&+\big(n-(r+s)c\big)T_{n,m+1}(x)+\big(n-r-s-(n+2c(r+s))x\big)T_{n,m}(x)
\end{aligned}
$$

Now, integration by parts and rearrangements of the terms gives the recurrence relation. □

Corollary 3.1. *From Lemma 3.5, and in view of $\alpha_{n,r+s}=O(1)$, it follows that*

$$
T_{n,2}(x) \;=\; \frac{\alpha_{n,r+s}}{n-c(r+s+1)}\,\frac{2(n-c)\varphi^2(x)+(r+s+1)(r+s+2)(1+2cx)^2}{\{n-c(r+s+1)\}\{n-c(r+s+1)\}}.
$$

This gives $T_{n,2}(x)\leqslant C\delta_n^2(x)$, where $\delta_n(x)=\varphi(x)+\frac{1}{\sqrt{n}}$.

Our next result is a Bernstein type lemma which we shall use in inverse theorem.

Lemma 3.6. *If $f\in L_B[0,\infty)$, $f^{(l-1)}\in AC_{loc}(0,\infty)$ and $l\in N$ then, there hold the inequality:*

$$
\big|V^{(l)}_{n,r,s}(f,x)\big|\leqslant M\varphi^{-\lambda l}(x)\big\|\varphi^{\lambda l}f^{(l)}\big\|,
$$

where $M=M(l)$ is a constant that depends on r but is independent of f and n.

Proof. By the assumption we can write $f(t)=\sum\limits_{\nu=0}^{l-1}\frac{f^{(\nu)}(x)(t-x)^\nu}{\nu!}+R_l(f,t;x)$, where $R_l(f,t;x)=\frac{1}{(\nu-1)!}\int\limits_x^t(t-u)^{l-1}f^{(s)}(u)\,du$. Since, from Lemma 3.5 it follows that $V_{n,r,s}\big((t-x)^\nu,x\big)$ are polynomials in x of degree ν so that $V^{(r)}_{n,r,s}\big((t-x)^\nu,x\big)=0$ for $\nu<r$, it is sufficient to consider $V^{(l)}_{n,r,s}\Big(R_l(f,t;x),x\Big)$.

Making use of $\left|\int\limits_x^t (t-u)^{l-1} f^{(l)}(u)\,du\right| \leqslant \frac{|t-x|^l \|\varphi^{\lambda l} f^{(s)}\|}{x^{\lambda l/2}}\left(\frac{1}{(1+cx)^{\lambda l/2}}+\frac{1}{(1+ct)^{\lambda l/2}}\right)$ we get,

$$\begin{aligned}
&|V^{(l)}_{n,r,s}(f,x)| \\
&\leqslant \frac{\|\varphi^{\lambda l} f^{(s)}\|}{(l-1)!}\alpha_{n,r+s}\sum_{\substack{2i+j\leqslant l\\ i,j\geqslant 0}}\sum_{k=0}^{\infty}(n+cr)^i|k-(n+cr)x|^j \times \\
&\times \frac{|q_{i,j,l}(x)|}{\varphi^{2l}(x)} p_{n+c(r+s),k}(x)\left[\int\limits_0^\infty p_{n-c(r+s),k+r+s}(t)\frac{|t-x|^l}{\varphi^{\lambda l}(x)}\,dt + \right.\\
&+ \left.\int\limits_0^\infty p_{n-c(r+s),k+r+s}(t)\frac{|t-x|^l}{x^{\lambda l/2}}\frac{1}{(1+ct)^{\lambda l/2}}\,dt\right] \\
&= I_1 + I_2 \text{ say.}
\end{aligned}$$

We write $M = \sup\limits_{\substack{2i+j\leqslant l\\ i,j\geqslant 0}} \|q_{i,j,l}(x)\|$ and make use of Hölder's inequalities for integration and summation, the value $\int\limits_0^\infty p_{n-c(r+s),k+r+s}(t) = \frac{1}{n+c(k-1)}$ and Lemma 3.1, Lemma 3.5 to obtain following estimates

$$\begin{aligned}
I_1 &\leqslant \frac{M\left\|\varphi^{\lambda l} f^{(l)}\right\|}{(l-1)!\varphi^{2l+2\lambda}(x)}\sqrt{\alpha_{n,r+s}}\sum_{\substack{2i+j\leqslant l\\ i,j\geqslant 0}}\left(\sum_{k=0}^{\infty}\Big(\frac{k}{n+cr}-x\Big)^{2j} p_{n+c(r+s),k}(x)\right)^{\frac12} \times \\
&\times \frac{(n+cr)^{i+j}}{\sqrt{n+c(k-1)}}\left(\alpha_{n,r+s}\sum_{k=0}^{\infty}p_{n+c(r+s),k}(x)\int\limits_0^\infty (t-x)^{2l}p_{n-c(r+s),k+r+s}(t)\,dt\right)^{\frac12} \\
&\leqslant M\frac{\left\|\varphi^{\lambda l} f^{(l)}\right\|}{(l-1)!\varphi^{2l+2\lambda}(x)}\frac{1}{\sqrt{n+c(k-1)}}\sum_{\substack{2i+j\leqslant l\\ i,j\geqslant 0}}(n+cr)^i\Big(n^{-j+1}\delta_n^{2j}(x)\Big)^{\frac12} n^{-l/2}\delta_n^l(x) \\
&\leqslant M\varphi^{-\lambda l}(x)\left\|\varphi^{\lambda l} f^{(l)}\right\|,
\end{aligned}$$

where we have used the equivalence $\delta_n(x) \sim \frac{1}{\sqrt{n}}$ for $x \in E_n$ and for $x \in E_n^c$, $\delta_n(x) \sim \varphi(x)$. Now it follows by direct calculations that $\int\limits_0^\infty p_{n-c(r+s),k+r+s}(t)(1+ct)^{-l\lambda}\,dt \leqslant M(1+cx)^{-l\lambda}$. Therefore, we get

$$\begin{aligned}
I_2 &\leqslant \varphi^{\lambda l}\frac{\|\varphi^{\lambda l} f\|}{x^{l\lambda/2}}\alpha_{n,r+s}\sum_{\substack{2i+j\leqslant l\\ i,j\geqslant 0}}\sum_{k=0}^{\infty}(n+cr)^i|k-(n+cr)x|^j\frac{|q_{i,j,l}(x)|}{\varphi^{2l}(x)} \times \\
&\times p_{n+c(r+s),k}(x)\int\limits_0^\infty p_{n-c(r+s),k+r+s}(t)|t-x|^l(1+ct)^{-l\lambda/2}\,dt
\end{aligned}$$

$$\leqslant \quad M\frac{\|\varphi^{\lambda l}f^{(l)}\|}{\varphi^{2l}(x)x^{l\lambda/2}}\alpha_{n,r+s}\sum_{\substack{2i+j\leqslant l\\ i,j\geqslant 0}}\sum_{k=0}^{\infty}(n+cr)^i|k-(n+cr)x|^j p_{n+c(r+s),k}(x)\times$$

$$\times \quad \left(\int_0^\infty p_{n-c(r+s),k+r+s}(t)(t-x)^{2l}\,dt\right)^{\frac{1}{2}}\left(\int_0^\infty p_{n-c(r+s),k+r+s}(t)(1+ct)^{-l\lambda}\,dt\right)^{\frac{1}{2}}$$

$$\leqslant \quad M\frac{\|\varphi^{\lambda l}f^{(l)}\|}{\varphi^{(2+\lambda)l}(x)}\sqrt{\alpha_{n,r+s}}\sum_{\substack{2i+j\leqslant l\\ i,j\geqslant 0}}(n+cr)^{i+j}\left(\sum_{k=0}^{\infty}\left(\frac{k}{n+cr}-x\right)^{2j}p_{n+c(r+s),k}(x)\right)^{\frac{1}{2}}$$

$$\times \quad \left(\alpha_{n,r+s}\sum_{k=0}^{\infty}p_{n+c(r+s),k}(x)\int_0^\infty (t-x)^{2l}p_{n-c(r+s),k+r+s}(t)\,dt\right)^{\frac{1}{2}}$$

$$\leqslant \quad M\|\varphi^{\lambda l}f^{(l)}\|.$$

Lemma 3.7. *If $f\in L_B[0,\infty)$ and $r\in N$ then, there hold the inequalities :*

$$\left|V_{n,r,s}^{(r)}(f,x)\right|\leqslant Mn^{r/2}\delta_n^r(x)\varphi^{-2r}(x)\|f\|,$$

where $M=M(r)$ is a constant that depends on r but is independent of f and n.

The proof of is similar to Lemma 3.6. □

4. Main Results

In this section we establish the direct and inverse theorems in simultaneous approximation by the operators $V_{n,r}(f,x)$.

Theorem 4.1. *If $f\in L_B[0,\infty)$, $f^{(s-1)}\in AC_{loc}(0,\infty)$, $0\leqslant\lambda\leqslant 1$, $0<\alpha<2$ and $\varphi(x)=\sqrt{x(1+cx)}$ then, we have*

$$\begin{aligned}\left|V_{n,r}^{(s)}(f,x)-f^{(s)}(x)\right| &\leqslant M\omega_{\varphi^\lambda}^2\left(f^{(s)},n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)\\ &+\ \omega\Big(f^{(s)},\frac{(n-c)\beta(n,r+s,c)(r+s+1)(1+2cx)}{\{n-c(r+s+1)\}\{n-c(r+s+2)\}}\Big).\end{aligned}$$

Proof. Let us take $g_{n,x,\lambda}=g\in W_{2,\lambda}$ such that

$$\|f^{(s)}-g\|+\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^2\|\varphi^{2\lambda}g''\|\leqslant 2\overline{K}_{2,\varphi^\lambda}\left(f^{(s)},\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^2\right). \quad (1)$$

We introduce the auxiliary operators $\widehat{V}_{n,r,s}$ defined by

$$\widehat{V}_{n,r,s}(f,x)=\frac{1}{C_{n,r}}\left[V_{n,r,s}(f,x)-f^{(s)}(x+z)+f^{(s)}(x)\right], \quad (2)$$

where $z=V_{n,r,s}(t-x,x)=\frac{(n-c)\beta(n,r+s,c)(r+s+1)(1+2cx)}{\{n-c(r+s+1)\}\{n-c(r+s+2)\}}$, $C_{n,r}=V_{n,r,s}(1,x)=(n-c)\beta(n,r+s,c)/\{n-c(r+s+1)\}$ and $x\in[0,\infty)$. The operators $\widehat{V}_{n,r,s}$ are linear and preserve the linear functions. Further, $\widehat{V}_{n,r,s}(1,x)=1$, $\widehat{V}_{n,r,s}(t-x,x)=0$ and from 2 it follows that $|\widehat{V}_{n,r,s}(f^s-g,x)|\leqslant M\|f^s-g\|$. Therefore,

$$\begin{aligned}&V_{n,r}^{(s)}(f,x)-f^{(s)}(x)\\ =\ & C_{n,r}\Big[\widehat{V}_{n,r,s}(f^s-g,x)+\{g(x)-f^s(x)\}\\ +\ & \widehat{V}_{n,r,s}(g,x)-g(x)\Big]+(C_{n,r}-1)f^{(s)}(x)+f^{(s)}(x+z)-f^{(s)}(x)\end{aligned}$$

Hence, in view of the limit $C_{n,r} \to 1$ as $n \to \infty$, we get

$$|V_n^{(s)}(f,x) - f(x)| \leqslant M\Big(4\|f^{(s)} - g\| + |\widehat{V}_{n,r,s}(g,x) - g(x)| + \omega\big(f^{(s)}, z\big)\Big).$$

Using the smoothness of g, and in view of $\widehat{V}_{n,r,s}(t-x,x) = 0$, we get

$$|\widehat{V}_{n,r,s}(g,x) - g(x)| \quad \leqslant \quad M\Big|V_{n,r,s}\big(R_2(g,t,x)\Big| + \Big|\int_x^{x+z}(x+z-u)g''(u)du\Big|.$$

where $R_2(g,t,x) = \int_x^t (t-u)g''(u)du$. Now following holds (see [4] p. 141.)

$$\begin{aligned} |R_2(g,t,x)| &\leqslant \frac{|t-x|}{x^\lambda}\left(\frac{1}{(1+cx)^\lambda} + \frac{1}{(1+ct)^\lambda}\right)\Big|\int_x^t \varphi^{2\lambda}(u)\big|g''(u)\big|du\Big| \\ &\leqslant \|\varphi^{2\lambda}g''\|(t-x)^2\left(\frac{1}{x^\lambda(1+cx)^\lambda} + \frac{1}{x^\lambda(1+ct)^\lambda}\right). \end{aligned}$$

Also it can be verified (cf. [6])that $V_{n,r+s}\big((1+ct)^{-m},x\big) \leqslant C(1+cx)^{-m}$ and $V_{n,r,s}\big((t-x)^4,x\big) \leqslant C\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2$. Therefore, we get

$$\begin{aligned} |V_{n,r,s}\big(R_2(g,t,x)\big)| &\leqslant \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)}V_{n,r,s}\Big((t-x)^2,x\Big) + \frac{\|\varphi^{2\lambda}g''\|}{x^\lambda}V_{n,r,s}\Big(\frac{(t-x)^2}{(1+ct)^\lambda},x\Big) \\ &\leqslant \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)}V_{n,r,s}\big((t-x)^2,x\big) \\ &+ \frac{\|\varphi^{2\lambda}g''\|}{x^\lambda}\big(V_{n,r,s}\big((t-x)^4,x\big)\big)^{1/2}\big(V_{n,r,s}\big((1+ct)^{-2\lambda},x\big)^{1/2} \\ &\leqslant M\|\varphi^{2\lambda}g''\|\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2. \end{aligned}$$

Since, $z \leqslant C\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2$ for all values of x, therefore we obtain

$$\Big|\int_x^{x+z}(x+z-u)g''(u)du\Big| \leqslant \big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^4\|g''\|.$$

Collecting these estimates, we get

$$|\widehat{V}_{n,r,s}(g;x) - g(x)| \leqslant M\|\varphi^{2\lambda}g''\|\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2 + \big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^4\|g''\|.$$

Therefore, we have

$$\begin{aligned} &|V_{n,r}^{(s)}(f,x) - f^{(s)}(x)| \\ &\leqslant M\left(\|f^{(s)} - g\| + \|\varphi^{2\lambda}g''\|\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2 + \big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^4\|g''\|\right) \\ &+ \omega\big(f^{(s)},z\big). \end{aligned}$$

This in view of equivalence of $\overline{K}_{2,\varphi^\lambda}(f,t^2)$ and $\omega^2_{\varphi^\lambda}(f,t)$ gives

$$\begin{aligned} |V_{n,r}^{(s)}(f,x) - f^{(s)}(x)| &\leqslant M\overline{K}_{2,\varphi^\lambda}\big(f,\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2\big) + \omega(f^{(s)},z) \\ &\leqslant M\omega^2_{\varphi^\lambda}\big(f,\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big) + \omega\big(f^{(s)},z\big). \end{aligned}$$

This completes the proof of the theorem. □

Corollary 4.1. *Now, using Lemma 2.3* [14], *it follows that* $\omega^2_{\varphi^\lambda}\big(f,t\big) = O(t^\alpha), 0 < \alpha 2$ *implies that* $\omega\big(f^{(s)},t\big) = O\big(t^{\alpha(1-\lambda)}\big)$ *for* $0 < 1-\lambda < \frac{2}{\alpha}$. *Therefore,* $\omega^2_{\varphi^\lambda}\big(f,t\big) = O(t^\alpha)$ *implies* $|V^{(s)}_{n,r}(f,x) - f^{(s)}(x)| = O\big(t^\alpha\big)$.

Theorem 4.2 (Inverse)**.** *Let* $f \in L_B[0,\infty)$, $0 \leqslant \lambda \leqslant 1$, $0 < \alpha < 2$ *and* $\varphi(x) = \sqrt{x(1+cx)}$. *Then, there holds the implication:*

$$\left|V^{(s)}_{n,r}(f,x) - f^{(s)}(x)\right| = O\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha \Rightarrow \omega^2_{\varphi^\lambda}(f,x) = O\left(t\right)^\alpha.$$

Proof. We have

$$\begin{aligned} & \left|\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)} f^{(s)}(x)\right| \\ \leqslant\ & \left|\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)}\left(f^{(s)}(x) - V^{(s)}_{n,r}(f,x)\right)\right| + \left|\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)} V_{n,r,s}(f^{(s)},x)\right| \\ \leqslant\ & M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha + \left|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} V''_{n,r}(f^{(s)}-g,x+u+v)du\,dv\right| \\ +\ & \left|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} V''_{n,r}(g,x+u+v)du\,dv\right|. \end{aligned}$$

Using Lemma 3.6, and Lemma 3.7, we obtain

$$\begin{aligned} \omega^2_{\varphi^\lambda}(f,h) &\leqslant M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha + (h\varphi^\lambda(x))^2 \times \\ &\times \left(\varphi^{-2\lambda}\left(n^{1/2}\,\delta_n^{-(1-\lambda)}(x)\right)^2\|f^{(s)}-g\| + \varphi^{-2\lambda}\|\varphi^{2\lambda}g''\|\right) \\ &\leqslant M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^+ \left(\frac{h}{n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)}\right)^2 \times \\ &\times \left(\|f^{(s)}-g\| + \left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^2\|\varphi^{2\lambda}g''\|\right) \\ \leqslant\ & M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha + \left(\frac{h}{n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)}\right)^2 \omega^2_{\varphi^\lambda}\left(f, n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right). \end{aligned}$$

Using Lemma 3.3 this implies $\omega^2_{\varphi^\lambda}(f,t) = O(t^\alpha)$. □

Remark 4.1. *Analogous to Theorem 1,* [6] *we can obtain the corresponding theorem for the range* $0 < \alpha < 1$ *while for* $s = 0$ *from Theorem 4.1 and Theorem 4.2 we obtain following theorem for the range* $0 < \alpha < 2$*:*

Theorem 4.3. *Let* $f \in L_B[0,\infty), \varphi(x) = \sqrt{x(1+cx)}, 0 < \lambda \leqslant 1$ *and* $0 < \alpha < 2$. *Then, there holds the implication (i)* $\Leftrightarrow$ *(ii) in the following statements:*
(i) $\left|V_{n,r}(f,t) - f(x)\right| = O\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^\alpha$
(ii) $\omega^2_{\varphi^\lambda}(f,t) = O(t^\alpha)$.

Remark 4.2. *We obtain following operators as the special cases of these operators: For* $c = 0, r = 0$ *and* $\phi_n(x) = e^{-nx}$, *we get the Szász-Mirakyan-Durrmeyer operators (see* [8], [9], [13]*).*
For $c = 1$, $r = 0$ *and* $\phi_n(x) = e^{-nx}$, *we obtain the Baskakov-Durrmeyer operators*

(see [15]*).*
For $c = 0$, *and* $\phi_n(x) = e^{-nx}$, *we get the Szász-Durrmeyer operators (see* [13]*).*
For $c > 1$, $r = 0$ *and* $\phi_n(x) = (1 + cx)^{-n/c}$, *we obtain general Baskakov-Durrmeyer operators (see* [11]*).*
For $c = -1$, $r = 0$ *and* $\phi_n(x) = (1 - x)^{-n}$, *we obtain Bernstein-Durrmeyer operators (see* [5], [12]*).*

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