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The Monotone Convergence Theorem for the Riemann Integral

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Abstract. We present a quick proof of the Monotone Convergence Theorem of Arzelà.

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The Dominated Convergence Theorem is a fundamental result in Real Analysis, often presented as one of the main features of Lebesgue integral. Due to the omnipresence of Lebesgue integral in real analysis one might think that nothing of this kind works in the context of Riemann integral. This is not true because the discovery by C. Arzelà [2] of the Bounded Convergence Theorem, preceded by more than a decade the famous work of Lebesgue.

Theorem 1 (The Bounded Convergence Theorem). *If* $f_n, f : [a, b] \to \mathbb{R}$ *are Riemann integrable functions* (*for* $n \in \mathbb{N}$) *such that*

i) $\lim_{n\to\infty} f_n(x) = f(x)$ *for every* x *except possibly for a Lebesgue negligible subset*; *and*

ii) $\sup_{x\in[a,b]} |f_n(x)| \leq M$ *for a suitable positive constant* M,

then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

An important step in the proof of this result is the Monotone Convergence Theorem, for which B. S. Thomson presented a new proof in a recent issue of this Monthly [9].

Theorem 2 (The Monotone Convergence Theorem). *If* $(f_n)_n$ *is a monotone decreasing sequence of Riemann integrable functions on the interval* $[a, b]$ *such that*

$$\lim_{n\to\infty} f_n(x) = 0 \text{ for every } x \text{ except possibly for a Lebesgue negligible subset,}$$

then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = 0.$$

The aim of the present note is to provide an even shorter argument for Theorem 2, based on the Lebesgue criterion of Riemann integrability. Precisely we will need only the necessity part, that is, the fact that every Riemann integrable function $f : [a, b] \to \mathbb{R}$ is bounded and the set of its discontinuities is Lebesgue negligible. A nice proof of this part can be found in the book of Apostol [1], pp. 171-172.

The main ingredient that allows us to avoid anything hard of Lebesgue measure is the use of tagged partitions. A *partition* of $[a, b]$ is any finite collection $\mathcal{P} = (I_k)_{k=1}^n$ of

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nonoverlapping closed intervals whose union is $[a,b]$. Usually the partition is ordered and the intervals are specified by their end points; thus $I_i = [x_{i-1}, x_i]$, where

$$a = x_0 < x_1 < ... < x_n = b.$$

A *tagged partition* of $[a,b]$ is a collection of ordered pairs $(I_k, t_k)_{k=1}^n$ consisting of intervals I_k that form a partition of $[a,b]$, and tags $t_k \in I_k$, for $k = 1, ..., n$. If δ is a *gauge* (that is, a positive function) on $[a,b]$ we say that a tagged partition $(I_k, t_k)_{k=1}^n$ is δ-*fine* if

$$I_k \subset (t_k - \delta(t_k), t_k + \delta(t_k))$$

for $k = 1, ..., n$. A result known as Cousin's Lemma asserts the existence of δ-fine tagged partitions for each $\delta : [a,b] \to (0, \infty)$. See [3], page 11, or [6].

As was noticed by R. A. Gordon [6], the existence of δ-fine tagged partitions simplifies considerably the sufficiency part of Lebesgue criterion of Riemann integrability.

Proof. (of the Monotone Convergence Theorem). Since the Riemann integrable functions are bounded, one may choose a constant $M > 0$ such that

$$0 \leq f_1 \leq M.$$

According to Lebesgue's criterion of Riemann integrability, for every $n \in \mathbb{N}$ there is a Lebesgue negligible subset $A_n \subset [a,b]$ such that f_n is continuous at each point of $[a,b] \backslash A_n$. Since the set

$$B = A \cup \left(\bigcup_{n=1}^{\infty} A_n \right)$$

is Lebesgue negligible, for $\varepsilon > 0$ arbitrarily fixed there is a countable set of pairwise disjoint open intervals (α_n, β_n) such that

$$B \subset \bigcup_n (\alpha_n, \beta_n)$$

and

$$\sum_{n=1}^{\infty} (\beta_n - \alpha_n) < \frac{\varepsilon}{4M}.$$

Each point $z \in B$ belongs to a unique interval (α_n, β_n) and this fact allows us to choose a number $\delta(z) > 0$ for which

$$(z - \delta(z), z + \delta(z)) \subset (\alpha_n, \beta_n).$$

Since $\lim_{n\to\infty} f_n(z) = 0$ for every $z \in [a,b] \backslash B$, we may choose a natural number $n(\varepsilon, z)$ such that for all $n \geq n(\varepsilon, z)$,

$$f_n(z) < \frac{\varepsilon}{4(b-a)}.$$

The continuity of $f_{n(\varepsilon,z)}$ at z yields a number $\delta(z) > 0$ such that

$$f_{n(\varepsilon,z)}(x) < \frac{\varepsilon}{4(b-a)} \tag{1}$$

for all $x \in (z - \delta(z), z + \delta(z)) \cap [a,b]$. Due to our hypothesis on monotonicity, the inequality (1) still works for all $n \geq n(\varepsilon, z)$.

The discussion above outlined the existence of a positive function $\delta : z \to \delta(z)$, defined on the entire interval $[a,b]$.

According to Cousin's lemma, one can choose a δ-fine tagged division

$$\{([x_{k-1}, x_k], z_k) : k = 1, ..., N\},$$

where

$$a = x_0 < x_1 < ... < x_N = b$$
$$z_k \in [x_{k-1}, x_k],\ k = 1, ..., N$$

and

$$[x_{k-1}, x_k] \subset (z_k - \delta(z_k), z_k + \delta(z_k)) \text{ for } k = 1, ..., N.$$

Put

$$n_\varepsilon = \max\{n(\varepsilon, z_k) : k = 1, ..., N\}$$

and

$$\Delta = ([x_{k-1}, x_k])_{k=1}^N.$$

Then for $n \geq n_\varepsilon$ and arbitrary tags $t_k \in [x_{k-1}, x_k]$ $(k = 1, ..., N)$ we have

$$\begin{aligned}\sigma_\Delta(f_n, (t_k)_k) &= \sum_{k=1}^N f_n(t_k)(x_k - x_{k-1}) \\ &= \sum_{\{k:z_k \in B\}} f_n(t_k)(x_k - x_{k-1}) + \sum_{\{k:z_k \notin B\}} f_n(t_k)(x_k - x_{k-1}) \\ &< M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{4(b-a)} \cdot (b-a) = \frac{\varepsilon}{2}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_a^b f_n(x)dx &\leq S_\Delta(f_n) = \sup\{\sigma_\Delta(f_n, (t_k)_k) : t_k \in [x_{k-1}, x_k] \text{ for } k = 1, ..., N\} \\ &\leq \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

whence $\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$. □

Unlike the case of Lebesgue integral, Theorem 1 (as well as the variant of Theorem 2 with a nonzero limit) provides only an instance when the limit and the integral permute. The (Riemann) integrability of the limit f cannot be eliminated from the hypothesis (and obtained as a consequence of $i)$ & $ii)$). In other words, these theorems *do not* provide criteria of integrability.

Finally it is worth to mention other papers which treat (under different degrees of generality) the subject of bounded convergence: [4], [6], [7], [8]. The last paper contains also a good summary of the history of this subject.

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