On $R$-$I$-open sets and $A_I^*$-sets in ideal topological spaces

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Abstract. In this paper, properties of $R$-$I$-open sets and $A_I^*$-sets in ideal topological spaces are discussed. The relationships between $R$-$I$-open sets, $A_I^*$-sets and the related sets in ideal topological spaces are investigated. Moreover, decompositions of $A_I^*$-continuous functions are established.

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1. Introduction

The notions of $R$-$I$-open sets and $A_I^*$-sets in ideal topological spaces are introduced by [11] and [5], respectively. In [11], the notion of $\delta$-$I$-open sets via $R$-$I$-open sets was studied. In [5], decompositions of continuity via $A_I^*$-sets in ideal topological spaces have been established. The aim of this paper is to investigate properties of $R$-$I$-open sets and $A_I^*$-sets in ideal topological spaces. The relationships between $R$-$I$-open sets, $A_I^*$-sets and the related sets in ideal topological spaces are discussed. Also, decompositions of $A_I^*$-continuous functions are provided.

In this paper, $(X, \tau)$ or $(Y, \sigma)$ denote a topological space with no separation properties assumed. $\text{Cl}(K)$ and $\text{Int}(K)$ denote the closure and interior of $K$ in $(X, \tau)$, respectively for a subset $K$ of a topological space $(X, \tau)$. An ideal $I$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies

(1) $V \in I$ and $U \subset V$ implies $U \in I$,
(2) $V \in I$ and $U \in I$ implies $V \cup U \in I$ [10].

Also, $(X, \tau, I)$ is called an ideal topological space or simply an ideal space if $I$ is an ideal on $(X, \tau)$. For a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $P(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^* : P(X) \to P(X)$, said to be a local function [10] of $N \subseteq X$ with respect to $\tau$ and $I$ is defined as follows:

$N^*(I, \tau)_x = \{x \in X : K \cap N \notin I \text{ for every } K \in \tau(x)\}$ where $\tau(x) = \{K \in \tau : x \in K\}$.

A Kuratowski closure operator $Cl^*$ for a topology $\tau^*(I, \tau)$, said to be the $*$-topology, finer than $\tau$, is defined by $Cl^*(N) = N \cup N^*(I, \tau)$ [9]. We simply write $N^*$ for $N^*(I, \tau)$ and $\tau^*$ for $\tau^*(I, \tau)$.

Definition 1.1. A subset $K$ of an ideal topological space $(X, \tau, I)$ is said to be


Lemma 1.1. ([8]) Let $K$ be a subset of an ideal topological space $(X, \tau, I)$. If $N$ is an open set, then $N \cap Cl^*(K) \subset Cl^*(N \cap K)$. 

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Definition 1.2. ([3, 4]) A subset $K$ of an ideal topological space $(X, \tau, I)$ is said to be
(1) semi$^*$-I-open if $K \subseteq Cl(Int^*(K))$.
(2) semi$^*$-I-closed if its complement is semi$^*$-I-open.

2. Properties of R-I-open sets and $A_I^*$-sets

Theorem 2.1. For an ideal topological space $(X, \tau, I)$ and a subset $K$ of $X$, the following properties are equivalent:
(1) $K$ is an R-I-closed set,
(2) $K$ is semi$^*$-I-open and closed.

Proof. (1) $\Rightarrow$ (2): Let $K$ be an R-I-closed set in $X$. Then we have $K = Cl(Int^*(K))$. It follows that $K$ is semi$^*$-I-open and closed.

(2) $\Rightarrow$ (1): Suppose that $K$ is a semi$^*$-I-open set and a closed set in $X$. It follows that $K \subseteq Cl(Int^*(K))$. Since $K$ is closed, then we have

$$Cl(Int^*(K)) \subseteq Cl(K) = K \subseteq Cl(Int^*(K)).$$

Thus, $K = Cl(Int^*(K))$ and hence $K$ is R-I-closed.

Theorem 2.2. For an ideal topological space $(X, \tau, I)$ and a subset $K$ of $X$, $K$ is an R-I-open set if and only if $K$ is semi$^*$-I-closed and open.

Proof. It follows from Theorem 4.

Theorem 2.3. ([4]) A subset $K$ of an ideal topological space $(X, \tau, I)$ is semi$^*$-I-open if and only if there exists $N \in \tau^*$ such that $N \subseteq K \subseteq Cl(N)$.

Theorem 2.4. For an ideal topological space $(X, \tau, I)$ and a subset $K$ of $X$, the following properties are equivalent:
(1) $K$ is an R-I-closed set,
(2) There exists a $*$-open set $L$ such that $K = Cl(L)$.

Proof. (2) $\Rightarrow$ (1): Suppose that there exists a $*$-open set $L$ such that $K = Cl(L)$. Since $L = Int^*(L)$, then we have $Cl(L) = Cl(Int^*(L))$. It follows that

$$Cl(Int^*(Cl(L))) = Cl(Int^*(Cl(Int^*(L)))) = Cl(Int^*(L)) = Cl(L).$$

This implies

$$K = Cl(L) = Cl(Int^*(Cl(L))) = Cl(Int^*(K)).$$

Thus, $K = Cl(Int^*(K))$ and hence $K$ is an R-I-closed set in $X$.

(1) $\Rightarrow$ (2): Suppose that $K$ is an R-I-closed set in $X$. We have $K = Cl(Int^*(K))$. We take $L = Int^*(K)$. It follows that $L$ is a $*$-open set and $K = Cl(L)$.

Theorem 2.5. For an ideal topological space $(X, \tau, I)$ and a subset $K$ of $X$, $K$ is semi$^*$-I-open if $K = L \cap M$ where $L$ is an R-I-closed set and Int$(M)$ is a $*$-dense set.

Proof. Suppose that $K = L \cap M$ where $L$ is an R-I-closed set and Int$(M)$ is a $*$-dense set. By Theorem 7, there exists a $*$-open set $N$ such that $L = Cl(N)$. We take $O = N \cap Int(M)$. It follows that $O$ is $*$-open and $O \subseteq K$. Moreover, we have
Cl(O) = Cl(N \cap Int(M)) and Cl(N \cap Int(M)) \subset Cl(N). Since Int(M) is \ast\text{-dense},
then we have
\[
N = N \cap Cl^*(Int(M)) \subset Cl^*(N \cap Int(M))
\subset Cl(N \cap Int(M)).
\]
It follows that Cl(N) \subset Cl(N \cap Int(M)). Furthermore, we have
\[
Cl(O) = Cl(N \cap Int(M)) \\
\subset Cl(N) = L \subset Cl(N \cap Int(M))
= Cl(O).
\]
Thus, O \subset K \subset L = Cl(O). Hence, by Theorem 6, K is a semi\textsuperscript{*}-I-open set in X.
\[\square\]

**Definition 2.1.** ([4]) The semi\textsuperscript{*}-I-closure of a subset K of an ideal topological space
(X, \tau, I), denoted by s\textsuperscript{*}I Cl(K), is defined by the intersection of all semi\textsuperscript{*}-I-closed sets
of X containing K.

**Theorem 2.7.** ([4]) For a subset K of an ideal topological space (X, \tau, I), s\textsuperscript{*}I Cl(K) = K \cup Int(Cl^*(K)).

**Definition 2.2.** Let (X, \tau, I) be an ideal topological space and K \subset X. K is called
\begin{enumerate}
\item[(1)] \textit{generalized semi\textsuperscript{*}-I-closed (gs\textsuperscript{*}I-closed) in (X, \tau, I) if }s\textsuperscript{*}I Cl(K) \subset O \text{ whenever } K \subset O \text{ and } O \text{ is an open set in } (X, \tau, I).
\item[(2)] \textit{generalized semi\textsuperscript{*}-I-open (gs\textsuperscript{*}I-open) in (X, \tau, I) if }X \setminus K \text{ is a gs\textsuperscript{*}I-closed set in } (X, \tau, I).
\end{enumerate}

**Theorem 2.8.** For a subset M of an ideal topological space (X, \tau, I), M is gs\textsuperscript{*}I-open
if and only if T \subset s\textsuperscript{*}I Int(M) whenever T \subset M and T is a closed set in (X, \tau, I),
where s\textsuperscript{*}I Int(M) = M \cap Cl(Int^*(M)).

**Proof.** (\Rightarrow) : Suppose that M is a gs\textsuperscript{*}I-open set in X. Let T \subset M and T be a
closed set in (X, \tau, I). It follows that X \setminus M is a gs\textsuperscript{*}I-closed set and X \setminus M \subset X \setminus T
where X \setminus T is an open set. Since X \setminus M is gs\textsuperscript{*}I-closed, then s\textsuperscript{*}I Cl(X \setminus M) \subset X \setminus T,
where s\textsuperscript{*}I Cl(X \setminus M) = (X \setminus M) \cup Int(Cl^*(X \setminus M)). Since (X \setminus M) \cup Int(Cl^*(X \setminus M)) =
(X \setminus M) \cup X \setminus Cl(Int^*(M)) = X \setminus (M \cap Cl(Int^*(M))), then
(X \setminus M) \cup Int(Cl^*(X \setminus M)) = X \setminus (M \cap Cl(Int^*(M))) = X \setminus s\textsuperscript{*}I Int(M). It follows that
s\textsuperscript{*}I Cl(X \setminus M) = X \setminus s\textsuperscript{*}I Int(M).
Thus, T \subset X \setminus s\textsuperscript{*}I Cl(X \setminus M) = s\textsuperscript{*}I Int(M) and hence T \subset s\textsuperscript{*}I Int(M).

(\Leftarrow) : The converse is similar.
\[\square\]

**Theorem 2.8.** Let (X, \tau, I) be an ideal topological space and N \subset X. The following
properties are equivalent:
\begin{enumerate}
\item[(1)] N is an R-I-open set,
\item[(2)] N is open and gs\textsuperscript{*}I-closed.
\end{enumerate}

**Proof.** (1) \Rightarrow (2) : Let N be an R-I-open set in X. Then we have N = Int(Cl^*(N)).
It follows that N is open and semi\textsuperscript{*}-I-closed in X. Thus, s\textsuperscript{*}I Cl(N) \subset K whenever
N \subset K and K is an open set in (X, \tau, I). Hence, N is a gs\textsuperscript{*}I-closed set in X.

(2) \Rightarrow (1) : Let N be open and gs\textsuperscript{*}I-closed in X. We have N \subset Int(Cl^*(N)).
Since N is gs\textsuperscript{*}I-closed and open, then we have s\textsuperscript{*}I Cl(N) \subset N. Since s\textsuperscript{*}I Cl(N) = N \cup Int(Cl^*(N)),
then s\textsuperscript{*}I Cl(N) = N \cup Int(Cl^*(N)) \subset N. Thus, Int(Cl^*(N)) \subset N
and N \subset Int(Cl^*(N)). Hence, N = Int(Cl^*(N)) and N is an R-I-open set in X.
\[\square\]

**Definition 2.3.** A subset K of an ideal topological space (X, \tau, I) is said to be
\begin{enumerate}
\item[(1)] an \mathcal{A}I-set [5] if K = L \cap M, where L is an open set and M = Cl(Int^*(M)).
\item[(2)] a locally closed set [1] if K = L \cap M where L is an open set and M is a closed
set in X.
\end{enumerate}
Remark 2.1. Let \((X, \tau, I)\) be an ideal topological space. Any open set and any \(R\)-I-closed set in \(X\) is an \(A^*_I\)-set in \(X\). The reverse of this implication is not true in general as shown in the following example.

Example 2.1. Let \(X = \{a, b, c, d\}\), \(\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}\) and \(I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}\). Then the set \(K = \{b, c, d\}\) is an \(A^*_I\)-set but it is not open. The set \(L = \{a, b, c\}\) is an \(A^*_I\)-set but it is not \(R\)-I-closed.

Remark 2.2. Let \((X, \tau, I)\) be an ideal topological space. Any \(A^*_I\)-set is a locally closed set in \(X\). The reverse implication is not true in general as shown in the following example.

Example 2.2. Let \(X = \{a, b, c, d\}\), \(\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}\) and \(I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}\). Then the set \(K = \{d\}\) is locally closed but it is not an \(A^*_I\)-set.

Theorem 2.9. Let \((X, \tau, I)\) be an ideal topological space, \(N \subseteq X\) and \(K \subseteq X\). If \(N\) is a semi*-I-open set and \(K\) is an open set, then \(N \cap K\) is semi*-I-open.

Proof. Suppose that \(N\) is a semi*-I-open set and \(K\) is an open set in \(X\). It follows that

\[N \cap K \subseteq Cl(Int^*(N)) \cap K \subseteq Cl(Int^*(N) \cap K) = Cl(Int^*(N \cap K)).\]

Thus, \(N \cap K \subseteq Cl(Int^*(N \cap K))\) and hence, \(N \cap K\) is a semi*-I-open set in \(X\). \(\square\)

Lemma 2.1. ([1]) For a subset \(A\) of a topological space \((X, \tau)\), \(A\) is locally closed if and only if \(A = U \cap Cl(A)\) for an open set \(U\).

Definition 2.4. ([6]) A subset \(K\) of an ideal topological space \((X, \tau, I)\) is said to be

1. \(\beta^*_I\)-open if \(K \subseteq Cl(Int^*(Cl(K)))\).
2. \(\beta^*_I\)-closed if \(X \setminus K\) is \(\beta^*_I\)-open.

Theorem 2.10. Let \((X, \tau, I)\) be an ideal topological space and \(K \subseteq X\). The following properties are equivalent:

1. \(K\) is an \(A^*_I\)-set,
2. \(K\) is semi*-I-open and locally closed,
3. \(K\) is a \(\beta^*_I\)-open set and a locally closed set.

Proof. (1) \(\Rightarrow\) (2) : Suppose that \(K\) is an \(A^*_I\)-set in \(X\). It follows that \(K = L \cap M\) where \(L\) is an open set and \(M = Cl(Int^*(M))\). Then \(K\) is locally closed. Since \(M\) is a semi*-I-open set, then by Theorem 19, \(K\) is a semi*-I-open set in \(X\).

(2) \(\Rightarrow\) (3) : It follows from the fact that any semi*-I-open set is \(\beta^*_I\)-open.

(3) \(\Rightarrow\) (1) : Let \(K\) be a \(\beta^*_I\)-open set and a locally closed set in \(X\). We have \(K \subseteq Cl(Int^*(Cl(K)))\). Since \(K\) is a locally closed set in \(X\), then there exists an open set \(L\) such that \(K = L \cap Cl(K)\). It follows that

\[K = L \cap Cl(K) \subseteq L \cap Cl(Int^*(Cl(K))) \subseteq L \cap Cl(K) = K\]

and then \(K = L \cap Cl(Int^*(Cl(K)))\). We take \(M = Cl(Int^*(Cl(K)))\). Then \(Cl(Int^*(M)) = M\). Thus, \(K\) is an \(A^*_I\)-set in \(X\). \(\square\)

Theorem 2.11. Let \((X, \tau, I)\) be an ideal topological space. If every subset of \((X, \tau, I)\) is an \(A^*_I\)-set, then \((X, \tau, I)\) is a discrete ideal topological space with respect to \(\tau^*\).
Proof. Suppose that every subset of \((X, \tau, I)\) is an \(A_I^*\)-set. It follows from Theorem 22 that \(\{x\}\) is semi*-I-open and locally closed for any \(x \in X\). We have \(\{x\} \subset Cl(\text{Int}^*(\{x\}))\). Thus, we have \(\text{Int}^*(\{x\}) = \{x\}\). Hence, \((X, \tau, I)\) is a discrete ideal topological space with respect to \(\tau^*\). \(\square\)

3. Decompositions of \(A_I^*\)-continuous functions

Definition 3.1. A function \(f : (X, \tau, I) \rightarrow (Y, \sigma)\) is said to be

(1) \(A_I^*\)-continuous \([5]\) if \(f^{-1}(T)\) is an \(A_I^*\)-set in \(X\) for each open set \(T\) in \(Y\).

(2) \(LC\)-continuous \([7]\) if \(f^{-1}(T)\) is a locally closed set in \(X\) for each open set \(T\) in \(Y\).

Remark 3.1. For a function \(f : (X, \tau, I) \rightarrow (Y, \sigma)\), the following diagram holds. The reverses of these implications are not true in general as shown in the following example.

\[
\begin{array}{ccc}
\text{LC-continuous} & \uparrow & \text{continuous} \\
\downarrow & & \downarrow \\
\text{\(A_I^*\)-continuous} & & \text{\(A_I^*\)-continuous}
\end{array}
\]

Example 3.1. Let \(X = \{a, b, c, d\}\), \(\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}\) and \(I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}\). The function \(f : (X, \tau, I) \rightarrow (X, \tau)\), defined by \(f(a) = a\), \(f(b) = b\), \(f(c) = b\), \(f(d) = c\) is \(A_I^*\)-continuous but it is not continuous. The function \(g : (X, \tau, I) \rightarrow (X, \tau)\), defined by \(g(a) = b\), \(g(b) = c\), \(g(c) = c\), \(g(d) = a\) is \(LC\)-continuous but it is not \(A_I^*\)-continuous.

Definition 3.2. A function \(f : (X, \tau, I) \rightarrow (Y, \sigma)\) is said to be

(1) semi*-I-continuous \([5]\) if \(f^{-1}(T)\) is a semi*-I-open in \(X\) for each open set \(T\) in \(Y\).

(2) \(\beta_I^*\)-continuous if \(f^{-1}(T)\) is a \(\beta_I^*\)-open set in \(X\) for each open set \(T\) in \(Y\).

Theorem 3.1. The following properties are equivalent for a function \(f : (X, \tau, I) \rightarrow (Y, \sigma)\):

(1) \(f\) is \(A_I^*\)-continuous,
(2) \(f\) is semi*-I-continuous and \(LC\)-continuous,
(3) \(f\) is \(\beta_I^*\)-continuous and \(LC\)-continuous.

Proof. It follows from Theorem 22. \(\square\)

References


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