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On R-I-open sets and $\mathcal{A}_I^*$-sets in ideal topological spaces

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Abstract. In this paper, properties of R-I-open sets and $\mathcal{A}_I^*$-sets in ideal topological spaces are discussed. The relationships between R-I-open sets, $\mathcal{A}_I^*$-sets and the related sets in ideal topological spaces are investigated. Moreover, decompositions of $\mathcal{A}_I^*$-continuous functions are established.

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1. Introduction

The notions of R-I-open sets and $\mathcal{A}_I^*$-sets in ideal topological spaces are introduced by [11] and [5], respectively. In [11], the notion of δ-I-open sets via R-I-open sets was studied. In [5], decompositions of continuity via $\mathcal{A}_I^*$-sets in ideal topological spaces have been established. The aim of this paper is to investigate properties of R-I-open sets and $\mathcal{A}_I^*$-sets in ideal topological spaces. The relationships between R-I-open sets, $\mathcal{A}_I^*$-sets and the related sets in ideal topological spaces are discussed. Also, decompositions of $\mathcal{A}_I^*$-continuous functions are provided.

In this paper, (X, τ) or (Y, σ) denote a topological space with no separation properties assumed. $Cl(K)$ and $Int(K)$ denote the closure and interior of K in (X, τ), respectively for a subset K of a topological space (X, τ). An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1) $V \in I$ and $U \subset V$ implies $U \in I$,

(2) $V \in I$ and $U \in I$ implies $V \cup U \in I$ [10].

Also, (X, τ, I) is called an ideal topological space or simply an ideal space if I is an ideal on (X, τ). For a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, said to be a local function [10] of $N \subset X$ with respect to τ and I is defined as follows:

$N^*(I, \tau) = \{x \in X : K \cap N \notin I \text{ for every } K \in \tau(x)\}$ where $\tau(x) = \{K \in \tau : x \in K\}$.

A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, said to be the $\star$-topology, finer than τ, is defined by $Cl^*(N) = N \cup N^*(I, \tau)$ [9]. We simply write N^* for $N^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$.

Definition 1.1. *A subset K of an ideal topological space (X, τ, I) is said to be*

(1) $\star$-dense [2] *if $Cl^*(K) = X$.*

(2) R-I-open [11] *if $K = Int(Cl^*(K))$.*

(3) R-I-closed [11] *if its complement is R-I-open.*

Lemma 1.1. ([8]) *Let K be a subset of an ideal topological space (X, τ, I). If N is an open set, then $N \cap Cl^*(K) \subset Cl^*(N \cap K)$.*

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Definition 1.2. *([3, 4]) A subset K of an ideal topological space (X, τ, I) is said to be*

(1) semi-*I*-open if $K \subset Cl(Int^*(K))$.*

(2) semi-*I*-closed if its complement is semi*-*I*-open.*

2. Properties of R-I-open sets and $\mathcal{A}_I^*$-sets

Theorem 2.1. *For an ideal topological space (X, τ, I) and a subset K of X, the following properties are equivalent:*

(1) K is an R-I-closed set,

(2) K is semi-*I*-open and closed.*

Proof. (1) ⇒ (2) : Let K be an R-I-closed set in X. Then we have $K = Cl(Int^*(K))$. It follows that K is semi*-I-open and closed.

(2) ⇒ (1) : Suppose that K is a semi*-I-open set and a closed set in X. It follows that $K \subset Cl(Int^*(K))$. Since K is closed, then we have

$$Cl(Int^*(K)) \subset Cl(K) = K \subset Cl(Int^*(K)).$$

Thus, $K = Cl(Int^*(K))$ and hence K is R-I-closed. □

Theorem 2.2. *For an ideal topological space (X, τ, I) and a subset K of X, K is an R-I-open set if and only if K is semi*-*I*-closed and open.*

Proof. It follows from Theorem 4. □

Theorem 2.3. *([4]) A subset K of an ideal topological space (X, τ, I) is semi*-*I*-open if and only if there exists $N \in \tau^*$ such that $N \subset K \subset Cl(N)$.*

Theorem 2.4. *For an ideal topological space (X, τ, I) and a subset K of X, the following properties are equivalent:*

(1) K is an R-I-closed set,

(2) There exists a $\star$-open set L such that $K = Cl(L)$.

Proof. (2) ⇒ (1) : Suppose that there exists a $\star$-open set L such that $K = Cl(L)$. Since $L = Int^*(L)$, then we have $Cl(L) = Cl(Int^*(L))$. It follows that

$$\begin{aligned} &Cl(Int^*(Cl(L))) = Cl(Int^*(Cl(Int^*(L)))) \\ &= Cl(Int^*(L)) = Cl(L). \end{aligned}$$

This implies

$$\begin{aligned} &K = Cl(L) = Cl(Int^*(Cl(L))) \\ &= Cl(Int^*(K)). \end{aligned}$$

Thus, $K = Cl(Int^*(K))$ and hence K is an R-I-closed set in X.

(1) ⇒ (2) : Suppose that K is an R-I-closed set in X. We have $K = Cl(Int^*(K))$. We take $L = Int^*(K)$. It follows that L is a $\star$-open set and $K = Cl(L)$. □

Theorem 2.5. *For an ideal topological space (X, τ, I) and a subset K of X, K is semi*-*I*-open if $K = L \cap M$ where L is an R-I-closed set and $Int(M)$ is a $\star$-dense set.*

Proof. Suppose that $K = L \cap M$ where L is an R-I-closed set and $Int(M)$ is a $\star$-dense set. By Teorem 7, there exists a $\star$-open set N such that $L = Cl(N)$. We take $O = N \cap Int(M)$. It follows that O is $\star$-open and $O \subset K$. Moreover, we have

$Cl(O) = Cl(N \cap Int(M))$ and $Cl(N \cap Int(M)) \subset Cl(N)$. Since $Int(M)$ is $\star$-dense, then we have

$$N = N \cap Cl^*(Int(M)) \subset Cl^*(N \cap Int(M)) \subset Cl(N \cap Int(M)).$$

It follows that $Cl(N) \subset Cl(N \cap Int(M))$. Furthermore, we have

$$Cl(O) = Cl(N \cap Int(M)) \subset Cl(N) = L \subset Cl(N \cap Int(M)) = Cl(O).$$

Thus, $O \subset K \subset L = Cl(O)$. Hence, by Theorem 6, K is a semi*-I-open set in X. □

Definition 2.1. *([4]) The semi*-I-closure of a subset K of an ideal topological space (X, τ, I), denoted by $s_I^*Cl(K)$, is defined by the intersection of all semi*-I-closed sets of X containing K.*

Theorem 2.6. *([4]) For a subset K of an ideal topological space (X, τ, I), $s_I^*Cl(K) = K \cup Int(Cl^*(K))$.*

Definition 2.2. *Let (X, τ, I) be an ideal topological space and $K \subset X$. K is called*

(1) *generalized semi*-I-closed (gs_I^*-closed) in (X, τ, I) if $s_I^*Cl(K) \subset O$ whenever $K \subset O$ and O is an open set in (X, τ, I).*
(2) *generalized semi*-I-open (gs_I^*-open) in (X, τ, I) if $X \backslash K$ is a gs_I^*-closed set in (X, τ, I).*

Theorem 2.7. *For a subset M of an ideal topological space (X, τ, I), M is gs_I^*-open if and only if $T \subset s_I^*Int(M)$ whenever $T \subset M$ and T is a closed set in (X, τ, I), where $s_I^*Int(M) = M \cap Cl(Int^*(M))$.*

Proof. $(\Rightarrow)$: Suppose that M is a gs_I^*-open set in X. Let $T \subset M$ and T be a closed set in (X, τ, I). It follows that $X \backslash M$ is a gs_I^*-closed set and $X \backslash M \subset X \backslash T$ where $X \backslash T$ is an open set. Since $X \backslash M$ is gs_I^*-closed, then $s_I^*Cl(X \backslash M) \subset X \backslash T$, where $s_I^*Cl(X \backslash M) = (X \backslash M) \cup Int(Cl^*(X \backslash M))$. Since $(X \backslash M) \cup Int(Cl^*(X \backslash M)) = (X \backslash M) \cup X \backslash Cl(Int^*(M)) = X \backslash (M \cap Cl(Int^*(M)))$, then $(X \backslash M) \cup Int(Cl^*(X \backslash M)) = X \backslash (M \cap Cl(Int^*(M))) = X \backslash s_I^*Int(M)$. It follows that $s_I^*Cl(X \backslash M) = X \backslash s_I^*Int(M)$. Thus, $T \subset X \backslash s_I^*Cl(X \backslash M) = s_I^*Int(M)$ and hence $T \subset s_I^*Int(M)$.

$(\Leftarrow)$: The converse is similar. □

Theorem 2.8. *Let (X, τ, I) be an ideal topological space and $N \subset X$. The following properties are equivalent:*

(1) N is an R-I-open set,
(2) N is open and gs_I^*-closed.

Proof. (1) $\Rightarrow$ (2) : Let N be an R-I-open set in X. Then we have $N = Int(Cl^*(N))$. It follows that N is open and semi*-I-closed in X. Thus, $s_I^*Cl(N) \subset K$ whenever $N \subset K$ and K is an open set in (X, τ, I). Hence, N is a gs_I^*-closed set in X.

(2) $\Rightarrow$ (1) : Let N be open and gs_I^*-closed in X. We have $N \subset Int(Cl^*(N))$. Since N is gs_I^*-closed and open, then we have $s_I^*Cl(N) \subset N$. Since $s_I^*Cl(N) = N \cup Int(Cl^*(N))$, then $s_I^*Cl(N) = N \cup Int(Cl^*(N)) \subset N$. Thus, $Int(Cl^*(N)) \subset N$ and $N \subset Int(Cl^*(N))$. Hence, $N = Int(Cl^*(N))$ and N is an R-I-open set in X. □

Definition 2.3. *A subset K of an ideal topological space (X, τ, I) is said to be*

(1) *an $\mathcal{A}_I^*$-set [5] if $K = L \cap M$, where L is an open set and $M = Cl(Int^*(M))$.*
(2) *a locally closed set [1] if $K = L \cap M$ where L is an open set and M is a closed set in X.*

Remark 2.1. *Let (X,τ,I) be an ideal topological space. Any open set and any R-I-closed set in X is an $\mathcal{A}_I^*$-set in X. The reverse of this implication is not true in general as shown in the following example.*

Example 2.1. *Let $X=\{a,b,c,d\}$, $\tau=\{X,\varnothing,\{a\},\{b,c\},\{a,b,c\}\}$ and $I=\{\varnothing,\{a\},\{d\},\{a,d\}\}$. Then the set $K=\{b,c,d\}$ is an $\mathcal{A}_I^*$-set but it is not open. The set $L=\{a,b,c\}$ is an $\mathcal{A}_I^*$-set but it is not R-I-closed.*

Remark 2.2. *Let (X,τ,I) be an ideal topological space. Any $\mathcal{A}_I^*$-set is a locally closed set in X. The reverse implication is not true in general as shown in the following example.*

Example 2.2. *Let $X=\{a,b,c,d\}$, $\tau=\{X,\varnothing,\{a\},\{b,c\},\{a,b,c\}\}$ and $I=\{\varnothing,\{a\},\{d\},\{a,d\}\}$. Then the set $K=\{d\}$ is locally closed but it is not an $\mathcal{A}_I^*$-set.*

Theorem 2.9. *Let (X,τ,I) be an ideal topological space, $N\subset X$ and $K\subset X$. If N is a semi*-I-open set and K is an open set, then $N\cap K$ is semi*-I-open.*

Proof. Suppose that N is a semi*-I-open set and K is an open set in X. It follows that

$$N\cap K\subset Cl(Int^*(N))\cap K$$
$$\subset Cl(Int^*(N)\cap K)=Cl(Int^*(N\cap K)).$$

Thus, $N\cap K\subset Cl(Int^*(N\cap K))$ and hence, $N\cap K$ is a semi*-I-open set in X. □

Lemma 2.1. *([1]) For a subset A of a topological space (X,τ), A is locally closed if and only if $A=U\cap Cl(A)$ for an open set U.*

Definition 2.4. *([6]) A subset K of an ideal topological space (X,τ,I) is said to be*
(1) *β_I^*-open if $K\subset Cl(Int^*(Cl(K)))$.*
(2) *β_I^*-closed if $X\backslash K$ is β_I^*-open.*

Theorem 2.10. *Let (X,τ,I) be an ideal topological space and $K\subset X$. The following properties are equivalent:*
(1) *K is an $\mathcal{A}_I^*$-set,*
(2) *K is semi*-I-open and locally closed,*
(3) *K is a β_I^*-open set and a locally closed set.*

Proof. (1) $\Rightarrow$ (2) : Suppose that K is an $\mathcal{A}_I^*$-set in X. It follows that $K=L\cap M$ where L is an open set and $M=Cl(Int^*(M))$. Then K is locally closed. Since M is a semi*-I-open set, then by Theorem 19, K is a semi*-I-open set in X.

(2) $\Rightarrow$ (3) : It follows from the fact that any semi*-I-open set is β_I^*-open.

(3) $\Rightarrow$ (1) : Let K be a β_I^*-open set and a locally closed set in X. We have $K\subset Cl(Int^*(Cl(K)))$. Since K is a locally closed set in X, then there exists an open set L such that $K=L\cap Cl(K)$. It follows that

$$K=L\cap Cl(K)$$
$$\subset L\cap Cl(Int^*(Cl(K)))$$
$$\subset L\cap Cl(K)=K$$

and then $K=L\cap Cl(Int^*(Cl(K)))$. We take $M=Cl(Int^*(Cl(K)))$.

Then $Cl(Int^*(M))=M$. Thus, K is an $\mathcal{A}_I^*$-set in X. □

Theorem 2.11. *Let (X,τ,I) be an ideal topological space. If every subset of (X,τ,I) is an $\mathcal{A}_I^*$-set, then (X,τ,I) is a discrete ideal topological space with respect to τ^*.*

Proof. Suppose that every subset of (X, τ, I) is an $\mathcal{A}_I^*$-set. It follows from Theorem 22 that $\{x\}$ is semi*-I-open and locally closed for any $x \in X$. We have $\{x\} \subset Cl(Int^*(\{x\}))$. Thus, we have $Int^*(\{x\}) = \{x\}$. Hence, (X, τ, I) is a discrete ideal topological space with respect to τ^*. □

3. Decompositions of $\mathcal{A}_I^*$-continuous functions

Definition 3.1. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be*
(1) *$\mathcal{A}_I^*$-continuous* [5] *if $f^{-1}(T)$ is an $\mathcal{A}_I^*$-set in X for each open set T in Y.*
(2) *LC-continuous* [7] *if $f^{-1}(T)$ is a locally closed set in X for each open set T in Y.*

Remark 3.1. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following diagram holds. The reverses of these implications are not true in general as shown in the following example.*

$$\begin{array}{ccc} & & LC\text{-}continuous \\ & & \Uparrow \\ continuous & \Rightarrow & \mathcal{A}_I^*\text{-}continuous \end{array}$$

Example 3.1. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\varnothing, \{a\}, \{d\}, \{a, d\}\}$. The function $f : (X, \tau, I) \to (X, \tau)$, defined by $f(a) = a$, $f(b) = b$, $f(c) = b$, $f(d) = c$ is $\mathcal{A}_I^*$-continous but it is not continuous. The function $g : (X, \tau, I) \to (X, \tau)$, defined by $g(a) = b$, $g(b) = c$, $g(c) = c$, $g(d) = a$ is LC-continous but it is not $\mathcal{A}_I^*$-continous.*

Definition 3.2. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be*
(1) semi*-I-continuous [5] *if $f^{-1}(T)$ is a semi*-I-open in X for each open set T in Y.*
(2) β_I^*-continuous if $f^{-1}(T)$ is a β_I^*-open set in X for each open set T in Y.

Theorem 3.1. *The following properties are equivalent for a function $f : (X, \tau, I) \to (Y, \sigma)$:*
(1) *f is $\mathcal{A}_I^*$-continuous,*
(2) *f is semi*-I-continuous and LC-continuous,*
(3) *f is β_I^*-continuous and LC-continuous.*

Proof. It follows from Theorem 22. □

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