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Algebraic templates of ω-trees, similarity and templates generated by semantic schemas

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Abstract. In this paper we introduce the concepts of algebraic template and similar templates. An algebraic template is the greatest equivalence class of ω-trees generated by the same nonterminal label and the split noetherian mapping ω. We show that the similarity relation is an equivalence one. Such templates can be generated by a semantic schema and we exemplify this case.

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1. Introduction

We consider a nonempty set L and a decomposition $L = L_N \cup L_T$ into disjoint sets. The elements of L_N are called *nonterminal labels* and those of L_T are called *terminal labels*. The elements of L are called *labels*. A **split mapping** on L is a function $\omega : L_N \longrightarrow L \times L$. An **$\omega$-tree** is a tuple $t = (A, D, h)$, where (A, D) is an ordered tree such that every element of D is of the form $[(i, i_1), (i, i_2)]$; $h : A \longrightarrow L$ is a mapping such that if $[(i, i_1), (i, i_2)] \in D$ then $h(i) \in L_N$, $\omega(h(i)) = (h(i_1), h(i_2))$. By $OBT(\omega)$ we denote the set of all ω-trees.

Let $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$ be two elements of $OBT(\omega)$ and an arbitrary mapping $\alpha : A_1 \longrightarrow A_2$. For every $u = [(i, i_1), (i, i_2)]$, where $i, i_1, i_2 \in A_1$, we denote $\overline{\alpha}(u) = [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))]$. We define the relation $t_1 \preceq t_2$ if there is a mapping $\alpha : A_1 \longrightarrow A_2$ such that:

$$u \in D_1 \Longrightarrow \overline{\alpha}(u) \in D_2$$
$$h_1(root(t_1)) = h_2(\alpha(root(t_1)))$$

where $root(t)$ denotes the root of t. Such a mapping α is an **embedding mapping** of t_1 into t_2 ([3]. An embedding mapping is injective ([3]). The relation $\preceq$ is reflexive and transitive, but is not antisymmetric ([3]).

We define the binary relation $\simeq$ on the set $OBT(\omega)$ as follows: $t_1 \simeq t_2$ if $t_1 \preceq t_2$ and $t_2 \preceq t_1$ ([4]). The binary relation $\simeq$ is an equivalence relation on the set $OBT(\omega)$ ([4]). Suppose that $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ and $t_1 \simeq t_2$. There is one and only one embedding mapping α of t_1 into t_2, α is bijective and α^{-1} is the unique embedding mapping of t_2 into t_1 ([4]).

We denote by $OBT(\omega)/\simeq$ the factor set, the set of all equivalence classes. The equivalence class of the element $t \in OBT(\omega)$ is denoted by $[t]$. Let us consider $[t_1] \in OBT(\omega)/_{\simeq}$ and $[t_2] \in OBT(\omega)/_{\simeq}$. We define the relation $[t_1] \sqsubseteq [t_2]$ if $t_1 \preceq t_2$. The relation $\sqsubseteq$ does not depend on representatives. The pair $(OBT(\omega)/_{\simeq}, \sqsubseteq)$ is a

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partial ordered set ([4]). For every $a \in L_N$ we consider the set

$$OBT_a(\omega) = \{\ t \in OBT(\omega) \mid t = (A, D, h), h(root(t)) = a\}$$

2. Algebraic templates and similarity

In this section we introduce the concept of ω-template and we study the algebraic properties of this structure. The binary relation ρ_ω generated by ω is the binary relation $\rho_\omega \subseteq L \times L$ defined as follows: $x\rho_\omega y$ if and only if there is $z \in L$ such that $\omega(x) = (y, z)$ or $\omega(x) = (z, y)$. Throughout in this section we suppose that ρ_ω is a noetherian binary relation. There is the greatest element of the set $(OBT_a(\omega)/\simeq, \sqsubseteq)$ and this element can be computed by means of an increasing operator defined in [6].

Definition 2.1. *The ω-**template generated** by $a \in L_N$, denoted by Ω_a, is the greatest element of the partial algebra $(OBT_a(\omega)/_{\simeq}, \sqsubseteq)$.*

It follows that if Ω_a is an ω-template then $\Omega_a = [t]$ for certain element $t \in OBT_a(\omega)$.

Definition 2.2. *Two ω-templates $\Omega_a = [t_1]$ and $\Omega_b = [t_2]$ are named **similar templates** if there is a bijective mapping $\gamma : A_1 \longrightarrow A_2$ such that*

$$\overline{\gamma}(D_1) = D_2 \tag{1}$$

*where $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. If this is the case then we write $\Omega_a \sim_s \Omega_b$. The relation $\sim_s$ is named **similarity relation**. The mapping γ is named (t_1, t_2)-**mapping of similarity**.*

Remark 2.1. *If γ is a (t_1, t_2)-mapping of similarity then γ^{-1} is a (t_2, t_1)-mapping of similarity.*

Remark 2.2. *In Definition 2.2 we supposed tacitly that $a \neq b$. This can be explained by the fact that for $a = b$ the definition gives a trivial case. Let us detail this case. If $[t_1] \in OBT_a(\omega)/_{\simeq}$ and $[t_2] \in OBT_a(\omega)/_{\simeq}$ are ω-templates then $[t_1] = [t_2]$ because both $[t_1]$ and $[t_2]$ is the greatest element of $(OBT_a(\omega)/_{\simeq}, \sqsubseteq)$. This means that $t_1 \simeq t_2$ and so there is a bijective mapping γ such that (1) is satisfied.*

Proposition 2.1. *The similarity relation does not depend on representatives.*

Proof. Suppose that $\Omega_a = [t_1]$, $\Omega_b = [t_2]$ and $\Omega_a \sim_s \Omega_b$. Denote by γ a (t_1, t_2)-mapping of similarity. Consider $t_3 \in [t_1]$ and $t_4 \in [t_2]$. Denote $t_i = (A_i, D_i, h_i)$ for $i = 1, 2, 3, 4$. We have to prove that there is a (t_3, t_4)-mapping of similarity.
We know (Corollary 3.1, [4]) that there are the bijective mappings $\beta_1 : A_3 \longrightarrow A_1$ and $\beta_2 : A_2 \longrightarrow A_4$ such that

$$\overline{\beta}_1(D_3) = D_1 \tag{2}$$

$$\overline{\beta}_2(D_2) = D_1 \tag{3}$$

The mapping $\beta_1 \circ \gamma \circ \beta_2 : A_3 \longrightarrow A_4$ is a bijective mapping. By the similarity relation we obtain (1). From (2) and (1) we obtain

$$\overline{\beta_1 \circ \gamma}(D_3) = D_2 \tag{4}$$

From (3) and (4) we obtain:

$$\overline{\beta_1 \circ \gamma \circ \beta_2}(D_3) = D_4$$

from which we conclude that $\beta_1 \circ \gamma \circ \beta_2$ is a (t_3, t_4)-mapping of similarity. □

Proposition 2.2. *If $\Omega_a = [t_1]$, $\Omega_b = [t_2]$, $\Omega_a \sim_s \Omega_b$ and γ is a (t_1, t_2)-mapping of similarity then*

$$(p_0, p_1, \dots, p_n) \in Path(t_1) \Leftrightarrow (\gamma(p_0), \gamma(p_1), \dots, \gamma(p_n)) \in Path(t_2) \tag{5}$$

where $Path(t)$ denotes the set of all paths of t.

Proof. Denote $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. We prove (5) by induction on n, where $n \geq 1$. Let us verify this property for $n = 1$. The following sentences are equivalent:

- $(p_0, p_1) \in Path(t_1)$
- Either $[(p_0, p_1), (p_0, q_1)] \in D_1$ or $[(p_0, q_1), (p_0, p_1)] \in D_1$ for some $q_1 \in A_1$.
- Either $[(\gamma(p_0), \gamma(p_1)), (\gamma(p_0), \gamma(q_1))] \in D_2$ or $[(\gamma(p_0), \gamma(q_1)), (\gamma(p_0), \gamma(p_1))] \in D_2$ for some $q_1 \in A_1$.
- $(\gamma(p_0), \gamma(p_1)) \in Path(t_2)$.

So (5) is true for $n = 1$. Suppose that (5) is true for every $n \in \{1, \dots, m\}$. The following sentences are equivalent:

- $(p_0, p_1, \dots, p_m, p_{m+1}) \in Path(t_1)$
- $(p_0, p_1, \dots, p_m) \in Path(t_1)$ and $(p_m, p_{m+1}) \in Path(t_1)$
- $(\gamma(p_0), \gamma(p_1), \dots, \gamma(p_m)) \in Path(t_2)$ and $(\gamma(p_m), \gamma(p_{m+1})) \in Path(t_2)$
- $(\gamma(p_0), \gamma(p_1), \dots, \gamma(p_{m+1})) \in Path(t_2)$

Thus (5) is proved for $n = m + 1$. □

Proposition 2.3. *If $[t_1] \sim_s [t_2]$ then $\gamma(root(t_1)) = root(t_2)$, where γ is the (t_1, t_2)-mapping of similarity.*

Proof. According to Proposition 2.2 we have

$$(root(t_1), p_1, \dots, p_n) \in Path(t_1) \Leftrightarrow (\gamma(root(t_1)), \gamma(p_1), \dots, \gamma(p_n)) \in Path(t_2)$$

Suppose that $\gamma(root(t_1)) = j$ and $j \neq root(t_2)$. There is a path $(root(t_2), q_1, \dots, q_r, j) \in Path(t_2)$. Applying again Proposition 2.2 we deduce that

$$(\gamma^{-1}(root(t_2)), \gamma^{-1}(q_1), \dots, \gamma^{-1}(q_r), \gamma^{-1}(j)) \in Path(t_1)$$

But $\gamma^{-1}(j) = root(t_1)$. It follows that there is a sequence

$$(\gamma^{-1}(root(t_2)), \gamma^{-1}(q_1), \dots, \gamma^{-1}(q_r), root(t_1)) \in Path(t_1)$$

But this property is not possible, therefore our assumption is false. Thus $\gamma(root(t_1)) = root(t_2)$. □

Proposition 2.4. *If $\Omega_a = [t_1]$ and $\Omega_b = [t_2]$ are similar ω-templates then there is only one (t_1, t_2)-mapping of similarity.*

Proof. Take $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. We consider that γ_1 and γ_2 are (t_1, t_2)-mappings of similarity. According to (1) we have $\gamma_1 : A_1 \longrightarrow A_2$, $\gamma_2 : A_1 \longrightarrow A_2$ and

$$\overline{\gamma}_1(D_1) = D_2 \tag{6}$$

$$\overline{\gamma}_2(D_1) = D_2 \tag{7}$$

Consider the tree $T_k(t_1) = (A_1^{(k)}, D_1^{(k)}, h_1^{(k)})$, where T_k is the slicing operator ([5]). We verify by induction on $k \geq 1$ that

$$i \in A_1^{(k)} \Rightarrow \gamma_1(i) = \gamma_2(i) \tag{8}$$

Let us verify first that for $k = 1$ the relation (8) is true. We have

$$T_1(t_1) = (\{root(t_1), i_1, i_2\}, \{[(root(t_1), i_1), (root(t_1), i_2)]\}, h_1^{(1)}\})$$

where

$$h_1^{(1)}(root(t_1)) = h_1(root(t_1)), h_1^{(1)}(i_1) = h_1(i_1), h_1^{(1)}(i_2) = h_1(i_2)$$

By Proposition 2.3 we have

$$\gamma_1(root(t_1)) = root(t_2) = \gamma_2(root(t_1)) \tag{9}$$

According to (6) and (7) we obtain for γ_1 and γ_2

$$[(\gamma_1(root(t_1)), \gamma_1(i_1)), (\gamma_1(root(t_1)), \gamma_1(i_2))] \in D_2$$

$$[(\gamma_2(root(t_1)), \gamma_2(i_1)), (\gamma_2(root(t_1)), \gamma_2(i_2))] \in D_2$$

Taking into account (9) we obtain now $\gamma_1(i_1) = \gamma_2(i_1)$ and $\gamma_1(i_2) = \gamma_2(i_2)$. Thus (8) is true for $k = 1$.

Suppose that (8) is true for $k = r$ and we verify this property for $k = r + 1$. Take $i \in A_1^{(r+1)}$. We have $A_1^{(r+1)} = A_1^{(r)} \cup (A_1^{(r+1)} \setminus A_1^{(r)})$. If $i \in A_1^{(r)}$ then by the inductive assumption we have $\gamma_1(i) = \gamma_2(i)$. It remains to consider the case $i \in A_1^{(r+1)} \setminus A_1^{(r)}$. There is a path $(root(t_1), p_1, \ldots, p_r, i) \in Path(t_1)$, therefore

$$(\gamma_1(root(t_1)), \gamma_1(p_1), \ldots, \gamma_1(p_r), \gamma_1(i)) \in Path(t_2) \tag{10}$$

$$(\gamma_2(root(t_1)), \gamma_2(p_1), \ldots, \gamma_2(p_r), \gamma_2(i)) \in Path(t_2) \tag{11}$$

From (10) we obtain that $(\gamma_1(root(t_1)), \gamma_1(p_1), \ldots, \gamma_1(p_r)) \in Path(t_2)$, therefore $(root(t_1), p_1, \ldots, p_r) \in Path(t_1)$. Thus $p_r \in A_1^{(r)}$. By the inductive assumption we have $\gamma_1(p_r) = \gamma_2(p_r)$. From (10) and (11) we deduce that

$$[(\gamma_1(p_r), \gamma_1(i)), (\gamma_1(p_r), \gamma_1(j))] \in D_2 \tag{12}$$

or

$$[(\gamma_1(p_r), \gamma_1(j)), (\gamma_1(p_r), \gamma_1(i))] \in D_2 \tag{13}$$

Suppose that we have (12). But γ_1 is a mapping of similarity and so from (12) we obtain

$$[(p_r, i), (p_r, j)] \in D_1 \tag{14}$$

Applying (7) we obtain

$$[(\gamma_2(p_r), \gamma_2(i)), (\gamma_2(p_r), \gamma_2(j))] \in D_2 \tag{15}$$

But $\gamma_1(p_r) = \gamma_2(p_r)$, therefore from (12) and (15) we obtain $\gamma_1(i) = \gamma_2(i)$. The same conclusion is obtained if we suppose that (13) is true.

□

Proposition 2.5. *The relation $\sim_s$ is an equivalence relation.*

Proof. Let us verify that $\Omega_a \sim_s \Omega_a$. If $t_1, t_2 \in \Omega_a$ then $t_1 \simeq t_2$. Suppose that $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. There is a bijective mapping $\gamma : A_1 \longrightarrow A_2$ such that $\overline{\gamma}(D_1) = D_2$. From Definition 2.2 we have $[t_1] \sim_s [t_2]$ and therefore $\Omega_a \sim_s \Omega_a$. Thus we verified the reflexivity of $\sim_s$.

Suppose that $\Omega_a \sim_s \Omega_b$ and $\Omega_b \sim_s \Omega_c$. Consider $t_1 = (A_1, D_1, h_1) \in \Omega_a$, $t_2 = (A_2, D_2, h_2) \in \Omega_b$ and $t_3 = (A_3, D_3, h_3) \in \Omega_c$. There is a bijective mapping $\alpha_1 : A_1 \longrightarrow A_2$ such that $\overline{\alpha}_1(D_1) = D_2$. There is also a bijective mapping $\alpha_2 : A_2 \longrightarrow A_3$ such that $\overline{\alpha}_2(D_2) = D_3$. It follows that $\overline{\alpha}_2(\overline{\alpha}_1(D_1)) = D_3$. So we verified that $\Omega_a \sim_s \Omega_c$ and therefore $\sim_s$ is transitive.

The symmetry of $\sim_s$ is obtained immediately from the fact that we have $\overline{\gamma}(D_1) = D_2$ if and only if $\overline{\gamma^{-1}}(D_2) = D_1$. □

Suppose that $t_1 = (A_1, D_1, h_1) \in \Omega_a$, $t_2 = (A_2, D_2, h_2) \in \Omega_b$ and $[t_1] \sim_s [t_2]$. Denote by $\gamma : A_1 \longrightarrow A_2$ the (t_1, t_2)-mapping of similarity. We denote also $T_k(t_1) = (A_1^{(k)}, D_1^{(k)}, h_1^{(k)})$, where T_k is the slicing operator.

Definition 2.3. *The structure $\gamma(T_k(t_1)) = (\gamma(A_1^{(k)}), \overline{\gamma}(D_1^{(k)}))$ is the **image** of $T_k(t_1)$ by the mapping γ.*

Proposition 2.6. *Suppose that $t_1 = (A_1, D_1, h_1) \in \Omega_a$, $t_2 = (A_2, D_2, h_2) \in \Omega_b$ and $\Omega_a \sim_s \Omega_b$. If $\gamma : A_1 \longrightarrow A_2$ is the (t_1, t_2)-mapping of similarity then $\gamma(T_k(t_1)) = T_k(t_2)$.*

Proof. We note $root(t_1) = r_1$ and $root(t_2) = r_2$. By Proposition 2.3 we have $\gamma(r_1) = r_2$. From (1) we obtain

$$\overline{\gamma}(D_1) = D_2 \tag{16}$$

For $k \geq 1$ consider $T_k(t_1) = (A_1^{(k)}, D_1^{(k)}, h_1^{(k)})$ and $T_k(t_2) = (A_2^{(k)}, D_2^{(k)}, h_2^{(k)})$. We prove the proposition by induction on $k \geq 1$.
Consider $T_1(t_1) = (A_1^{(1)}, D_1^{(1)}, h_1^{(1)})$ and $T_1(t_2) = (A_2^{(1)}, D_2^{(1)}, h_2^{(1)})$, where

$$A_1^{(1)} = \{r_1, i_1, i_2\}, D_1^{(1)} = \{[(r_1, i_1), (r_1, i_2)]\}$$
$$A_2^{(1)} = \{r_2, j_1, j_2\}, D_2^{(1)} = \{[(r_2, j_1), (r_2, j_2)]\}$$

But $[(r_2, j_1), (r_2, j_2)] \in D_2$ and $r_2 = \gamma(r_1)$. From (16) we deduce $\gamma(i_1) = j_1$ and $\gamma(i_2) = j_2$. So we have $A_2^{(1)} = \{r_2, j_1, j_2\} = \{\gamma(r_1), \gamma(i_1), \gamma(i_2)\} = \gamma(A_1^{(1)})$. We have also $D_2^{(1)} = \{[(r_2, j_1), (r_2, j_2)]\} = \{[(\gamma(r_1), \gamma(i_1)), (\gamma(r_2), \gamma(i_2))]\} = \overline{\gamma}(D_1^{(1)})$. Thus we have $\gamma(T_1(t_1)) = T_1(t_2)$ and the proposition is proved for $k = 1$.
Suppose that $\gamma(T_k(t_1)) = T_k(t_2)$. Let us prove that $\gamma(T_{k+1}(t_1)) = T_{k+1}(t_2)$. We denote by

$$Path_n(r_1) = \{i \in A_1 \mid \exists (r_1, p_1, \ldots, p_{n-1}, i) \in Path(t_1)\}$$

where $n \geq 1$. Analogously we consider the set $Path_n(r_2)$ for t_2. We remark that

$$A_1^{(k+1)} \setminus A_1^{(k)} = Path_{k+1}(r_1)$$
$$A_2^{(k+1)} \setminus A_2^{(k)} = Path_{k+1}(r_2)$$

The following sentences are equivalent:

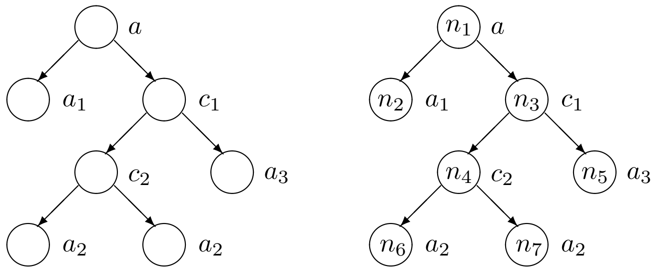
- $i \in A_1^{(k+1)} \setminus A_1^{(k)}$;
- $i \in Path_{k+1}(r_1)$;
- There is $(r_1, p_1, \ldots, p_k, i) \in Path(t_1)$;
- There is $(\gamma(r_1), \gamma(p_1), \ldots, \gamma(p_k), \gamma(i)) \in Path(t_2)$;
- There is $(r_2, \gamma(p_1), \ldots, \gamma(p_k), \gamma(i)) \in Path(t_2)$;
- $\gamma(i) \in Path_{k+1}(r_2)$;
- $\gamma(i) \in A_2^{(k+1)} \setminus A_2^{(k)}$.

From these relations we deduce that $\gamma(A_1^{(k+1)} \setminus A_1^{(k)}) = A_2^{(k+1)} \setminus A_2^{(k)}$. Thus $\gamma(A_1^{(k+1)}) = \gamma(A_1^{(k)} \cup (A_1^{(k+1)} \setminus A_1^{(k)})) = \gamma(A_1^{(k)}) \cup \gamma(A_1^{(k+1)} \setminus A_1^{(k)}) = A_2^{(k)} \cup (A_2^{(k+1)} \setminus A_2^{(k)}) = A_2^{(k+1)}$.
We remark that

$$D_1^{(k+1)} = \{[(i, i_1), (i, i_2)] \in D_1 \mid i \in Path_k(r_1)\}$$

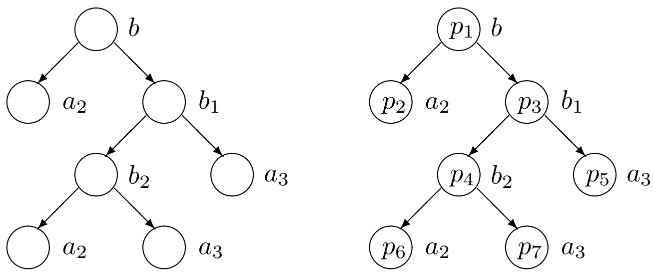
and the following sentences are equivalent:

- $[(i, i_1), (i, i_2)] \in D_1^{(k+1)}$;
- $[(i, i_1), (i, i_2)] \in D_1$ and $i \in Path_k(r_1)$;
- $[(\gamma(i), \gamma(i_1)), (\gamma(i), \gamma(i_2))] \in D_2$ and $\gamma(i) \in Path_k(r_2)$;
- $[(\gamma(i), \gamma(i_1)), (\gamma(i), \gamma(i_2))] \in D_2^{(k+1)}$



Ω-template Ω_a A representative t_1 of Ω_a

FIGURE 1. $t_1 \in \Omega_a$



Ω-template Ω_b A representative t_2 of Ω_b

FIGURE 2. $t_2 \in \Omega_b$

It follows that $D_2^{(k+1)} = \overline{\gamma}(D_1^{(k+1)})$ and the proposition is proved. $\square$

In order to exemplify these concepts we consider $L_N = \{a, b, c_1, c_2, b_1, b_2\}$ and $L_T = \{a_1, a_2, a_3, a_4\}$. Consider the mapping $\omega : L_N \longrightarrow L \times L$ defined as follows:

$\omega(a) = (a_1, c_1)$; $\omega(c_1) = (c_2, a_3)$; $\omega(c_2) = (a_2, a_2)$;

$\omega(b) = (a_2, b_1)$; $\omega(b_1) = (b_2, a_3)$; $\omega(b_2) = (a_2, a_3)$;

In Figure 1 we represented the template Ω_a in the left side and a representative t_1 of Ω_a in the right side. Based on the same mapping ω we represented in Figure 2 another template Ω_b and one of its representative denoted by t_2. We defined two equivalence relations: one relation for templates and another relation for ω-trees. We remark that $\Omega_a \sim_s \Omega_b$, but $t_1 \not\approx t_2$.

3. Algebraic templates generated by semantic schemas

In this section we show that semantic schemas can generate algebraic templates. First we recall the concept of semantic schema.

Definition 3.1. ([2]) *A* θ**-semantic schema** *is a system* $\mathcal{S} = (X, A_0, A, R)$*, where*

- X *is a finite nonempty set and its elements are named* ***object*** *symbols.*
- A_0 *is a finite nonempty set, its elements are named* ***label*** *symbols and* $A_0 \subseteq A \subseteq \overline{A}_0$*, where* $\overline{A}_0$ *is the Peano* θ*-algebra generated by* A_0

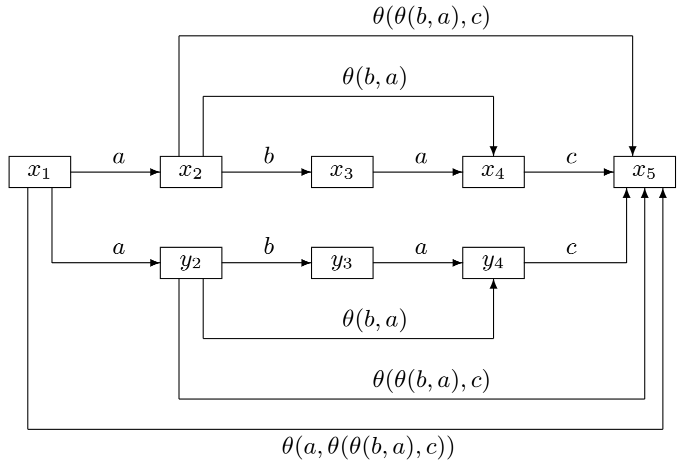


FIGURE 3. A semantic schema

- $R \subseteq X \times A \times X$ *is a nonempty set and its elements satisfy the following conditions:*

$$(x, \theta(u, v), y) \in R, u \in \overline{A}_0, v \in \overline{A}_0 \Longrightarrow \exists z \in X : (x, u, z) \in R, (z, v, y) \in R \tag{17}$$

$$\theta(u, v) \in A, (x, u, z) \in R, (z, v, y) \in R \Longrightarrow (x, \theta(u, v), y) \in R \tag{18}$$

$$u \in A \Longleftrightarrow \exists (x, u, y) \in R \tag{19}$$

where $\overline{A}_0$ the *Peano θ-algebra generated by* A_0 ([1]). This means that $\overline{A}_0 = \bigcup_{n\geq 0} A_n$, where A_n is defined recursively as follows ([1]):

$$A_{n+1} = A_n \cup \{ \ \theta(u, v) \ \ | \ \ u, v \in A_n\}, \quad n \geq 0$$

We shall use the notation $R_0 = R \cap (X \times A_0 \times X)$.

3.1. Algebraic templates over R. We consider $L = R$, $L_T = R_0$ and $L_N = R \setminus R_0$. The relation (17) allows us to define the concept of ω_R mapping.

Definition 3.2. *A split mapping of R is a mapping $\omega_R : R \setminus R_0 \longrightarrow R \times R$ such that if $\omega_R(x, \theta(u, v), y) = ((x, u, z_1), (z_2, v, y))$ then $z_1 = z_2$.*

Remark 3.1. *For a given semantic schema at least one split mapping can be built. Really, based on (17) for every $(x, \theta(u, v), y) \in R \setminus R_0$ we can choose $z \in X$ such that $(x, u, z) \in R$ and $(z, v, y) \in R$.*

We can exemplify this concept by considering the case of the semantic schema represented in Figure 3. For this case we have:

- $R_0 = \{(x_1, a, x_2), (x_2, b, x_3), (x_3, a, x_4), (x_4, c, x_5), (x_1, a, y_2), (y_2, b, y_3),$
  $(y_3, a, y_4), (y_4, c, x_5)\}$
- $R = R_0 \cup \{(x_2, \theta(b, a), x_4), (x_2, \theta(\theta(b, a), c), x_5), (x_1, \theta(a, \theta(\theta(b, a), c)), x_5),$
  $(y_2, \theta(b, a), y_4), (y_2, \theta(\theta(b, a), c), x_5)\}$

Two ω_R split mappings can be defined:

(1) $\omega^1_R(x_1, \theta(a, \theta(\theta(b, a), c)), x_5) = ((x_1, a, x_2), (x_2, \theta(\theta(b, a), c), x_5));$
$\omega^1_R(x_2, \theta(\theta(b, a), c), x_5) = ((x_2, \theta(b, a), x_4), (x_4, c, x_5));$
$\omega^1_R(x_2, \theta(b, a), x_4) = ((x_2, b, x_3), (x_3, a, x_4));$

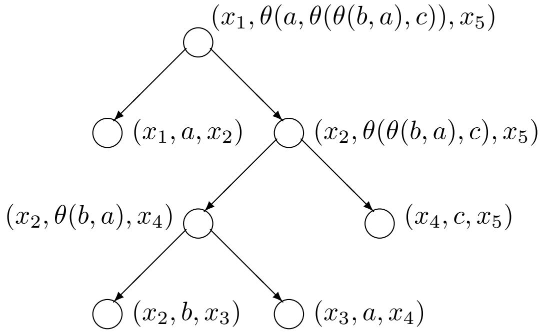


FIGURE 4. ω_R^1-template

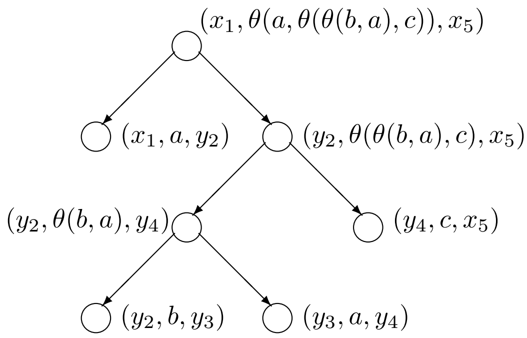


FIGURE 5. ω_R^2-template

(2) $\omega_R^2(x_1, \theta(a, \theta(\theta(b,a),c)), x_5) = ((x_1, a, y_2), (y_2, \theta(\theta(b,a),c), x_5))$;
$\omega_R^2(y_2, \theta(\theta(b,a),c), x_5) = ((y_2, \theta(b,a), y_4), (y_4, c, x_5))$;
$\omega_R^2(y_2, \theta(b,a), y_4) = ((y_2, b, y_3), (y_3, a, y_4))$;

Accordingly we can build ω_R^1-trees and ω_R^2-trees. Moreover, we can obtain ω_R^1-templates and ω_R^2-templates. Such structures are represented in Figure 4 and Figure 5. As we see below the relation ρ_{ω_R} is a noetherian relation (Proposition 3.2).

3.2. Algebraic templates over A. We consider $L = A$, $L_T = A_0$ and $L_N = A \setminus A_0$. We consider the split mapping defined as follows:

$$\omega_A : L_N \to L \times L$$

$$\omega_A(\theta(u,v)) = (u,v)$$

Consider the binary relation $\rho_{\omega_A} \subseteq A \times A$ generated by ω_A. From the properties satisfied by a θ-Peano algebra, particularly $\overline{A_0}$, we know that if $\theta(u,v) \in \overline{A_0}$, $u \in \overline{A_0}$ and $v \in \overline{A_0}$ then u and v are uniquely determined. It follows that if $\theta(u,v)\rho_{\omega_A} x$ then $x = u$ or $x = v$.

We define $|u| = 1$ if $u \in A_0$ and $|\theta(u,v)| = |u| + |v|$.

Remark 3.2. *If $u \rho_{\omega_A} v$ then $|u| > |v|$.*

Proposition 3.1. *ρ_{ω_A} is a noetherian binary relation.*

Proof. If we consider an infinite sequence $u_1\rho_{\omega_A}u_2$, $u_2\rho_{\omega_A}u_3$, ... then we obtain an infinite decreasing sequence of natural numbers $|u_1| > |u_2| > \ldots$. But this property is not possible. □

Proposition 3.2. *The relation $\rho_{\omega_R} \subseteq R \times R$ generated by the split mapping ω_R is a noetherian relation.*

Proof. We observe that if $(x_1, u_1, y_1)\rho_{\omega_R}(x_2, u_2, y_2)$ then $u_1\rho_{\omega_A}u_2$. But ρ_{ω_A} is a noetherian relation, so ρ_{ω_R} is also a noetherian relation. □

4. Similar templates generated by distinct split mappings

In the previous sections we presented the case of similar templates generated by the same split mapping. In this section we show that the case of similar templates generated by distinct split mappings is possible.

We consider two split mappings

$$\omega_1 : L^1_N \longrightarrow L^1 \times L^1$$
$$\omega_2 : L^2_N \longrightarrow L^2 \times L^2$$

where $L^1 \cap L^2 = \emptyset$, $L^1 = L^1_N \cup L^1_T$, $L^1_N \cap L^1_T = \emptyset$ and $L^2 = L^2_N \cup L^2_T$, $L^2_N \cap L^2_T = \emptyset$.

We consider $L_N = L^1_N \cup L^2_N$, $L_T = L^1_T \cup L^2_T$, $L = L_N \cup L_T$ and the mapping $\omega : L_N \longrightarrow L \times L$ defined by

$$\omega(x) = \begin{cases} \omega_1(x)\ if\ x \in L^1_N \\ \\ \omega_2(x)\ if\ x \in L^2_N \end{cases}$$

The mapping ω is named the **union mapping** of ω_1 and ω_2.

Proposition 4.1. *Suppose that ω is the union mapping of ω_1 and ω_2. If $x\rho_\omega y$ and $x \in L^j_N$, where $j \in \{1, 2\}$, then $y \in L^j$ and $x\rho_{\omega_j}y$.*

Proof. If $x \in L^j_N$ and $x\rho_\omega y$ then there is $z \in L$ such that $\omega(x) = (y, z)$ or $\omega(x) = (z, y)$. But if $x \in L^j_N$ then $\omega(x) = \omega_j(x) \in L^j \times L^j$. It follows that $y \in L^j$ and $x\rho_{\omega_j}y$. □

Proposition 4.2. *If ρ_{ω_1} and ρ_{ω_2} are noetherian relations and ω is the union mapping of ω_1 and ω_2 then ρ_ω is a noetherian relation.*

Proof. Suppose by contrary that ρ_ω is not a noetherian relation. There is an infinite sequence $x_1\rho_\omega x_2$, $x_2\rho_\omega x_3$, ... of elements from L. Because $L = L^1 \cup L^2$ we have two cases. If $x_1 \in L^j$ then by Proposition 4.1 we obtain $x_2, x_3, \ldots \in L^j$ and $x_1\rho_{\omega_j}x_2$, $x_2\rho_{\omega_j}x_3$, ... This shows that ρ_{ω_j} is not a noetherian relation, which is not true. □

Proposition 4.3. *The following properties are satisfied:*
(1) $OBT(\omega_1) \cap OBT(\omega_2) = \emptyset$
(2) $OBT(\omega) = OBT(\omega_1) \cup OBT(\omega_2)$

Proof. Suppose that $OBT(\omega_1) \cap OBT(\omega_2) \neq \emptyset$ and take $t \in OBT(\omega_1) \cap OBT(\omega_2)$. It follows that $label(root(t)) \in L^1_N \cap L^2_N$, which is not possible because $L^1_N \cap L^2_N = \emptyset$. Obviously we have $OBT(\omega_1) \subseteq OBT(\omega)$ and $OBT(\omega_2) \subseteq OBT(\omega)$. Suppose that $t = (A, D, h) \in OBT(\omega)$. It follows that $label(root(t)) \in L^1_N \cup L^2_N$. If $[(i, i_1), (i, i_2)] \in D$ then the following two properties are satisfied:

- $h(i) \in L_N$
- $\omega(h(i)) = (h(i_1), h(i_2))$

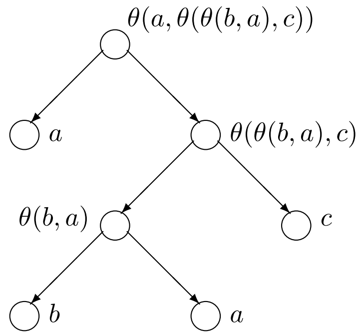


FIGURE 6. ω_R^1-template

We use this property for $r_0 = root(t)$. If $h(r_0) \in L_N^j$, where $j \in \{1, 2\}$, and $[(r_0, i_1), (r_0, i_2)] \in D$ then $\omega(h(r_0)) = (h(i_1), h(i_2))$. But $\omega(h(r_0)) = \omega_j(h(r_0))$ and $\omega_j(h(r_0)) \in L^j \times L^j$. It follows that $h(i_1) \in L^j$ and $h(i_2) \in L^j$. We reiterate this reasoning and we deduce that all labels of t belong to L^j. Thus $t \in OBT(\omega_j)$. □

Proposition 4.4. *Suppose that ω is the union mapping of ω_1 and ω_2. For $j \in \{1, 2\}$, if $t_1 \in OBT(\omega_j)$, $t_2 \in OBT(\omega)$ and $t_2 \simeq t_1$ then $t_2 \in OBT(\omega_j)$.*

Proof. Suppose that $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. There is a bijective mapping $\gamma : A_1 \longrightarrow A_2$ such that

$\gamma(root(t_1)) = root(t_2)$

$\overline{\gamma}(D_1) = D_2$

$h_1(root(t_1) = h_2(root(t_2))$

If $t_1 \in OBT(\omega_j)$ then $h_1(root(t_1)) \in L^j$, therefore $h_2(root(t_2)) \in L^j$. From Proposition 4.3 we deduce that either $t_2 \in OBT(\omega_j)$ or $t_2 \in OBT(\omega_{3-j})$. If $t_2 \in OBT(\omega_{3-j})$ then $h_2(root(t_2)) \in L^{3-j}$ and this is not possible because $h_2(root(t_2)) \in L^j$ and $L^j \cap L^{3-j} = \emptyset$. □

Proposition 4.5. *Suppose that ω is the union mapping of ω_1 and ω_2. Every ω-template is either an ω_1-template or an ω_2-template.*

Proof. An ω-template is an equivalence class of elements from $OBT(\omega)$. Let be Ω an ω-template. Take an element $t_0 \in \Omega$. If $t_0 \in OBT(\omega_j)$ and $t \in \Omega$ then $t \simeq t_0$, therefore $t \in OBT(\omega_j)$ by Proposition 4.4. It follows that Ω is an ω_j-template. □

In order to exemplify this case we consider the set A given by the semantic schema depicted in Figure 3. We take:

$A_0 = \{a, b, c\}$

$A = A_0 \cup \{\theta(b, a), \theta(\theta(b, a), c), \theta(a, \theta(\theta(b, a), c))\}$

$L_N^2 = A \setminus A_0$, $L_T^2 = A_0$, $\omega_2 : L_N^2 \to L^2 \times L^2$, $\omega_2(\theta(u, v) = (u, v)$

We denote by ω the union mapping of ω_R^1 and ω_2. The templates depicted in Figure 4 and Figure 6 are similar templates.

5. Conclusions

In this paper we considered the equivalence classes generated by the same nonterminal label. The greatest equivalence class of this set is an algebraic template. We defined the concept of similar templates. We showed that the similarity relation is an equivalence relation. We exemplified these concepts as templates generated by semantic schemas. Finally we studied the case of two similar templates generated by two distinct split mappings. In a shortcoming paper we study the use of these concepts to characterize the formal computations in a semantic schema.

References

[1] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, Lukasiewicz-Moisil Algebra, *Annals of Discrete Mathematics* **49** (1991), North-Holland.

[2] N. Ţăndăreanu, Semantic Schemas and Applications in Logical Representation of Knowledge, *Proceedings of the 10th Int. Conf. on CITSA* **III** (2004), July 21-25 2004, 82–87.

[3] N. Ţăndăreanu and C. Zamfir, Algebraic properties of ω-trees (I), *Annals of the University of Craiova, Mathematics and Computer Science Series* **37** (2010), no. 1, 80–89.

[4] N. Ţăndăreanu and C. Zamfir, Algebraic properties of ω-trees (II), *Annals of the University of Craiova, Mathematics and Computer Science Series* **37** (2010), no. 2, 7–17.

[5] N. Ţăndăreanu and C. Zamfir, Slices and extensions of ω-trees, *Annals of the University of Craiova, Mathematics and Computer Science Series* **38** (2011), no. 1, 72–82.

[6] N. Ţăndăreanu and C. Zamfir, Local Greatest Equivalence Classes of ω-trees, *Annals of the University of Craiova, Mathematics and Computer Science Series* **38** (2011), no. 2, 32–42.

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