Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 38(3), 2011, Pages 12–21
ISSN: 1223-6934, Online 2246-9958

Maximal elements in the partial algebra of accepted structured paths of a semantic schema

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Abstract. The concept of semantic schema was introduced in [2] as a structure for knowledge representation. This structure is based on graph theory and universal algebras. New computational aspects in such structures were discussed in [3]. The intuitive aspect of this computation in a semantic schema $\mathcal{S}$ is given by the set of accepted structured paths $ASP(\mathcal{S})$ of $\mathcal{S}$. For this reason in this paper we study this set. We organize the set $ASP(\mathcal{S})$ as a partial algebra. We discuss the maximal elements of this partial algebra.

2000 *Mathematics Subject Classification.* Primary 05C20; Secondary 18A10.
Key words and phrases. θ-semantic schema, structured paths, accepted structured paths, maximal elements.

1. Introduction

Today there are two major implications of the mathematical results into the domain of computer science given by the theory of universal algebras and the domain of graph theory. The Peano algebras and graph theory were applied successfully in knowledge representation and various applications.

The concept of semantic schema was introduced in [2] as a structure for knowledge representation. This structure is based on graph theory and universal algebras. Two kinds of computations can be performed in a semantic schema: *syntactic computations* and *semantic computations*. The intuitive aspect of the syntactic computation in a semantic schema $\mathcal{S}$ is given by the set of accepted structured paths $ASP(\mathcal{S})$ of $\mathcal{S}$. In this paper we define the set $ASP(\mathcal{S})$, we organize this set as a partial algebra and we study the maximal elements of this algebra.

This paper is organized as follows: In Section 2 we recall the notion of a θ-semantic schema. In Section 3 we define the set $STR(\mathcal{S})$ of structured paths over $\mathcal{S}$. We decompose this set into disjoint layers and we study the minimal and the maximal elements for each layer. We define the set $ASP(\mathcal{S})$ of all accepted structured paths of $\mathcal{S}$. We show that $ASP(\mathcal{S})$ is a finite set, but $STR(\mathcal{S})$ can be an infinite one. In Section 4 we identify a subset $\mathcal{H}^{us}$ of the Peano σ-algebra generated by the elementary arcs of a semantic schema $\mathcal{S}$, named useful elements for the inference process of $\mathcal{S}$. We study several properties connected by this set and finally we show that this is a finite one. Section 5 contains the conclusions and future work.

2. Semantic schemas

We consider a symbol θ of arity 2 and a finite and nonempty set A_0. We denote by $\overline{A}_0$ the *Peano θ-algebra generated by* A_0 ([1]). This means that $\overline{A}_0 = \bigcup_{n\geq 0} A_n$,

Received June 20, 2011. Revision received August 16, 2011.

where A_n is defined recursively as follows:

$$A_{n+1} = A_n \cup \{ \theta(u, v) \mid u, v \in A_n\}, \quad n \geq 0 \tag{1}$$

We observe that $\overline{A}_0 = \bigcup_{n\geq 0} B_n$, where

$$\begin{cases} B_0 = A_0 \\ B_{n+1} = A_{n+1} \setminus A_n, \quad n \geq 0 \end{cases} \tag{2}$$

Definition 2.1. ([2]) *A θ-* **schema** *(or a* ***semantic schema****) is a system $\mathcal{S} = (X, A_0, A, R)$, where*

- *X is a finite nonempty set and its elements are named* ***object*** *symbols.*
- *A_0 is a finite nonempty set, its elements are named* ***label*** *symbols and $A_0 \subseteq A \subseteq \overline{A}_0$, where $\overline{A}_0$ is the Peano θ-algebra generated by A_0*
- *$R \subseteq X \times A \times X$ is a nonempty set and its elements satisfy the following conditions:*

$$(x, \theta(u, v), y) \in R, u \in \overline{A}_0, v \in \overline{A}_0 \Longrightarrow \exists z \in X : (x, u, z) \in R, (z, v, y) \in R \tag{3}$$

$$\theta(u, v) \in A, (x, u, z) \in R, (z, v, y) \in R \Longrightarrow (x, \theta(u, v), y) \in R \tag{4}$$

$$u \in A \Longleftrightarrow \exists (x, u, y) \in R \tag{5}$$

We denote $R_0 = R \cap (X \times A_0 \times X)$. If $\mathcal{S} = (X, A_0, A, R)$ is a θ-schema then we denote by $\mathcal{L}_n(X)$ the set of all lists of n elements of the set X. Consider also the set $\mathcal{L}(X) = \bigcup_{n\geq 1} \mathcal{L}_n(X)$ of all non empty lists obtained by means of X. On the set $\mathcal{L}(X) \times \overline{A_0}$ we define a partial binary operation:

$$([x_1, \ldots, x_p], u) \circledast ([x_p, \ldots, x_q], v) = ([x_1, \ldots, x_q], \theta(u, v))$$

This is a *partial* operation because two elements $([x_1, \ldots, x_p], u)$ and $([y_1, \ldots, y_s], v)$ can be combined if and only if $x_p = y_1$. In this manner the pair $(\mathcal{L}(X) \times \overline{A_0}, \circledast)$ becomes a *partial algebra*.

3. Structured paths and accepted structured paths

Definition 3.1. *Consider the set $B_0 = \{([x, y], a) \mid (x, a, y) \in R_0\}$ and denote by $STR(\mathcal{S})$ the closure of B_0 in $(\mathcal{L}(X) \times \overline{A_0}, \circledast)$. An element of the set $STR(\mathcal{S})$ is a* ***structured path*** *over $\mathcal{S}$.*

This means that if we build the sequence

$$\begin{cases} B_0 = \{([x, y], a) \mid (x, a, y) \in R_0\} \\ B_{n+1} = B_n \cup \{\gamma \mid \exists \alpha, \beta \in B_n : \gamma = \alpha \circledast \beta\} \end{cases} \tag{6}$$

then

$$STR(\mathcal{S}) = \bigcup_{n\geq 0} B_n \tag{7}$$

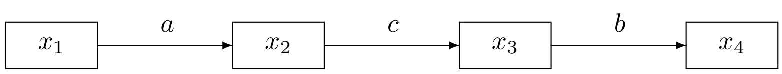

FIGURE 1. R_0 for finite case

The set $STR(\mathcal{S})$ can be a finite or an infinite set. In order to exemplify the finite case we consider the set R_0 described in Figure 1. Applying (6) we obtain the following computations:

$B_0 = \{([x_1, x_2], a), ([x_2, x_3], c), ([x_3, x_4], b)\}$
$B_1 = B_0 \cup \{([x_1, x_2, x_3], \theta(a, c)), ([x_2, x_3, x_4], \theta(c, b))\}$
$B_2 = B_1 \cup \{([x_1, x_2, x_3, x_4], \theta(\theta(a, c)), b), ([x_1, x_2, x_3, x_4], \theta(a, \theta(c, b)))\}$
$B_3 = B_2$

It follows that $STR(\mathcal{S}) = B_2$ and this is a finite set.

In order to exemplify the infinite case we consider the set R_0 represented in Figure 2. The computations in this case can be described as follows:

$B_0 = \{([x_1, x_2], a), ([x_2, x_3], c), ([x_3, x_4], b), ([x_4, x_2], a))\}$
$B_1 = B_0 \cup \{([x_1, x_2, x_3], \theta(a, c)), ([x_2, x_3, x_4], \theta(c, b)), ([x_3, x_4, x_2], \theta(b, a)),$
$([x_4, x_2, x_3], \theta(a, c))\}$
$B_2 = B_1 \cup \{([x_1, x_2, x_3, x_4], \theta(a, \theta(c, b))), ([x_2, x_3, x_4, x_2], \theta(c, \theta(b, a))),$
$([x_3, x_4, x_2, x_3], \theta(b, \theta(a, c))), ([x_4, x_2, x_3, x_4], \theta(a, \theta(c, b)))$
$([x_1, x_2, x_3, x_4], \theta(\theta(a, c), b)), ([x_2, x_3, x_4, x_2], \theta(\theta(c, b), a)),$
$([x_3, x_4, x_2, x_3], \theta(\theta(b, a), c)), ([x_4, x_2, x_3, x_4], \theta(\theta(a, c), b))\}$
$\ldots$

We denote by $[[x_2, x_3, x_4]^n, x_2]$ the list obtained by taking n times the sequence of nodes x_2, x_3, x_4 and x_2 is the last element: $[x_2, x_3, x_4, \ldots, x_2, x_3, x_4, x_2]$. It is not difficult to observe that $([[x_2, x_3, x_4]^n, x_2], \theta(u_{3n-2}, a)) \in STR(\mathcal{S})$, where $u_1 = \theta(c, b)$ and $u_{3k+1} = \theta(\theta(\theta(u_{3k-2}, a), c), b)$ for every $k \geq 1$. It follows that in this case the set $STR(\mathcal{S})$ is an infinite one.

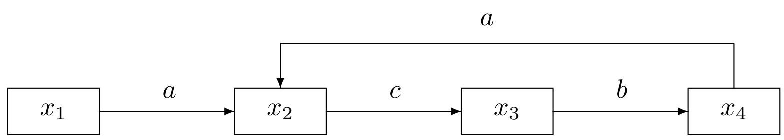

FIGURE 2. R_0 for infinite case

We can write also

$$STR(\mathcal{S}) = \bigcup_{n \geq 0} C_n \tag{8}$$

where

$$\begin{cases} C_0 = B_0 \\ C_{n+1} = B_{n+1} \setminus B_n, \ n \geq 0 \end{cases} \tag{9}$$

From (9) we have $C_j \cap C_r = \emptyset$ for $j \neq r$ and therefore (8) gives a decomposition of $STR(\mathcal{S})$ into disjoint sets. For every $p \geq 0$ we have also

$$B_p = \bigcup_{j=0}^{p} C_j \tag{10}$$

We observe that

$$C_{n+1} = \{\alpha \circledast \beta \mid \alpha \in C_n, \beta \in B_n\} \cup \{\alpha \circledast \beta \mid \alpha \in B_{n-1}, \beta \in C_n\} \tag{11}$$

We define the mapping $trace : \overline{A_0} \longrightarrow \bigcup_{n \geq 1} \mathcal{L}_n(A_0)$ as follows:

- If $a \in A_0$ then $trace(a) = [a]$;

- If $trace(u) = [a_1, \dots, a_p]$ and $trace(v) = [b_1, \dots, b_q]$ then

$$trace(\theta(u, v)) = [a_1, \dots, a_p, b_1, \dots, b_q]$$

Remark 3.1. *If $u \in A$ and $trace(u) = [a_1, \dots, a_p]$ then we write $|u| = p$.*

We define the following entities:
- $Min(C_k)$ gives the least number n such that $([x_1, \dots, x_n], u) \in C_k$;
- $Max(C_k)$ denotes the greatest number n such that $([x_1, \dots, x_n], u) \in C_k$;

By an abuse of language we shall say that an element $([x_1, \dots, x_n], u) \in C_k$ is a minimal (maximal) element if $n = Min(C_k)$ $(n = Max(C_k))$.

Proposition 3.1. *The following valuations are true for C_k, where $k \geq 0$:*

$$Min(C_k) = k + 2 \tag{12}$$

$$Max(C_k) = 2^k + 1 \tag{13}$$

Proof. A minimal element of C_{k+1} is obtained as a product $e_1 \circledast e_2$, where e_1 is a minimal element in C_k and e_2 is from C_0. It follows that $Min(C_{k+1}) = 1 + Min(C_k)$. From this recursive relation we obtain $Min(C_{k+1}) = 2 + Min(C_{k-1}) = \ldots = k + 1 + Min(C_0) = k + 3$ because $Min(C_0) = 2$. Thus (12) is proved.
A maximal element of C_{k+1} is obtained if we take $e_1 \circledast e_2$, where e_1 and e_2 are maximal elements in C_k. If $e_1 = ([y_1, \dots, y_s], u) \in C_k$, $e_2 = ([z_1, \dots, z_s], v) \in C_k$ and $y_s = z_1$, where $s = Max(C_k)$), then $e_1 \circledast e_2 = ([y_1, \dots, y_s, z_2, \dots, z_s], \theta(u, v))$. The list $[y_1, \dots, y_s, z_2, \dots, z_s]$ contains $2s - 1$ elements, therefore $Max(C_{k+1}) = 2Max(C_k) - 1$. Using this recursive relation we find $Max(C_{k+1}) = 2(2Max(C_{k-1}) - 1) - 1 = 2^2 Max(C_{k-1}) - 2 - 1 = \ldots = 2^{k+1} Max(C_0) - 2^k - 2^{k-1} - \ldots - 1 = 2^{k+2} - (1 + 2 + \ldots + 2^k) = 2^{k+2} - 2^{k+1} + 1 = 2^{k+1} + 1$. The relation (13) is proved. $\square$

We recall the concept of path in a semantic schema. A pair $([x_1, \dots, x_{n+1}], [a_1, \dots, a_n]) \in \mathcal{L}(X) \times \mathcal{L}(A_0)$ is a path in $\mathcal{S} = (X, A_0, A, R)$ if $(x_i, a_i, x_{i+1}) \in R_0$ for every $i \in \{1, \dots, n\}$. We denote by $Path(\mathcal{S})$ the set of all paths of $\mathcal{S}$.

Proposition 3.2. *If $([x_1, \dots, x_p], u) \in STR(\mathcal{S})$ then*

$$([x_1, \dots, x_p], trace(u)) \in Path(\mathcal{S})$$

Proof. We prove by induction on $k \geq 0$ the following property: if $([x_1, \dots, x_p], u) \in C_k$ then $([x_1, \dots, x_p], trace(u)) \in Path(\mathcal{S})$. If $k = 0$ then from $([x_1, x_2], u) \in C_0$ we deduce that $(x_1, u, x_2) \in R_0$. Thus $u \in A_0$ and $trace(u) = [u]$. But $([x_1, x_2], [u]) \in Path(\mathcal{S})$, therefore the sentence is true for $k = 0$.

Suppose the sentence is true for every $k \leq n$. Let us consider an element $\gamma = ([x_1, \dots, x_p], \theta(u, v)) \in C_{n+1}$. There are $\alpha, \beta \in \bigcup_{k \leq n} C_k$ such that

$$\gamma = \alpha \circledast \beta \tag{14}$$

Suppose that $\alpha = ([y_1, \dots, y_s], u)$, $\beta = ([z_1, \dots, z_r], v)$ and $y_s = z_1$. From (14) we have $[x_1, \dots, x_p] = [y_1, \dots, y_s, z_2, \dots, z_r]$, therefore

$p = s + r - 1$
$x_j = y_j$ for $j \in \{1, \dots, s\}$
$x_j = z_{j-s+1}$ for $j \in \{s + 1, \dots, s + r - 1\}$

Because $\alpha, \beta \in \bigcup_{k \leq n} C_k$, we can use the inductive assumption both for α and β. It follows that

$$([y_1, \dots, y_s], trace(u)) \in Path(\mathcal{S}) \tag{15}$$

$$([z_1, \dots, z_r], trace(v)) \in Path(\mathcal{S}) \tag{16}$$

Let us denote $trace(u) = [a_1, \ldots, a_{s-1}]$ and $trace(v) = [b_1, \ldots, b_{r-1}]$. It follows that $trace(\theta(u, v)) = [a_1, \ldots, a_{s-1}, b_1, \ldots, b_{r-1}]$. From (15) and (16) we obtain:

$(y_j, a_j, y_{j+1}) \in R_0$ for $j \in \{1, \ldots, s-1\}$

$(y_s, b_1, z_2) \in R_0$ because $y_s = z_1$ and $(z_1, b_1, z_2) \in R_0$

$(z_j, b_j, z_{j+1}) \in R_0$ for $j \in \{2, \ldots, r-1\}$

It follows that $([y_1, \ldots, y_s, z_2, \ldots, z_r], [a_1, \ldots, a_{s-1}, b_1, \ldots, b_{r-1}]) \in Path(\mathcal{S})$. Equivalently we can write $([x_1, \ldots, x_p], trace(\theta(u, v))) \in Path(\mathcal{S})$. The proposition is proved. □

Proposition 3.3. *The following properties are satisfied by the elements of $STR(\mathcal{S})$:*

(1) *Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in STR(\mathcal{S})$. If $\alpha_1 \circledast \beta_1 = \alpha_2 \circledast \beta_2$ then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

(2) *If $\gamma = ([x_1, \ldots, x_{p+1}], \theta(u_1, u_2)) \in STR(\mathcal{S})$, where $u_1, u_2 \in \overline{A_0}$, then there is a number $s \in \{2, \ldots, p\}$ and only one, such that $\alpha = ([x_1, \ldots, x_s], u_1) \in STR(\mathcal{S})$, $\beta = ([x_s, \ldots, x_{p+1}],\ u_2) \in STR(\mathcal{S})$ and $\gamma = \alpha \circledast \beta$.*

Proof. We can suppose that

$$\alpha_1 = ([x_1, \ldots, x_q], u_1), u_1 \in \overline{A_0} \tag{17}$$

$$\alpha_2 = ([y_1, \ldots, y_r], u_2), u_2 \in \overline{A_0} \tag{18}$$

$$\beta_1 = ([x_q, \ldots, x_{q+s}], v_1), v_1 \in \overline{A_0} \tag{19}$$

$$\beta_2 = ([y_r, \ldots, y_{r+k}], v_2), v_2 \in \overline{A_0} \tag{20}$$

From $\alpha_1 \circledast \beta_1 = \alpha_2 \circledast \beta_2$ we obtain

$$([x_1, \ldots, x_{q+s}], \theta(u_1, v_1)) = ([y_1, \ldots, y_{r+k}], \theta(u_2, v_2))$$

therefore

$$([x_1, \ldots, x_{q+s}] = [y_1, \ldots, y_{r+k}] \tag{21}$$

and $\theta(u_1, v_1) = \theta(u_2, v_2)$. Based on the properties of a Peano algebra we deduce that $u_1 = u_2$ and $v_1 = v_2$. By Proposition 3.2 we have

$$([x_1, \ldots, x_q], trace(u_1)) \in Path(\mathcal{S}) \tag{22}$$

$$([y_1, \ldots, y_r], trace(u_2)) \in Path(\mathcal{S}) \tag{23}$$

If $trace(u_1) = [a_1, \ldots, a_m]$ then from (22) we have $q = m - 1$. From $u_1 = u_2$ we have $trace(u_2) = [a_1, \ldots, a_m]$ and from (23) we obtain $r = m - 1$. It follows that $q = r$. From (21) we have $q + s = r + k$, therefore $s = k$. From the same relation we obtain $x_j = y_j$ for $j \in \{1, \ldots, q + s\}$. Now, from (17) and (18) we obtain $\alpha_1 = \alpha_2$ and from (19) and (20) we obtain $\beta_1 = \beta_2$.

We prove now the second part of this proposition. Suppose that $\gamma = ([x_1, \ldots, x_{p+1}], \theta(u_1, u_2)) \in STR(\mathcal{S})$ and $u_1 \in \overline{A_0}$, $u_2 \in \overline{A_0}$. We can suppose that there is $\alpha = ([y_1, \ldots, y_{r+1}], v_1) \in C_n$ and $\beta = ([z_1, \ldots, z_{q+1}], v_2) \in \bigcup_{j=0}^{n} C_j$ such that $\gamma = \alpha \circledast \beta$. Using these notations we can write $\alpha \circledast \beta = ([y_1, \ldots, y_{r+1}, z_2, \ldots, z_{q+1}], \theta(v_1, v_2))$ and because there exists the product $\alpha \circledast \beta$ we have $y_{r+1} = z_1$. From $\gamma = \alpha \circledast \beta$ we obtain

$$\theta(u_1, u_2) = \theta(v_1, v_2) \tag{24}$$

$$r + q + 1 = p + 1 \ \&\ [x_1, \ldots, x_{p+1}] = [y_1, \ldots, y_{r+1}, z_2, \ldots, z_{q+1}] \tag{25}$$

Take $s = r + 1$ and we show that this number satisfies the second part of the proposition. From (24) we obtain $u_1 = v_1$ and $u_2 = v_2$. From (25) we have $y_1 = x_1$, …, $y_{r+1} = x_{r+1}$. But $r + 1 = s$, therefore $y_1 = x_1$, …, $y_s = x_s$. This property allows to write $\alpha = ([x_1, \ldots, x_s], u_1)$. The same relation (25) gives also $x_{r+2} = z_2$, …, $x_{p+1} = z_{q+1}$, therefore $\beta = ([x_{r+1}, \ldots, x_{p+1}], u_2) = ([x_s, \ldots, x_{p+1}], u_2)$.

It remains to show that $s \in \{2, \ldots, p\}$. We have $q \geq 1$ and $p = r + q$, therefore $p \geq r + 1$. But $r + 1 = s$ and thus $s \leq p$. On the other hand, from $s = r + 1$ and $r \geq 1$ we obtain $s \geq 2$. The uniqueness of s is obtained from the first sentence of the proposition. The proposition is proved. □

As we stated before a structured path is a pair $([x_1, \ldots, x_{n+1}], u)$ such that $u \in \overline{A_0}$ and $([x_1, \ldots, x_{n+1}], trace(u))$ is a path in $\mathcal{S}$. From this set we retain only some subset of useful paths as we specify in the next definition.

Definition 3.2. *An element $([x_1, \ldots, x_n], u) \in STR(\mathcal{S})$ such that $u \in A$ is an **accepted structured path** of $\mathcal{S}$. We denote by $ASP(\mathcal{S})$ the set of all accepted structured paths of $\mathcal{S}$.*

Proposition 3.4. *If $\gamma = ([x_1, \ldots, x_{p+1}], \theta(u_1, u_2)) \in ASP(\mathcal{S})$, where $u_1, u_2 \in \overline{A_0}$, then there is a number $s \in \{2, \ldots, p\}$ and only one, such that $\alpha = ([x_1, \ldots, x_s], u_1) \in ASP(\mathcal{S})$, $\beta = ([x_s, \ldots, x_{p+1}], u_2) \in ASP(\mathcal{S})$ and $\gamma = \alpha \circledast \beta$.*

Proof. We apply Proposition 3.3. If $\theta(u_1, u_2) \in A$ and $u_1, u_2 \in \overline{A_0}$ then $u_1, u_2 \in A$. It follows that $\alpha \in ASP(\mathcal{S})$ and $\beta \in ASP(\mathcal{S})$. □

Proposition 3.5. *If $u \in A$ and $|u| = n$ then there is an element $([x_1, \ldots, x_{n+1}], u) \in ASP(\mathcal{S})$.*

Proof. Consider the sets (2) and denote $R_n = R \cap (X \times B_n \times X)$, where $n \geq 0$. We prove by induction on n the following property $P(n)$: *if $(x, u, z) \in R_n$ then there is an element $([x, \ldots, z], u) \in ASP(\mathcal{S})$.* The property $P(0)$ is true because if $(x, u, z) \in R_0$ then $u \in A_0$, $([x, z], u) \in STR(\mathcal{S})$, therefore $([x, z], u) \in ASP(\mathcal{S})$. Suppose that $P(k)$ is true for every $k \in \{0, \ldots, n\}$. Consider an element $(x, u, z) \in R_{n+1}$. There is $u_1, v_1 \in \bigcup_{p=0}^{n} B_p$ such that $n = \theta(u_1, v_1)$. From $(x, u, z) \in R_{n+1}$ and (3) we deduce that there is $y \in X$ such that $(x, u_1, y) \in \bigcup_{p=0}^{n} R_p$ and $(y, u_2, z) \in \bigcup_{p=0}^{n} R_p$. Applying the inductive assumption we deduce that there are the elements $d_1 = ([x, \ldots, y], u_1) \in ASP(\mathcal{S})$ and $d_2 = ([y, \ldots, z], v_1) \in ASP(\mathcal{S})$. But $d_1 \circledast d_2$ satisfies $P(n+1)$ and therefore $P(n)$ is true for every $n \geq 0$. But $A = pr_2 R$ and $R = \bigcup_{j \geq 0} R_n$. It follows that if $u \in A$ then there is an element $(x, u, y) \in R_j$ for some $j \geq 0$. Applying property $P(j)$ we deduce that there is an element $([x, \ldots, y], u) \in ASP(\mathcal{S})$. But $ASP(\mathcal{S}) \subseteq STR(\mathcal{S})$, therefore $([x, \ldots, y], trace(u)) \in Path(\mathcal{S})$. Thus, if $|u| = n$ then the list of nodes $[x, \ldots, y]$ has $n + 1$ elements. □

Proposition 3.6. *If $m = max\{|u| \mid u \in A\}$ then*
(1) $ASP(\mathcal{S}) \subseteq \bigcup_{k=0}^{m-1} C_k$.
(2) $C_k \cap ASP(\mathcal{S}) = \emptyset$ *for every* $k \geq m$.

Proof. Take $d = ([x_1, \ldots, x_p], v) \in ASP(\mathcal{S})$. We have $ASP(\mathcal{S}) \subseteq STR(\mathcal{S}) = \bigcup_{k \geq 0} C_k$. There is $k \geq 0$ such that $d \in C_k$. By Proposition 3.1 we obtain $k + 2 \leq p$. But $p - 1 = length(d) = |v|$ and $|v| \leq m$. It follows that $p - 1 \leq m$, therefore $k + 1 \leq m$. In conclusion $d \in C_k$ and $k \leq m - 1$.
In order to prove the second sentence we suppose by contrary, $C_{k_0} \cap ASP(\mathcal{S}) \neq \emptyset$ for some $k_0 \geq m$. Take $p \in C_{k_0} \cap ASP(\mathcal{S})$. But $p \in ASP(\mathcal{S})$ and $ASP(\mathcal{S}) \subseteq \bigcup_{k=0}^{m-1} C_k$. There is $k_1 \leq m - 1$ such that $p \in C_{k_1}$. It follows that $C_{k_0} \cap C_{k_1} \neq \emptyset$, which is not true because $k_0 \neq k_1$. □

Corollary 3.1. *$ASP(\mathcal{S})$ is a finite set.*

Proof. $\bigcup_{k=0}^{m-1} C_k$ is a finite set because C_k is finite for every $k \geq 0$. □

4. Maximal elements in $ASP(\mathcal{S})$

We consider a symbol h of arity 3, a symbol σ of arity 2 and denote $M = \{h(x,a,y) \mid (x,a,y) \in R_0\}$. Consider the Peano σ-algebra $\mathcal{H}$ generated by M. This is an infinite set. We define the mapping $fst : \mathcal{H} \longrightarrow X$ and $lst : \mathcal{H} \longrightarrow X$ as follows:

$$\begin{cases} fst(h(x,a,y)) = x \text{ if } (x,a,y) \in R_0 \\ fst(\sigma(m_1,m_2)) = fst(m_1) \text{ if } m_1 \in \mathcal{H}, m_2 \in \mathcal{H} \end{cases}$$

$$\begin{cases} lst(h(x,a,y)) = y \text{ if } (x,a,y) \in R_0 \\ lst(\sigma(m_1,m_2)) = lst(m_2) \text{ if } m_1 \in \mathcal{H}, m_2 \in \mathcal{H} \end{cases}$$

We consider a path $d = ([x_1,\ldots,x_{n+1}],[a_1,\ldots,a_n]) \in Path(\mathcal{S})$. This means that $(x_j,a_j,x_{j+1}) \in R_0$ for each $j \in \{1,\ldots,n\}$. We define the sequence $\{(H_k^d, U_k^d)\}_{k\geq 0}$, where $H_k^d \subseteq \mathcal{H}$ and $U_k^d : H_k^d \to \overline{A_0}$, as follows:

$$\begin{cases} H_0^d = \{h(x_j,a_j,x_{j+1}) \mid j = 1,\ldots,n\} \\ U_0^d : H_0^d \longrightarrow A_0,\ U_0^d(h(x_j,a_j,x_{j+1})) = a_j \end{cases}$$

$$\begin{cases} H_{k+1}^d = H_k^d \cup \{\sigma(w_1,w_2) \mid w_1,w_2 \in H_k^d, lst(w_1) = fst(w_2), \\ \qquad\qquad\qquad\qquad\qquad \theta(U_k^d(w_1),U_k^d(w_2)) \in A\} \\ U_{k+1}^d(w) = U_k^d(w)\ for\ w \in H_k^d \\ U_{k+1}^d(\sigma(w_1,w_2)) = \theta(U_k^d(w_1),U_k^d(w_2))\ if\ \sigma(w_1,w_2) \in H_{k+1}^d \setminus H_k^d \end{cases}$$

We consider the set $\mathcal{H}^{us} = \bigcup_{d\in Path(S)} \mathcal{H}(d)$ and the elements of this set are named **useful elements** for the inference process of $\mathcal{S}$.

If $d \in Path(S)$ then we can write

$$\mathcal{H}(d) = \bigcup_{k\geq 0} E_k^d$$

where $E_0^d = H_0^d$ and $E_{k+1}^d = H_{k+1}^d \setminus H_k^d$ for $k \geq 0$. Moreover, we can relieve the following property

$$\begin{aligned} E_{k+1}^d = \{\sigma(w_1,w_2) \mid w_1 \in E_k^d, w_2 \in H_k^d, Q_k(w_1,w_2)\} \cup \\ \{\sigma(w_1,w_2) \mid w_2 \in E_k^d, w_1 \in H_k^d, Q_k(w_1,w_2)\} \end{aligned} \tag{26}$$

where $Q_k(w_1,w_2)$ represents the condition

$$lst(w_1) = fst(w_2), \theta(U_k^d(w_1),U_k^d(w_2)) \in A$$

Proposition 4.1. *Let be $m = max\{|u| \mid u \in A\}$. If $d \in Path(\mathcal{S})$ then $E_k^d = \emptyset$ for every $k \geq m$.*

Proof. We prove first that $E_m^d = \emptyset$. Suppose by contrary, that $E_m^d \neq \emptyset$. Take an element $w \in E_m^d$. We use the extractive mappings f of $\mathcal{S}$ ([3]). From Proposition 3.4 proved in [3] we have $f(w) = ([x_1,\ldots,x_n],u) \in C_m \cap ASP(\mathcal{S})$. From Proposition 3.1 we obtain $m+2 \leq n$. But $|u| = n-1$, therefore $|u| \geq m+1$. From $f(w) \in ASP(\mathcal{S})$ we have $u \in A$. This fact is not possible because $m = max\{|u| \mid u \in A\}$. Now from (26) we obtain $E_{m+1}^d = E_{m+2}^d = \ldots = \emptyset$. □

Consider a path $d = ([x_1,\ldots,x_{n+1}],[a_1,\ldots,a_n]) \in Path(\mathcal{S})$. For $1 \leq i < j \leq n+1$ the path $d_1 = ([x_i,\ldots,x_j],[a_i\ldots,a_{j-1}])$ is a **subpath** of d. We write in this case $d_1 \trianglelefteq d$. We denote by $SPath(d)$ the set of all subpaths of the path d. This relation satisfies the following properties:

- $d \trianglelefteq d$ for every $d \in Path(\mathcal{S})$
- If $d_1 \trianglelefteq d_2$ and $d_2 \trianglelefteq d_1$ then $d_1 = d_2$.
- If $d_1 \trianglelefteq d_2$ and $d_2 \trianglelefteq d_3$ then $d_1 \trianglelefteq d_3$.

It follows that $(Path(\mathcal{S}), \trianglelefteq)$ is a partial ordered set.

Proposition 4.2. *If $p \trianglelefteq d$ then for every $k \geq 0$:*
$E_k^p \subseteq E_k^d$
$U_k^p(w) = U_k^d(w)$ *for every* $w \in E_k^p$

Proof. Obviously $E_0^p \subseteq E_0^d$ and $U_0^p(w) = U_0^d(w)$ for every $w \in E_0^p$. Suppose that for every $k \in \{0, \ldots, n\}$ we have $E_k^p \subseteq E_k^d$ and $U_k^p(w) = U_k^d(w)$ for every $w \in E_k^p$. Take $w \in E_{n+1}^p$. There are $w_1 \in E_n^p$ and $w_2 \in \bigcup_{j=0}^n E_j^p$ such that $w = \sigma(w_1, w_2)$, $lst(w_1) = fst(w_2)$ and $\theta(U_n^p(w_1), U_n^p(w_2)) \in A$. By the inductive assumption we have $w_1 \in E_n^d$, $w_2 \in E_n^d$, $U_n^p(w_1) = U_n^d(w_1)$ and $U_n^p(w_2) = U_n^d(w_2)$. Thus $\theta(U_n^d(w_1), U_n^d(w_2)) \in A$, therefore $w = \sigma(w_1, w_2) \in E_{n+1}^d$. Similarly we proceed for the case $w_2 \in E_n^p$ and $w_1 \in \bigcup_{j=0}^n E_j^p$. We have also $U_{n+1}^p(w) = \theta(U_n^p(w_1), U_n^p(w_2)) = \theta(U_n^p(w_1), U_n^p(w_2)) = U_{n+1}^d(w)$. ☐

Definition 4.1. *For $p = ([x_1, \ldots, x_{n+1}], u) \in ASP(\mathcal{S})$ we denote $p_t = ([x_1, \ldots, x_{n+1}], trace(u))$. Consider $d \in Path(\mathcal{S})$. We denote*

$$ASP(d) = \{p \in ASP(\mathcal{S}) \mid p_t \trianglelefteq d\}$$

$$ACC(d) = \{p_t \mid p \in ASP(d)\}$$

The set of all maximal elements of the partial algebra $(ACC(d), \trianglelefteq)$ is denoted by $MAX_a(d)$.

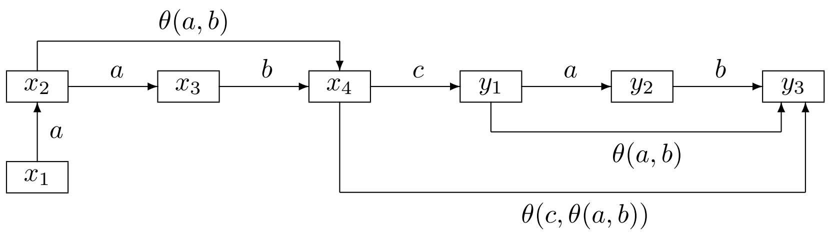

FIGURE 3. A part of a θ-schema

In order to exemplify these concepts we consider the path $d = ([x_1, x_2, x_3, x_4, y_1, y_2, y_3], [a, a, b, c, a, b])$ represented in Figure 3. For this case we have

$$MAX_a(d) = \{([x_1, x_2], [a]), ([x_2, x_3, x_4], [a, b]), ([x_4, y_1, y_2, y_3], [c, a, b])\}$$

If we denote by d_1, d_2 and d_3 the elements of $MAX_a(d)$ we observe the following properties:

(1) $E_0^{d_1} = \{h(x_1, a, x_2)\}$; $E_k^{d_1} = \emptyset$ for $k \geq 1$;
(2) $E_0^{d_2} = \{h(x_2, a, x_3), h(x_3, b, x_4)\}$; $E_1^{d_2} = \{\sigma(h(x_2, a, x_3), h(x_3, b, x_4))\}$; $E_k^{d_2} = \emptyset$ for $k \geq 2$;
(3) $E_0^{d_3} = \{h(x_4, c, y_1), h(y_1, a, y_2), h(y_2, b, y_3)\}$; $E_1^{d_3} = \{\sigma(h(y_1, a, y_2), h(y_2, b, y_3))\}$; $E_2^{d_3} = \{\sigma(h(x_4, c, y_1), \sigma(h(y_1, a, y_2), h(y_2, b, y_3)))\}$; $E_k^{d_3} = \emptyset$ for $k \geq 3$;
(4) $E_k^d = E_k^{d_1} \cup E_k^{d_2} \cup E_k^{d_3}$ for $k \geq 0$.

The last property is not a particular one. This is stated in the next proposition.

Proposition 4.3. *If $d \in Path(\mathcal{S})$ then for every $k \geq 0$ we have*

$$E_k^d = \bigcup_{p \in MAX_a(d)} E_k^p \tag{27}$$

Proof. If $p \in MAX_a(d)$ then $p \trianglelefteq d$. Applying Proposition 4.2 we obtain $E_k^p \subseteq E_k^d$ for every $k \geq 0$. It follows that

$$\bigcup_{p \in MAX_a(d)} E_k^p \subseteq E_k^d \tag{28}$$

We prove now the following property $P(k)$: *if $w \in E_k^d$ then there is $p \in MAX_a(d)$ such that $w \in E_k^p$.*
If $w = h(x, a, y) \in E_0^d$ then we have two cases:

- $([x, y], [a]) \in MAX_a(d)$
  In this case we take $p = ([x, y], [a])$. We obtain $w \in E_0^p$.
- $([x, y], [a]) \notin MAX_a(d)$
  In this case there is $d_1 = ([x_1, \ldots, x_{n+1}], u) \in ASP(\mathcal{S})$ such that $([x, y], [a]) \triangleleft p_1$, where $p_1 = ([x_1, \ldots, x_{n+1}], trace(u))$. We can suppose that $p_1 \in MAX_a(d)$ because otherwise we reiterate this procedure. This situation can not be iterated infinitely times because the set $SPath(d)$ is a finite set. We have $w \in E_0^{p_1}$.

It follows that $P(0)$ is true. Consider an element $w \in E_k^d$ for some $k \geq 1$. There are $w_1 \in E_{k-1}^d$ and $w_2 \in \bigcup_{j=0}^{k-1} E_j^d$ such that $w = \sigma(w_1, w_2)$ or $w = \sigma(w_2, w_1)$. Suppose we have the first case, the second case is discussed in a similar manner. We apply the inductive assumption. There is $p_1 \in MAX_a(d)$ such that $w_1 \in E_{k-1}^{p_1}$. There is also $p_2 \in MAX_a(d)$ such that $w_2 \in E_j^{p_2}$, where $j \in \{0, \ldots, k-1\}$. It follows that $p_1 = p_2$ and $w \in E_k^{p_1}$. Thus the property $P(k)$ is true and from this property we deduce that

$$E_k^d \subseteq \bigcup_{p \in MAX_a(d)} E_k^p \tag{29}$$

Now, from (28) and (29) we deduce (27). $\square$

Proposition 4.4. *Let be $m = max\{|u| \mid u \in A\}$. If $p \in ASP(\mathcal{S})$ then $H_k^{p_t} = H_{m-1}^{p_t}$ for every $k \geq m$. Moreover, $\mathcal{H}^{us}$ is a finite set:*

$$\mathcal{H}^{us} = \bigcup_{k=0}^{m-1} \bigcup_{p \in ASP(\mathcal{S})} E_k^{p_t} \tag{30}$$

Proof. By Proposition 4.1 we have $E_k^{p_t} = \emptyset$ for every $k \geq m$. It follows that for $k \geq m$ we can write $H_k^{p_t} = \bigcup_{j=0}^{k} E_j^{p_t} = \bigcup_{j=0}^{m-1} E_j^{p_t} = H_{m-1}^{p_t}$.
We prove by double inclusion the following equality:

$$\bigcup_{p \in ASP(\mathcal{S})} E_k^{p_t} = \bigcup_{d \in Path(S)} \bigcup_{j=0}^{k} E_j^d \tag{31}$$

Take $w \in \bigcup_{p \in ASP(\mathcal{S})} E_k^{p_t}$. There is $p \in ASP(\mathcal{S})$ such that $w \in E_k^{p_t}$. But $p_t \in Path(\mathcal{S})$ therefore $w \in \bigcup_{d \in Path(S)} \bigcup_{j=0}^{k} E_j^d$. So we proved the inclusion

$$\bigcup_{p \in ASP(\mathcal{S})} E_k^{p_t} \subseteq \bigcup_{d \in Path(S)} \bigcup_{j=0}^{k} E_j^d \tag{32}$$

We prove now the converse inclusion. Take $w \in \bigcup_{d \in Path(S)} \bigcup_{j=0}^{k} E_j^d$. There is $d \in Path(S)$ such that $w \in \bigcup_{j=0}^{k} E_j^d$. It follows that there is $j \in \{0, \ldots, k\}$ such that $w \in E_j^d$. By Proposition 4.3 we have $E_j^d = \bigcup_{p \in MAX_a(d)} E_j^p$, therefore there is $p \in$

$MAX_a(d)$ such that $w \in E_j^p$. From Definition 4.1 we deduce that there is $q \in ASP(\mathcal{S})$ such that $p = q_t$, therefore $w \in \bigcup_{p\in ASP(\mathcal{S})} E_k^{p_t}$. Thus we have the inclusion

$$\bigcup_{p\in ASP(\mathcal{S})} E_k^{p_t} \supseteq \bigcup_{d\in Path(S)} \bigcup_{j=0}^{k} E_j^d \tag{33}$$

Now, from (32) and (33) we obtain (31). From Proposition 4.1 we have $E_k^d = \emptyset$ for every $k \geq m$. It follows that $\bigcup_{k=0}^{m-1} \bigcup_{p\in ASP(\mathcal{S})} E_k^{p_t} = \bigcup_{k=0}^{m-1} \bigcup_{d\in Path(S)} \bigcup_{j=0}^{k} E_j^d$ $= \mathcal{H}^{us}$. Based on (30) and Corollary 3.1 we deduce that $\mathcal{H}^{us}$ is a finite set. □

5. Conclusions

In this paper we discussed the set $ASP(\mathcal{S})$ of all accepted structured paths of $\mathcal{S}$, we organize this set as a partial ordered set and we discussed the maximal elements of this algebra. The results presented in this paper can be considered as closely related by the results presented in [3], where a revised formal computation in a semantic schema based on the elements of $\mathcal{H}^{us}$ is considered. The main results can be summarized as follows: the set $ASP(\mathcal{S})$ is a finite one even if the set $STR(\mathcal{S})$ is an infinite set; the maximal elements of $ASP(\mathcal{S})$ are fully implied to compute the elements of the set $\mathcal{H}^{us}$; the new formal computation, as was described in [3], is based on the elements of $\mathcal{H}^{us}$, which is a finite one. In a future paper we will discuss the new computational aspects and we will apply it to a master-slave systems of semantic schemas.

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