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On the second derivative of the sums of trigonometric series

Xhevat Z. Krasniqi

Abstract. Some representations for the second derivatives of the sums of the cosine or sine trigonometric series are found in terms of the second differences of their coefficients. If for the cosine series we denote its sum by $f(x)$, then it is proved that under certain conditions the function $f'(x)-(a_1-2a_2)x$ is concave or convex on $(0,\pi]$, which demonstrates an adherence of those representations. Also, we have obtained some estimates of the integrals of the absolute values of those derivatives in terms of the coefficients of such series.

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1. Introduction

Let us consider the trigonometric series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos kx \tag{1}$$

and

$$\sum_{k=1}^{\infty} a_k \sin kx, \tag{2}$$

whose coefficients tend to zero, in other words

$$\lim_{k\to\infty} a_k = 0. \tag{3}$$

A numerical sequence $\{a_k\}$ is said to be quasi-convex, if

$$\sum_{k=1}^{\infty} k|\triangle^2 a_k| < \infty, \tag{4}$$

where $\triangle a_k = a_k - a_{k+1}$, $\triangle^2 a_k = \triangle(\triangle a_k)$.

It is a well-known fact that conditions (3) and (4) are satisfied if and only if the sequence $\{a_k\}$ can be expressed as a difference of two convex sequences ($\triangle^2 a_k \geq 0$) that tend to zero (see [6], paragraph 5.7.1). Moreover, it is known that the series (1) and (2) with convex coefficients that tend to zero, converge uniformly on each interval $[\varepsilon,\pi]$, $\varepsilon > 0$, and their sums are continuously differentiable on $(0,\pi]$ (see [6], paragraph 5.7.6). So, under conditions (3) and (4), the series (1) and (2) possess these characteristics too. We shall denote by $f(x)$ and $g(x)$ the sums of the series (1) and (2), respectively.

S. A. Telyakovskiĭ [4] has investigated the estimates of integrals of $|f'(x)|$ and $|g'(x)|$ on intervals that are inside the interval $(0,\pi]$. Firstly, he studied the aspects of how the integrals of $|f'(x)|$ and $|g'(x)|$ increase on intervals $[\varepsilon,\pi]$ when $\varepsilon \to +0$, if

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these functions are integrable over their period, and secondly he studied the aspects of how these integrals decrease on intervals $[0, \varepsilon]$ when $\varepsilon \to +0$, if these functions are integrable over their period. In fact, he studied these integrals over the intervals of the form $[\pi/(m+1), \pi/\ell]$, where $1 \le \ell \le m$, and $\ell, m \in \mathbb{N}$. Then putting $\ell = 1$ and letting $m \to \infty$ respectively, he obtained the estimates mentioned above.

So far, the results of Telyakovskiĭ are extended in two main directions: by Gembarskaya has been extended for functions of two variables defined by trigonometric cosine series obtaining some estimates for their variations in the Hardy–Vitali sense (see [2]), and also by present author has been extended those results using the concept of the quasi-convex sequences of higher order (see [3]).

We say that $\{a_k\}$ is a quasi-convex sequence of order r, $r = 1, 2, \ldots$, if it tends to zero and satisfies condition

$$\sum_{k=1}^{\infty} k^r |\triangle^{r+1} a_k| < \infty, \tag{5}$$

where $\triangle^r a_k = \triangle \left(\triangle^{r-1} a_k\right)$.

Note that for $r = 1$ the concept of quasi-convexity of order r reduces to the standard concept of quasi-convexity of a sequence.

In this paper we are concerned regarding to the following question: Under what conditions the second derivatives of the sums $f(x)$ and $g(x)$ can be represented in a similar form as those of Telyakovskiĭ ? In order to give an answer for this question, which is the main aim of this paper, we have deduced that conditions (3) and

$$\sum_{k=1}^{\infty} k^3 |\triangle^2 a_k| < \infty, \tag{6}$$

are sufficient conditions so that the second derivatives of the sums $f(x)$ and $g(x)$ are continuous functions on the interval $(0, \pi]$. It should be noted here that (5) does not imply (6), but the converse obviously is true $(r = 1)$.

Throughout this paper $O-$symbol contain positive constants, generally speaking, different in different estimates.

The rest of the paper is organized as follows. Section 2, contains some helpful lemmas that are needed to prove main results. In section 3 we have proved the main results, until we finalize with Section 4, where we have verified a few corollaries.

2. Auxiliary Lemmas

Till to the end of the paper we denote

$$w_k := k^2 a_k, \quad (k = 1, 2, \ldots).$$

Lemma 2.1. *Let a_k be real numbers such that*

$$\triangle a_k \to 0 \quad as \quad k \to \infty. \tag{7}$$

If the condition

$$\sum_{k=1}^{\infty} k^3 |\triangle^2 a_k| < \infty \tag{8}$$

holds, then the condition

$$\sum_{k=1}^{\infty} |\triangle^2 w_k| < \infty \tag{9}$$

holds as well.

Proof. We have

$$\begin{aligned}\triangle w_k &= w_k - w_{k+1}\\ &= k^2 a_k - (k+1)^2 a_{k+1}\\ &= k^2 \triangle a_k - (2k+1) a_{k+1}, \end{aligned} \qquad (10)$$

and

$$\begin{aligned}\triangle^2 w_k &= k^2 \triangle a_k - (2k+1)a_{k+1} - \left[(k+1)^2 \triangle a_{k+1} - (2k+3)a_{k+2}\right]\\ &= k^2 \triangle^2 a_k - 2(2k+1)\triangle a_{k+1} + 2a_{k+2}\\ &= k^2 \triangle^2 a_k - 4k\triangle a_{k+1} - 2\triangle a_{k+1} + 2a_{k+2}.\end{aligned}$$

Consequently, since $\triangle a_k \to 0$ then

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\triangle^2 w_k\right| &\le \sum_{k=1}^{\infty} k^2\left|\triangle^2 a_k\right| + 4\sum_{k=1}^{\infty} k\left|\triangle a_{k+1}\right|\\ &\quad +2\sum_{k=1}^{\infty}\left|\triangle a_{k+1}\right| + 2\sum_{k=1}^{\infty}\left|a_{k+2}\right|\\ &\le \sum_{k=1}^{\infty} k^2\left|\triangle^2 a_k\right| + 4\sum_{k=1}^{\infty} k\sum_{i=k+1}^{\infty}\left|\triangle^2 a_i\right|\\ &\quad +2\sum_{k=1}^{\infty} k\sum_{i=k+1}^{\infty}\left|\triangle^2 a_i\right| + 2\sum_{k=1}^{\infty}\sum_{i=k+2}^{\infty}\left|\triangle a_i\right|\\ &\le 19\sum_{k=1}^{\infty} k^2\left|\triangle^2 a_k\right| + 6\sum_{k=1}^{\infty} k^2\left|\triangle a_k\right|\\ &\le 19\sum_{k=1}^{\infty} k^2\left|\triangle^2 a_k\right| + 6\sum_{k=1}^{\infty} k^2\sum_{i=k}^{\infty}\left|\triangle^2 a_i\right|\\ &= 19\sum_{k=1}^{\infty} k^2\left|\triangle^2 a_k\right| + \sum_{k=1}^{\infty} k(k+1)(2k+1)\left|\triangle^2 a_k\right|\\ &\le 19\sum_{k=1}^{\infty} k^3\left|\triangle^2 a_k\right| + 6\sum_{k=1}^{\infty} k^3\left|\triangle^2 a_k\right|\\ &\le 25\sum_{k=1}^{\infty} k^3\left|\triangle^2 a_k\right|.\end{aligned}$$

So (9) clearly implies (8). □

Lemma 2.2. *Let a_k be real numbers that satisfy conditions (3) and (6). Then*

$$\triangle w_k = \triangle\left(k^2 a_k\right) \to 0 \quad \textit{as} \quad k \to \infty. \qquad (11)$$

Proof. We note that

$$\left|k^2 \triangle a_k\right| = \left|k^2 \sum_{i=k}^{\infty}\left(\triangle^2 a_i\right)\right| \le \sum_{i=k}^{\infty} i^3\left|\triangle^2 a_i\right| \to 0 \quad \text{as} \quad k \to \infty, \qquad (12)$$

and

$$\begin{aligned}
|ka_{k+1}| &= \left| k \sum_{i=k+1}^{\infty} (\triangle a_i) \right| \\
&\leq \sum_{i=k+1}^{\infty} (i-1) \left|\triangle a_i\right| \\
&= \sum_{i=k+1}^{\infty} (i-1) \left| \sum_{j=i}^{\infty} (\triangle^2 a_j) \right| \\
&\leq \sum_{i=k+1}^{\infty} (i-1) \sum_{j=i}^{\infty} \left|\triangle^2 a_j\right| \\
&= \sum_{j=k+1}^{\infty} \left|\triangle^2 a_j\right| \sum_{i=k+1}^{j} (i-1) \\
&\leq \sum_{j=k+1}^{\infty} (j-1)(j-k) \left|\triangle^2 a_j\right| \\
&\leq \sum_{j=k+1}^{\infty} j^3 \left|\triangle^2 a_j\right| \to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Therefore, using (10) and the above estimates we immediately obtain (11). □

We pass now to the main results of the paper.

3. Main Results

We establish the following statement.

Theorem 3.1. *If the coefficients of the series (1) satisfy conditions (3) and (6), then for the second derivative of its sum the following equality holds*

$$f''(x) = -\sum_{k=0}^{\infty} \triangle^2 \left(k^2 a_k\right) F_k(x), \qquad 0 < x \leq \pi. \tag{13}$$

Proof. Since the condition $\sum\limits_{k=1}^{\infty} k^3 |\triangle^2 a_k| < \infty$ implies $\sum\limits_{k=1}^{\infty} k |\triangle^2 a_k| < \infty$, then the series (1) converges uniformly on $[\varepsilon, \pi]$, $\varepsilon > 0$, therefore its Cesàro means

$$\sigma_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) a_k \cos kx$$

converge uniformly as well, on $[\varepsilon, \pi]$.

Let us prove that $\sigma_n''(f;x)$ converges uniformly at $f''(x)$ on $[\varepsilon, \pi]$. Indeed, we denote

$$\beta_k := \left(1 - \frac{k}{n+1}\right) w_k, \quad k = 0, 1, \ldots, n+1,$$

then

$$\sigma_n''(f;x) = -\frac{\beta_0}{2} - \sum_{k=1}^{n} \beta_k \cos kx, \quad (\beta_0 = \beta_{n+1} = 0).$$

Applying successively two times the summation by parts we obtain

$$\begin{aligned}\sigma_n''(f;x) &= -\sum_{k=0}^{n} \triangle\beta_k D_k(x)\\ &= -\sum_{k=0}^{n-1} \triangle^2\beta_k F_k(x) - \triangle\beta_n F_n(x)\\ &= -\sum_{k=0}^{n-1} \triangle^2\beta_k F_k(x) - \frac{n^2}{n+1} a_n F_n(x).\end{aligned}$$

It is easily shown that

$$\triangle^2\beta_k = \left(1-\frac{k}{n+1}\right)\triangle^2 w_k + \frac{2}{n+1}\triangle w_{k+1}, \quad (k+1\le n),$$

therefore

$$\sigma_n''(f;x) = -\sum_{k=0}^{n-1}\triangle^2 w_k F_k(x) + q_n(x),$$

where

$$\begin{aligned}q_n(x) &= \frac{1}{n+1}\sum_{k=1}^{n-1} k\triangle^2 w_k F_k(x)\\ &\quad -\frac{2}{n+1}\sum_{k=0}^{n-1}\triangle w_{k+1}F_k(x) - \frac{n^2}{n+1}a_n F_n(x).\end{aligned}$$

Since, by Lemma 2.1, the series $\sum\limits_{k=0}^{\infty}|\triangle^2 w_k|$ converges and $F_k(x)\le \frac{C}{x^2}$, where C is a constant independent of k and x, then the series

$$\sum_{k=0}^{\infty}\triangle^2 w_k F_k(x)$$

converges uniformly on each interval $[\varepsilon,\pi]$, $\varepsilon>0$. So, our theorem will be proved if we show that $q_n(x)$ uniformly tends to zero on $[\varepsilon,\pi]$.

For $x\in[\varepsilon,\pi]$, we have

$$|q_n(x)| \le \frac{C}{\varepsilon^2}\left\{\frac{1}{n+1}\sum_{k=1}^{n-1}k|\triangle^2 w_k| + \frac{2}{n+1}\sum_{k=0}^{n-1}|\triangle w_{k+1}| + n|a_n|\right\}.$$

Since $\sum\limits_{k=0}^{\infty}|\triangle^2 w_k|<\infty$, then

$$\frac{1}{n+1}\sum_{k=1}^{n-1}k|\triangle^2 w_k|\to 0, \quad \text{as} \quad n\to\infty.$$

This follows by standard arguments. For an arbitrary fixed N one has:

$$\frac{1}{n+1}\sum_{k=1}^{n-1}k|\triangle^2 w_k| \le \frac{1}{n+1}\sum_{k=1}^{N}k|\triangle^2 w_k| + \sum_{k=N+1}^{\infty}|\triangle^2 w_k|.$$

We choose, for a given $\varepsilon > 0$, a number $N = N(\varepsilon)$ so that

$$\sum_{k=N+1}^{\infty} |\triangle^2 w_k| < \frac{\varepsilon}{2}.$$

So, for all sufficiently large n we obtain

$$\frac{1}{n+1}\sum_{k=1}^{n-1} k|\triangle^2 w_k| < \varepsilon.$$

Also, with help of Lemma 2.2 we obtain

$$\frac{2}{n+1}\sum_{k=0}^{n-1} |\triangle w_{k+1}| \to 0,$$

and as in the proof of the Lemma 2.2 we obtain

$$n|a_n| \to 0, \quad \text{as} \quad n \to \infty,$$

which finished the proof of (13). □

Now we shall prove briefly a similar result regarding to the sine series. For $x \in (0, \pi]$ and $k = 0, 1, 2, \dots$, we denote

$$\varphi_k(x) := -\frac{1}{2}\cot\frac{x}{2} + \sum_{i=1}^{k} \sin ix,$$

$$\psi_k(x) := \sum_{i=0}^{k} \varphi_i(x) = -\frac{\sin(k+1)x}{4\sin^2\frac{x}{2}}.$$

Theorem 3.2. *If the coefficients of the series (2) satisfy conditions (3) and (6), then for the second derivative of its sum the following equality holds*

$$g''(x) = -\sum_{k=0}^{\infty} \triangle^2\left(k^2 a_k\right)\psi_k(x), \qquad 0 < x \le \pi.$$

Proof. Similarly, as in the proof of the Theorem 3.1, the series (2) converges uniformly on $[\varepsilon, \pi]$, $\varepsilon > 0$, therefore its Cesàro means

$$\sigma_n(g; x) := \sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right) a_k \sin kx$$

converge uniformly, as well, on $(0, \pi]$.

Keeping same notations as in theorem 3.1, and applying two times the summation by parts to the equality

$$\sigma_n''(g; x) = -\sum_{k=1}^{n} \beta_k \sin kx$$

we obtain

$$\begin{aligned}
\sigma_n''(g;x) &= -\sum_{k=1}^{n-1} \triangle\beta_k \left(\varphi_k(x)+\frac{1}{2}\cot\frac{x}{2}\right) - \beta_n \left(\varphi_n(x)+\frac{1}{2}\cot\frac{x}{2}\right) \\
&= \beta_1\varphi_0(x) - \sum_{k=1}^{n-1} \triangle\beta_k\varphi_k(x) - \beta_n\varphi_n(x) \\
&= -\beta_0\varphi_0(x)+\beta_1\varphi_0(x) - \sum_{k=1}^{n-1} \triangle\beta_k\varphi_k(x) - \beta_n\varphi_n(x)+\beta_{n+1}\varphi_n(x), \\
&= -\sum_{k=0}^{n} \triangle\beta_k\varphi_k(x), \quad (\beta_0=\beta_{n+1}=0), \\
&= -\sum_{k=0}^{n-1} \triangle^2\beta_k\psi_k(x) - \triangle\beta_n\psi_n(x) \\
&= -\sum_{k=0}^{n-1} \left[\left(1-\frac{k}{n+1}\right)\triangle^2 w_k + \frac{2}{n+1}\triangle w_{k+1}\right]\psi_k(x) - \frac{n^2}{n+1}a_n\psi_n(x) \\
&= -\sum_{k=0}^{n-1} \triangle^2 w_k\psi_k(x) + p_n(x),
\end{aligned}$$

where

$$p_n(x) := \frac{1}{n+1}\sum_{k=0}^{n-1} k\triangle^2 w_k\psi_k(x) - \frac{2}{n+1}\sum_{k=0}^{n-1} \triangle w_{k+1}\psi_k(x) - \frac{n^2}{n+1}a_n\psi_n(x).$$

Since $\psi_k(x) \le \frac{C}{x^2}$, ($C$ is a constant independent of k and x), then repeating the same reasoning as in the proof of Theorem 3.1 we can show that $p_n(x) \to 0$ as $n \to \infty$ (we omit the details). With this we have finished the proof of the theorem. □

We note that

$$\triangle^2 w_0 F_0(x) = (-2a_1+4a_2)\cdot\frac{1}{2} = -a_1+2a_2,$$

therefore (13) can be written as

$$f''(x) - a_1 + 2a_2 = -\sum_{k=1}^{\infty} \triangle^2\left(k^2 a_k\right) F_k(x).$$

Since the functions $F_k(x)$ are nonnegative for $x \in (0,\pi]$, then the following corollary is a direct result of theorem 3.1.

Corollary 3.1. *Let the sequence $\{a_k\}$ satisfies conditions (3) and (6). Then*

(1) *If for all $k = 1,2,\ldots,$*
$$\triangle^2\left(k^2 a_k\right) \ge 0,$$
then the function
$$f'(x) - (a_1 - 2a_2)x$$
is concave on $(0,\pi]$.

(2) *If for all $k = 1,2,\ldots,$*
$$\triangle^2\left(k^2 a_k\right) \le 0,$$
then the function
$$f'(x) - (a_1 - 2a_2)x$$

is convex on $(0, \pi]$.

From the above corollary we conclude that the results of this paper essentially extend the results of Telyakovskiĭ in the sense that until his results provide the monotonicity condition of a function that involves the sum function of the cosine series, our results provide the convexity or concavity condition of a function that involves the first derivative of the sum function of the such series.

4. Local integrability

The next theorem gives an estimate of the integral of $|f''(x)|$ on the intervals $[\pi/(m+1), \pi/\ell]$, $1 \le \ell \le m$.

Theorem 4.1. *Let the sequence* $\{a_k\}$ *satisfies conditions (3) and (6). Then*

$$\int_{\pi/(m+1)}^{\pi/\ell} |f''(x)|dx = O\left(\frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{k+1}{\ell}|\triangle\left(k^2 a_k\right)|\right) + \tag{14}$$

$$+O\left(\sum_{k=\ell}^{\infty}\min(k+1-\ell,\, m+1-\ell)|\triangle^2\left(k^2 a_k\right)|\right).$$

Proof. By Theorem 3.1 and Lemma 2.2, we have

$$\begin{aligned} f''(x) &= -\sum_{k=0}^{i-1}\triangle^2\left(k^2 a_k\right)F_k(x) - \sum_{k=i}^{\infty}\triangle^2\left(k^2 a_k\right)F_k(x) \\ &= -\sum_{k=0}^{i-1}\triangle\left(k^2 a_k\right)D_k(x) - \sum_{k=i}^{\infty}\triangle^2\left(k^2 a_k\right)\left[F_k(x) - F_{i-1}(x)\right]. \end{aligned}$$

The integral (20) can be written as

$$\int_{\pi/(m+1)}^{\pi/\ell}|f''(x)|dx = \sum_{i=\ell}^{m}\int_{\pi/(i+1)}^{\pi/i}|f''(x)|dx. \tag{15}$$

Therefore

$$\begin{aligned} \int_{\pi/(i+1)}^{\pi/i}|f''(x)|dx &\le \int_{\pi/(i+1)}^{\pi/i}\sum_{k=0}^{i-1}|\triangle\left(k^2 a_k\right)||D_k(x)|dx + \\ &\quad + \int_{\pi/(i+1)}^{\pi/i}\sum_{k=i}^{\infty}|\triangle^2\left(k^2 a_k\right)||F_k(x) - F_{i-1}(x)|dx. \end{aligned}$$

Using the estimates $|D_k(x)| \le C(k+1)$ and $0 \le F_k(x) \le C/x^2$, $(0 < x \le \pi)$, we obtain

$$\int_{\pi/(i+1)}^{\pi/i}|f''(x)|dx \le C\sum_{k=0}^{i-1}|\triangle\left(k^2 a_k\right)|\frac{k+1}{i(i+1)} + C\sum_{k=i}^{\infty}|\triangle^2\left(k^2 a_k\right)|. \tag{16}$$

Consequently, from (15) and (16) we get

$$\int_{\pi/(m+1)}^{\pi/\ell}|f''(x)|dx \le C\sum_{i=\ell}^{m}\sum_{k=0}^{i-1}|\triangle\left(k^2 a_k\right)|\frac{k+1}{i(i+1)} + C\sum_{i=\ell}^{m}\sum_{k=i}^{\infty}|\triangle^2\left(k^2 a_k\right)|. \tag{17}$$

For the first term of the right-hand of (17) we have

$$
\begin{aligned}
&\sum_{i=\ell}^{m}\sum_{k=0}^{i-1}|\triangle\left(k^2a_k\right)|\frac{k+1}{i(i+1)}= \\
= \quad & \sum_{i=\ell}^{m}\sum_{k=0}^{\ell-1}|\triangle\left(k^2a_k\right)|\frac{k+1}{i(i+1)}+\sum_{i=\ell+1}^{m}\sum_{k=\ell}^{i-1}|\triangle\left(k^2a_k\right)|\frac{k+1}{i(i+1)} \\
= \quad & \sum_{k=0}^{\ell-1}(k+1)|\triangle\left(k^2a_k\right)|\left(\frac{1}{\ell}-\frac{1}{m+1}\right) \\
& +\sum_{k=\ell}^{m-1}(k+1)|\triangle\left(k^2a_k\right)|\left(\frac{1}{k+1}-\frac{1}{m+1}\right) \\
\leq \quad & \frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{k+1}{\ell}|\triangle\left(k^2a_k\right)|+\sum_{k=\ell}^{m}\sum_{j=k}^{\infty}|\triangle\left(k^2a_k\right)|. \qquad (18)
\end{aligned}
$$

The second term in (17) and (18) can then be written as follows

$$
\sum_{i=\ell}^{m}\sum_{k=i}^{\infty}|\triangle\left(k^2a_k\right)|=\sum_{i=\ell}^{m}\sum_{k=i}^{m}|\triangle\left(k^2a_k\right)|+\sum_{i=\ell}^{m}\sum_{k=m+1}^{\infty}|\triangle\left(k^2a_k\right)|=
$$

$$
=\sum_{k=\ell}^{m}(k+1-\ell)|\triangle\left(k^2a_k\right)|+(m+1-\ell)\sum_{k=m+1}^{\infty}|\triangle\left(k^2a_k\right)|. \qquad (19)
$$

Now the proof of theorem is an immediate result of (17), (18) and (19). □

We shall now verify briefly a result regarding to the sine series which distinguishes from Theorem 4.1. For this we first denote

$$
d_k:=\frac{1}{2}\int_{\pi/(k+1)}^{\pi/k}\cot\frac{x}{2}dx=\log\frac{\sin\frac{\pi}{2k}}{\sin\frac{\pi}{2(k+1)}}, \quad (k=1,2,\dots).
$$

Theorem 4.2. *Let the sequence $\{a_k\}$ satisfies conditions (3) and (6). Then*

$$
\begin{aligned}
\int_{\pi/(m+1)}^{\pi/\ell}|g''(x)|dx \quad = \quad & \sum_{k=\ell}^{m}k^2|a_k|d_k \qquad (20) \\
& +O\left(\frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{k^2}{\ell^2}|\triangle\left(k^2a_k\right)|\right) \\
& +O\left(\sum_{k=\ell}^{\infty}\min(k+1-\ell,\, m+1-\ell)|\triangle^2\left(k^2a_k\right)|\right).
\end{aligned}
$$

Proof. According to the theorem 3.2 we can write

$$
\sum_{k=0}^{i-1}\triangle^2\left(k^2a_k\right)\psi_k(x)=\sum_{k=0}^{i-1}\triangle\left(k^2a_k\right)\varphi_k(x)-\triangle\left(i^2a_i\right)\psi_{i-1}(x).
$$

Since

$$
\varphi_k(x)=-\frac{1}{2}\cot\frac{x}{2}+\widetilde{D}_k(x),
$$

where $\widetilde{D}_k(x)$ is the conjugate Dirichelt kernel, then for $x \in (0,\pi]$ and an arbitrary positive integer number i we have

$$g_k''(x) = -\frac{i^2 a_i}{2}\cot\frac{x}{2} - \sum_{k=1}^{i-1}\triangle\left(k^2 a_k\right)\widetilde{D}_k(x) - \sum_{k=i}^{\infty}\triangle^2\left(k^2 a_k\right)\left[\psi_k(x)-\psi_{i-1}(x)\right].$$

Hence,

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell}|g''(x)|dx &= \sum_{i=\ell}^{m}\int_{\pi/(i+1)}^{\pi/i}|g''(x)|dx\\ &\le \sum_{i=\ell}^{m} i^2|a_i|d_i\\ &\quad +\sum_{i=\ell}^{m}\int_{\pi/(i+1)}^{\pi/i}\sum_{k=1}^{i-1}|\triangle\left(k^2a_k\right)||\widetilde{D}_k(x)|dx\\ &\quad +\sum_{i=\ell}^{m}\int_{\pi/(i+1)}^{\pi/i}\sum_{k=i}^{\infty}|\triangle^2\left(k^2a_k\right)||\psi_k(x)-\psi_{i-1}(x)|dx.\end{aligned}$$

Quite similarly as in the proof of Theorem 2 of the paper of Telyakovskiĭ (see [5], page 268), one can finish the proof of the theorem without any difficulty. For this we omit the details. □

Note that the estimates proved in theorems 4.1 and 4.2 can be written in a shorter way i.e., only in terms of differences of second order of the sequence $\{k^2a_k\}$. In fact, the following corollaries hold true.

Corollary 4.1. *Let the sequence $\{a_k\}$ satisfies conditions (3) and (6). Then for $1 \le \ell \le m$ the following estimates hold*

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell}|f''(x)|dx &= O\bigg(\frac{m+1-\ell}{m}\times\\ &\qquad\times\sum_{k=0}^{\infty}\min\left(\frac{(k+1)^2}{\ell},k+1,m\right)|\triangle^2\left(k^2a_k\right)|\bigg),\end{aligned}$$

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell}|g''(x)|dx &= \sum_{k=\ell}^{m}k^2|a_k|d_k\\ &\quad +O\left(\frac{m+1-\ell}{m}\sum_{k=1}^{\infty}\min\left(\frac{k^3}{\ell^2},k,m\right)|\triangle^2\left(k^2a_k\right)|\right).\end{aligned}$$

Moreover, since under the conditions of theorems 3.1 and 3.2 the second derivatives of the functions f and g are continuous, then the above corollary can be formulated in terms of total variations on intervals $[\pi/(m+1),\pi/\ell]$ of the first derivative of the functions f and g.

Definition 4.1. *The total variation of a real-valued function h, defined on an interval $[a,b]\subset\mathbb{R}$ is the quantity*

$$\bigvee_a^b(h) = \sup_{\mathcal{P}}\sum_{i=0}^{n_P-1}|f(x_{i+1}-f(x_i))|,$$

where the supremum runs over the set of all partitions

$$\mathcal{P} = \left\{P = \{x_0, x_1, \ldots, x_{n_P}\} \mid P \text{ is a partition of } [a,b]\right\}$$

of the given interval.

Corollary 4.2. *Let the sequence $\{a_k\}$ satisfies conditions (3) and (6). Then for $1 \le \ell \le m$ the following estimates hold*

$$\bigvee_{\frac{\pi}{m+1}}^{\frac{\pi}{\ell}} (f') = O\Bigg(\frac{m+1-\ell}{m} \times \sum_{k=0}^{\infty} \min\left(\frac{(k+1)^2}{\ell}, k+1, m\right) |\triangle^2 \left(k^2 a_k\right)|\Bigg),$$

$$\bigvee_{\frac{\pi}{m+1}}^{\frac{\pi}{\ell}} (g') = \sum_{k=\ell}^{m} k^2 |a_k| d_k + O\left(\frac{m+1-\ell}{m} \sum_{k=1}^{\infty} \min\left(\frac{k^3}{\ell^2}, k, m\right) |\triangle^2 \left(k^2 a_k\right)|\right).$$

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(Xhevat Z. Krasniqi) Department of Mathematics and Computer Sciences, University of Prishtina, Avenue "Mother Theresa " 5, Prishtinë 10000, KOSOVË
E-mail address: xheki00@hotmail.com