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Entropy solution to an elliptic problem with nonlinear boundary conditions

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Abstract. In this paper, we consider the equation $b(u) - div\ a(u, Du) = f$ in a bounded domain with nonlinear boundary conditions of the form $-a(u, Du).\eta \in \beta(x, u)$. We introduce a notion of entropy solution for this problem and prove existence and uniqueness of this solution for general L^1-data.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and $1 < p < N$. Consider the nonlinear elliptic problem

$$(E_b)(f) \begin{cases} b(u) - div\ a(u, Du) = f \text{ in } \Omega \\ -\langle a(u, Du), \eta\rangle \in \beta(x, u) \text{ on } \partial\Omega, \end{cases}$$

where η is the unit outward normal vector on $\partial\Omega$, $f \in L^1(\Omega)$, Du denotes the gradient of u, $b : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, nondecreasing and surjective with $b(0) = 0$ and, for a.e. $x \in \partial\Omega, \beta(x, r) = \partial j(x, r)$ is the subdifferential of a function $j : \partial\Omega \times \mathbb{R} \longrightarrow [0, \infty]$ which is convex, lower semicontinuous (l.s.c. for short) in $r \in \mathbb{R}$ for $\sigma-$a.e. $x \in \partial\Omega$, measurable with respect to the $(N - 1)-$ dimensional Hausdorff measure σ on $\partial\Omega$ and such that $j(., 0) = 0$. The vector-valued function $a : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is continuous and satisfying the following classical Leray-Lions-type conditions:
$(H_1)-$ Monotonicity in $\xi \in \mathbb{R}^N$:

$$(a(r, \xi) - a(r, \eta)).(\xi - \eta) \geq 0\ \forall r \in \mathbb{R},\ \forall \xi, \eta \in \mathbb{R}^N.$$

$(H_2)-$ Coerciveness: $\exists\lambda_0 > 0$ such that

$$(a(r, \xi) - a(r, 0)).\xi \geq \lambda_0|\xi|^p\ \forall r \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

$(H_3)-$ Growth restriction: there exists a continuous function $\Lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$|a(r, \xi)| \leq \Lambda(|r|)(1 + |\xi|^{p-1})\ \forall r \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

$(H_4)-$ There exists $C : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ continuous such that

$$|a(r, \xi) - a(s, \xi)| \leq C(r, s)|r - s|(1 + |\xi|^{p-1})\ \forall r, s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

A typical example of a function a satisfying these hypotheses is $a(r, \xi) = |\xi|^{p-2}\xi + F(r)$, where $F : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a locally Lipschitz function.

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Many results are known as regards to elliptic problems in the variational setting for Dirichlet or Dirichlet-Neumann problems (cf. [1, 3, 4, 5, 6, 15, 16, 20, 25, 26, 29, 31]). In the L^1-setting, for elliptic and parabolic equations in divergence form, new equivalent notions of entropy and renormalized solutions have been introduced. (cf. [2, 7, 13, 17]). In particular, in [7], a notion of entropy solution have been introduced for the following nonlinear problem

$$\begin{cases} u - div\ a(x, Du) = f \text{ in } \Omega \\ -a(x, Du).\eta \in \beta(u) \text{ on } \partial\Omega, \end{cases}$$

with a being independent of u and the graph β being independent of the space variable. Under a regularity assumption on a and for particular graphs β, the authors proved existence and uniqueness of this entropy solution for arbitrary L^1-data.

In [27], the authors used and extended the methods introduced in [7] to study the problem

$$\begin{cases} u - div\ a(u, Du) = f \text{ in } \Omega \\ -a(u, Du).\eta \in \beta(x, u) \text{ on } \partial\Omega, \end{cases}$$

where a is a divergentiel operator depending on u and β depending on u and also on the space variable x.

In the present paper, we use and extend the methods introduced in [7, 27] to study the problem

$$\begin{cases} b(u) - div\ a(u, Du) = f \text{ in } \Omega \\ -a(u, Du).\eta \in \beta(x, u) \text{ on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$, with b not necessarily invertible. Clearly, $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, nondecreasing and surjective with $b(0) = 0$.

The paper is organized as follows. In the next section we make precise the notations which will be used in the sequel and recall some facts on measures and capacities. In section 3, we study the problem $(E_b)(f)$ by variational methods. We introduce an accretive operator $A_{\delta,b}$ related to problem $(E_b)(f)$ and show that $A_{\delta,b}$ is T-accretive in $L^1(\Omega)$, verify that $D(A_{\delta,b})$ is dense in $L^1(\Omega)$ and $R(I+\alpha A_{\delta,b}) \supset L^\infty(\Omega)$ for all $\alpha > 0$. In section 4, we introduce the notion of entropy solution and prove the existence and uniqueness (in the sense of $b(u)$) of this solution. In order to do this, we characterize $\mathcal{A}_b$, the limit of the operator $A_{\delta,b}$ in $L^1(\Omega)$.

2. Preliminary

In this section, we introduce some notations and definitions used in this paper. We denote $|.|$ and $d\sigma$ respectively the $N-$dimensional Lebesgue measure in $\mathbb{R}^N$ and the $(N-1)-$dimensional Hausdorff measure of $\partial\Omega$.

The norm in $L^p(\Omega)$ is denoted by $||.||_p$, $1 \leq p \leq \infty$. $W^{1,p}(\Omega)$ denotes the classical Sobolev space endowed with the usual norm denoted $||.||_{1,p}$. It is well-known (cf. [23, 24]) that if $u \in W^{1,p}(\Omega)$, it is possible to define the trace of u on $\partial\Omega$, where the continuous linear trace operator $\tau : W^{1,p}(\Omega) \longrightarrow W^{-\frac{1}{p'},p}(\partial\Omega)$ is surjective.

For $0 < q < \infty$, $\mathcal{M}^q(\Omega)$ is the Marcinkiewicz space (cf. [12]) defined as the set of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that

$$|\{x \in \Omega : |f(x)| > k\}| \leq ck^{-q}, \text{ where } 0 < c < \infty.$$

As usual, for $k > 0$, we denote by T_k, the truncation function at height $k \geq 0$ defined by

$$T_k(u) = \min\left\{k, \max\{u, -k\}\right\} = \begin{cases} -k \text{ if } u < -k \\ u \text{ if } |u| \leq k \\ k \text{ if } u > k. \end{cases}$$

Let γ be a maximal monotone operator defined on $\mathbb{R}$. We recall the definition of the main section γ_0 of γ:

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s) \text{ if } \gamma(s) \neq \phi \\ +\infty \text{ if } [s,+\infty) \cap D(\gamma) = \phi \\ -\infty \text{ if } (-\infty, s] \cap D(\gamma) = \phi. \end{cases}$$

We denote by $\bar{u}$ the average of u, i.e. $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx$.

We define the set $\mathcal{P} = \{S \in C^1(\mathbb{R})/S(0) = 0, 0 \leq S' \leq 1, \text{supp}(S') \text{ is compact}\}$.
Let $\mathcal{A}$ be a multi-valued operator in $L^1(\Omega)$. Recall that $\mathcal{A}$ is said to be accretive in $L^1(\Omega)$ if $||u - \tilde{u}||_1 \leq ||u - \tilde{u} + \alpha(v - \tilde{v})||_1$ for any $(u, v), (\tilde{u}, \tilde{v}) \in \mathcal{A}, \alpha > 0$ i.e.; for any $\alpha > 0$, the resolvent of $\mathcal{A}$, $(I + \alpha\mathcal{A})^{-1}$ is a single-valued operator and a contraction in L^1-norm. $\mathcal{A}$ is called $T-$accretive if $||(u - \tilde{u})^+||_1 \leq ||(u - \tilde{u} + \alpha(v - \tilde{v}))^+||_1$ for any $(u, v), (\tilde{u}, \tilde{v}) \in \mathcal{A}$ and for any $\alpha > 0$. Finally, $\mathcal{A}$ is called $m-$accretive (resp. $m - T-$accretive) in $L^1(\Omega)$ if $\mathcal{A}$ is accretive ($T-$accretive) and moreover, $R(I + \alpha\mathcal{A}) = L^1(\Omega)$ for any $\alpha > 0$ (cf. [9, 10, 14] for more details about accretive operators and nonlinear semigroups).

Now, let us introduce some notations and recall some facts about capacities and measures used throughout this paper (cf. [18, 19, 21, 22]). Let G be an arbitrary fixed bounded open subset of $\mathbb{R}^N$ with $\bar{\Omega} \subset G$. Given a compact subset $K \subseteq G$, we define the $p-$capacity of K by:

$$C_{1,p}(K) := \inf\{||\varphi||_{1,p}; \varphi \in C_c^\infty(G), \varphi \geq \chi_K\}.$$

The $p-$capacity of an open set $O \subset G$ is then defined by

$$C_{1,p}(O) := \sup\{C_{1,p}(K); K \subset O, K \text{ is compact}\}$$

which reveals to be equal to the quantity

$$\inf\{||\varphi||_{1,p}; \varphi \in W_0^{1,p}(G), \varphi \geq \chi_O \text{ a.e. on } G\}.$$

Finally, the $p-$capacity of an arbitrary subset $E \subseteq G$ is defined by

$$C_{1,p}(E) := \inf\{C_{1,p}(O); O \text{ open, } E \subseteq O\}.$$

It is well-known that $C_{1,p}$ is an outer measure on G. Recall also that any function $u \in W^{1,p}(\Omega)$ admits a cap-quasi-continuous representative on G. In particular, as Ω is smooth, any function $v \in W^{\frac{1}{p'},p}(\partial\Omega)$ is the trace of a function $\hat{v} \in W_0^{1,p}(G)$ such that $\hat{v}|_{\partial\Omega} = v$, where G is an arbitrary fixed open subset of $\mathbb{R}^N$ such that $\bar{\Omega} \subset G$. A function u defined on Ω is said to be cap-quasi-continuous if for every $\varepsilon > 0$, there exists an open set $B \subseteq G$ with $C_{1,p}(B) < \varepsilon$ such that the restriction of u to $G \setminus B$ is continuous. It is well-known that every function in $W_0^{1,p}(G)$ has a cap-quasi-continuous representative, i.e. a function $\tilde{u} : G \longrightarrow \mathbb{R}$ such that $u = \tilde{u}$ a.e. on G and $\tilde{u}$ is cap-quasi-continuous. In particular, by the remarks above, any function $v \in W^{\frac{1}{p'},p}(\partial\Omega)$ has a cap-quasi-continuous representative $\tilde{v}$. Indeed, $\exists \hat{v} \in W^{1,p}(G)$ such that $\widetilde{\hat{v}}$ is a quasi-continuous representative of $\hat{v}$ on G and $\widetilde{\hat{v}}|_{\partial\Omega} = v$ a.e. on $\partial\Omega$. As usual, a property will be said to hold cap-quasi everywhere (q.e. for short) if it holds everywhere except on a set of zero capacity.

Let $\mathcal{M}_b(\partial\Omega)$ be the space of all Radon measures on $\partial\Omega$ with bounded total variation. For $\mu \in \mathcal{M}_b(\partial\Omega)$, denote by μ^+, μ^- and $|\mu|$ the positive part, negative part and the total variation of the measure μ, respectively, and denote by $\mu = \mu_r d\sigma + \mu_s$ the Radon-Nikodym decomposition of μ relatively to the $(N-1)-$dimensional Hausdorff measure $d\sigma$.

We denote by $\mathcal{M}_b^p(\partial\Omega)$ the set of Radon measures μ which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq \partial\Omega$ such that $C_{1,p}(B) = 0$, i.e. the Radon measures which do not charge sets of 0-capacity.

We denote $\mathcal{J}_0(\partial\Omega) = \{j/j : \partial\Omega \times \mathbb{R} \longrightarrow [0;+\infty]\}$, such that $j(.,r)$ is $\sigma-$ measurable $\forall r \in \mathbb{R}$ and $j(x,.)$ is convex, l.s.c. satisfying $j(x,0) = 0$ for a.e. $x \in \partial\Omega\}$. For a.e. $x \in \partial\Omega$, we define

$$\begin{aligned} \mathcal{J} : W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega) &\longrightarrow [0,\infty] \\ u &\longmapsto \int_{\partial\Omega} j(.,u)d\sigma. \end{aligned}$$

Note that $\mathcal{J}$ naturally extends to a functional $\hat{\mathcal{J}}$ on $W_0^{1,p}(G) \cap L^\infty(G)$ as follows: $\hat{\mathcal{J}}(u) = \displaystyle\int_{\partial\Omega} j(.,\tau(u))d\sigma$ for any $u \in W_0^{1,p}(G)$. We recall that the closure of $D(\hat{\mathcal{J}})$ in $W_0^{1,p}(G)$ is a convex bilateral set, so according to [8], there exist unique (in the sense q.e.) functions γ_+, γ_- which are cap-quasi-l.s.c. and cap-quasi-u.s.c. respectively, such that $\overline{D(\mathcal{J})}^{||.||_{\frac{1}{p'},p}} = \{u \in W^{\frac{1}{p'},p}(\partial\Omega); \gamma_-(x) \leq \tilde{u}(x) \leq \gamma_+(x) \text{ q.e. on } \partial\Omega\}$.

Moreover, $\gamma_-(x) = \inf\limits_n \tilde{u}_n(x) = \lim\limits_n \inf\limits_{1\leq k\leq n} \tilde{u}_k(x)$ q.e. $x \in \partial\Omega$ (respectively the corresponding analogue for γ_+) for any $||.||_{\frac{1}{p'},p}-$dense sequence $(u_n)_n$ in $D(\mathcal{J})$.

We define the subdifferential operator:
$\partial\mathcal{J} \subseteq \left(W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)\right) \times \left(W^{\frac{-1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^*\right)$ by

$$\mu \in \partial\mathcal{J}(u) \Longleftrightarrow \begin{cases} u \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega), \mu \in W^{\frac{-1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^* \\ \\ \text{and } \mathcal{J}(w) \ \geq \ \mathcal{J}(u) + \langle \mu, w-u \rangle \ \forall w \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega), \end{cases}$$

where, here as in the following, if not explicitly stated otherwise, $\langle .,. \rangle$ denotes the duality between $W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and its dual.

3. Variational approach

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $1 < p < N$, $a : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ a mapping satisfying the assumptions $(H_1)-(H_4)$ and β is such that $\beta(x,.) = \partial j(x,.)$ a.e. on $\partial\Omega$, where $j \in \mathcal{J}_0(\partial\Omega)$.

To apply the classical variational approach, we need an $L^\infty-$estimate on u (since b is onto, it is equivalent to the L^∞-estimate of $b(u)$), which is not evident to obtain directly in our problem. The obstacle which we encounter is that we cannot get rid of the term with $a(u,0)$. To overcome this difficulty, following [27], we first redefine and extend the function Λ which appears in hypothesis (H_3), on an odd monotone function ψ on $\mathbb{R}$ such that $\left|\dfrac{a(k,0)}{\psi(k)}\right| \longrightarrow 0$ as $k \longrightarrow \infty$. This will be possible by setting $\Lambda(r) := \sup\limits_{|z|\leq r} \left\{\psi(|z|), |z||a(z,0)|\right\}$ for $r \geq 0$. Secondly, we add a penalization term $\delta\psi(u)$ on the boundary for a fixed δ. This allows us to compensate the term with $a(u,0)$ by choosing k sufficiently large such that $\left|\dfrac{a(k,0)}{\psi(k)}\right| < \delta$.

In the next section, we tend δ to zero and the penalization term disappears. Consequently we obtain the entropy solution of our initial problem $(E_b)(f)$.

Now, we define the operator $A_{\delta,b}$ as follows:
$(b(u), f) \in A_{\delta,b}$ if and only if $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$; $f \in L^1(\Omega)$ and there exists a measure $\mu \in \mathcal{M}_b^p(\partial\Omega)$ with $\mu_r(x) \in \partial j(x,u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x))$ a.e. $x \in \partial\Omega$ such that for

all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(u,Du).D(u-\phi)dx + \delta\int_{\partial\Omega}\psi(u)(u-\phi)d\sigma \le \int_\Omega f(u-\phi)dx - \int_{\partial\Omega}(\tilde{u}-\tilde{\phi})\,d\mu,$$

$$\tilde{u} = \gamma_+ \ \mu_s^+ - \text{a.e. on } \partial\Omega,\ \tilde{u} = \gamma_- \ \mu_s^- - \text{a.e. on } \partial\Omega, \tag{3.1}$$

where for given interval $[a,b] \subset \mathbb{R}$, $I_{[a,b]}$ denotes the convex l.s.c. functional on $\mathbb{R}$ defined by 0 on $[a,b]$, $+\infty$ otherwise.

Remark 3.1. *As the measure $\mu \in \mathcal{M}_b^p(\partial\Omega)$, $|\mu|$ does not charge the sets of $0-$capacity. From $|\mu_s| \le |\mu|$, it follows that $|\mu_s|$ does not charge the sets of $0-$capacity. Consequently, the condition (3.1) is meaningful.*

We can now state the first main result.

Theorem 3.1. *The operator $A_{\delta,b}$ satisfies the following properties:*
i) $A_{\delta,b}$ is $T-$accretive in $L^1(\Omega)$,
ii) $L^\infty(\Omega) \subset R(I+\alpha A_{\delta,b})$ for any $\alpha > 0$,
iii) $D(A_{\delta,b})$ is dense in $L^1(\Omega)$.

Proof. i) Let u, v such that

$$f \in b(u) + A_{\delta,b}u \text{ and } g \in b(v) + A_{\delta,b}v. \tag{3.2}$$

We must show that

$$\int_\Omega (b(u)-b(v))^+dx \le \int_\Omega (f-g)^+dx. \tag{3.3}$$

Taking $\phi_1 = u - \frac{1}{k}T_k(u-v)^+$ and $\phi_2 = v + \frac{1}{k}T_k(u-v)^+$ as test functions in (3.2) respectively, we get after adding inequalities

$$\begin{aligned}
&\frac{1}{k}\int_{\{(u-v)^+<k\}}[a(u,Du)-a(v,Dv)].D(u-v)^+dx\\
&\qquad +\frac{1}{k}\delta\int_{\partial\Omega}(\psi(u)-\psi(v))T_k(u-v)^+d\sigma\\
&\le \frac{1}{k}\int_\Omega (f-b(u)-g+b(v))T_k(u-v)^+dx\\
&\quad -\frac{1}{k}\Big(\int_{\partial\Omega}T_k(\tilde{u}-\tilde{v})^+\,d\mu_1 - \int_{\partial\Omega}T_k(\tilde{u}-\tilde{v})^+\,d\mu_2\Big).
\end{aligned} \tag{3.4}$$

Denote by I_1 respectively I_2 the first, respectively the second integral in the left hand side of (3.4). Using hypothesis (H_1), (H_4) and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
I_1 &\ge \frac{1}{k}\int_{\{(u-v)^+<k\}}[a(u,Dv)-a(v,Dv)].D(u-v)^+dx\\
&\ge -\frac{1}{k}\int_{\{(u-v)^+<k\}}C(u,v)(u-v)^+(1+|Dv|^{p-1})|D(u-v)^+|dx\\
&\ge -\frac{C_1k}{k}\int_{\{(u-v)^+<k\}}(1+|Dv|^{p-1})|D(u-v)^+|dx\\
&\ge -C_1\int_\Omega (1+|Dv|^{p-1})|D(u-v)^+|\chi_{\{(u-v)^+<k\}}dx\\
&\longrightarrow 0 \text{ as } k \longrightarrow 0.
\end{aligned}$$

Note that the properties of the measures μ_1 and μ_2 guarantee to us that the second term in the brackets in the right hand side of (3.4) is nonnegative. Indeed, these integrals can be written as $\int_{\partial\Omega}T_k(\tilde{u}-\tilde{v})(\mu_{r,1}-\mu_{r,2}) + \int_{\partial\Omega}T_k(\gamma_+-\tilde{v})\,d(\mu_{s,1})^+ - \int_{\partial\Omega}T_k(-\gamma_+ +$

$\tilde{u})\, d(\mu_{s,2})^+ - \int_{\partial\Omega} T_k(\gamma_- - \tilde{v})\, d(\mu_{s,1})^- - \int_{\partial\Omega} T_k(-\gamma_- + \tilde{u})\, d(\mu_{s,2})^-$ which are, clearly, nonnegative by properties of μ_1, μ_2 and $\gamma_{+/-}$.
$I_2 \geq 0$ (thanks to the monotonicity of ψ); we get after passing to the limit in (3.4) with $k \longrightarrow 0$ and using Lebesgue dominated convergence theorem

$$\lim_{k\longrightarrow 0} \frac{1}{k} \int_\Omega (b(u) - b(v)) T_k(u-v)^+ dx \leq \lim_{k\longrightarrow 0} \frac{1}{k} \int_\Omega (f-g)^+ T_k(u-v)^+ dx$$

$$\Longrightarrow \int_\Omega (b(u) - b(v))^+ dx \leq \int_\Omega (f-g)^+ dx.$$

Therefore, (3.3) holds.

ii) It will be no restriction to assume that $\alpha = 1$. In order to prove that $L^\infty(\Omega) \subset R(I + A_{\delta,b})$, we approximate the problem $(E_b)(f)$ by problems of the form

$$\begin{cases} b\big(T_l(u_\lambda)\big) + \lambda |T_l(u_\lambda)|^{p-2} T_l(u_\lambda) - div\ a(T_l(u_\lambda), Du_\lambda) = f \text{ in } \Omega \\ \\ -a(T_l(u_\lambda), Du_\lambda).\eta = \beta_\lambda(x, T_l(u_\lambda)) + \delta T_l(\psi(u_\lambda)) \text{ on } \partial\Omega, \end{cases}$$

where $k \geq (b^{-1})_0\big(||f||_\infty + 1\big)$ satisfies $\Big|\dfrac{a(k,0)}{\psi(k)}\Big| < \delta$, with $(b^{-1})_0$ the main section of b^{-1}, $l > \max\{k, \psi(k)\}$. Here for every $\lambda \in \mathbb{N}, \beta_\lambda(x,.)$ is the Yosida approximation of $\beta(x,.)$, i.e.

$$\beta_\lambda(x,.) = \frac{1}{\lambda}\big(I - (I + \lambda\beta(x,.))\big)^{-1}.$$

Consider the operator $A_{\delta,\lambda,b} : W^{1,p}(\Omega) \longrightarrow [W^{1,p}(\Omega)]^*$ defined by

$$\begin{aligned} \langle A_{\delta,\lambda,b}\ u_\lambda, \phi\rangle = & \int_\Omega b\big(T_l(u_\lambda)\big)\phi dx + \lambda \int_\Omega |T_l(u_\lambda)|^{p-2} T_l(u_\lambda)\phi dx + \\ \int_\Omega a(T_l(u_\lambda), Du_\lambda).D\phi dx + & \int_{\partial\Omega} \beta_\lambda(., T_l(u_\lambda))\phi d\sigma + \delta \int_{\partial\Omega} T_l(\psi(u_\lambda))\phi d\sigma, \end{aligned}$$

for all $\phi \in W^{1,p}(\Omega)$.
Here, $\langle ., .\rangle$ denotes the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$.
We have the following result:

Lemma 3.1. *The operator $A_{\delta,\lambda,b}$ is bounded, coercive and verifies the $(M)-$property.*

Proof. • **The operator $A_{\delta,\lambda,b}$ is bounded.**
Taking $\phi = u_\lambda$ as test function in the definition of $A_{\delta,\lambda,b}$, we obtain

$$\begin{aligned} \langle A_{\delta,\lambda,b}\ u_\lambda, u_\lambda\rangle = & \int_\Omega b(T_l(u_\lambda)) u_\lambda dx + \lambda \int_\Omega |T_l(u_\lambda)|^{p-2} T_l(u_\lambda) u_\lambda dx + \\ \int_\Omega a(T_l(u_\lambda), Du_\lambda).Du_\lambda dx + & \int_{\partial\Omega} \beta_\lambda(., T_l(u_\lambda)) u_\lambda dx + \delta \int_{\partial\Omega} T_l(\psi(u_\lambda)) u_\lambda dx. \end{aligned}$$

It follows that

$$\begin{aligned} |\langle A_{\delta,\lambda,b}\ u_\lambda, u_\lambda\rangle| \leq & \int_\Omega |b(T_l(u_\lambda)) u_\lambda| dx + \lambda \int_\Omega |T_l(u_\lambda)|^{p-1} |u_\lambda| dx + \\ \int_\Omega |a(T_l(u_\lambda), Du_\lambda).Du_\lambda| dx + & \int_{\partial\Omega} |\beta_\lambda(., T_l(u_\lambda)) u_\lambda| d\sigma + \delta \int_{\partial\Omega} |T_l(\psi(u_\lambda)) u_\lambda| d\sigma. \end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned}\int_\Omega |b(T_l(u_\lambda))u_\lambda| &\leq \Big(\int_\Omega |b(T_l(u_\lambda))|^{p'}\Big)^{\frac{1}{p'}}\Big(\int_\Omega |u_\lambda|^p\Big)^{\frac{1}{p}}\\ &\leq C_1||u_\lambda||_p \text{ (since } b \text{ is continuous and } \Omega \text{ bounded)}\\ &\leq C_1||u_\lambda||_{1,p};\end{aligned} \tag{3.5}$$

$$\begin{aligned}\int_{\partial\Omega} |\beta_\lambda(.,T_l(u_\lambda))u_\lambda| &\leq \Big(\int_{\partial\Omega} |\beta_\lambda(.,T_l(u_\lambda))|^{p'}\Big)^{\frac{1}{p'}}\Big(\int_{\partial\Omega} |u_\lambda|^p\Big)^{\frac{1}{p}}\\ &\leq C_2||u_\lambda||_{1,p},\end{aligned} \tag{3.6}$$

(since β_λ is nondecreasing and $\beta_\lambda(.,0)=0$);

$$\begin{aligned}\int_\Omega |T_l(u_\lambda)|^{p-1}|u_\lambda| &\leq \Big(\int_\Omega (l)^{p'(p-1)}\Big)^{\frac{1}{p'}}\Big(\int_\Omega |u_\lambda|^p\Big)^{\frac{1}{p}}\\ &\leq C_3||u_\lambda||_p\\ &\leq C_3||u_\lambda||_{1,p}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\delta\int_{\partial\Omega} |T_l(\psi(u_\lambda))u_\lambda| &\leq \delta\Big(\int_{\partial\Omega} |T_l(\psi(u_\lambda))|^{p'}\Big)^{\frac{1}{p}}\Big(\int_{\partial\Omega} |u_\lambda|^p\Big)^{\frac{1}{p}}\\ &\leq C_4||u_\lambda||_{1,p}.\end{aligned} \tag{3.8}$$

By Hölder inequality and the hypothesis (H_3), we have

$$\begin{aligned}\int_\Omega |a(T_l(u_\lambda),Du_\lambda).Du_\lambda| &\leq \int_\Omega \Lambda(|T_l(u_\lambda)|)(1+|Du_\lambda|^{p-1})|Du_\lambda|\\ &\leq \int_\Omega \big(C+C|Du_\lambda|^{p-1}\big)|Du_\lambda|\\ &\leq \int_\Omega C|Du_\lambda| + \int_\Omega C|Du_\lambda|^p\\ &\leq C_5||Du_\lambda||_p + C_6||Du_\lambda||_p^p\\ &\leq C_6||u_\lambda||_{1,p} + C_7||u_\lambda||_{1,p}^p.\end{aligned} \tag{3.9}$$

From (3.5)-(3.9), it follows that $|\langle A_{\delta,\lambda,b}\ u_\lambda, u_\lambda\rangle| \leq C\big(||u_\lambda||_{1,p} + ||u_\lambda||_{1,p}^p\big) < +\infty$ if $u_\lambda \in W^{1,p}(\Omega)$.

- **The operator $A_{\delta,\lambda,b}$ is coercive.**

We have to show that $\dfrac{\langle A_{\delta,\lambda,b}\ u_\lambda, u_\lambda\rangle}{||u_\lambda||_{1,p}} \longrightarrow +\infty$ as $||u_\lambda||_{1,p} \longrightarrow +\infty$.

We have

$$\begin{aligned}\langle A_{\delta,\lambda,b}\ u_\lambda, u_\lambda\rangle = &\int_\Omega b(T_l(u_\lambda))u_\lambda dx\\ &+\lambda\int_\Omega |T_l(u_\lambda)|^{p-2}T_l(u_\lambda)u_\lambda dx+\int_\Omega a(T_l(u_\lambda),Du_\lambda).Du_\lambda dx\\ &+\int_{\partial\Omega}\beta_\lambda(.,T_l(u_\lambda))u_\lambda dx+\delta\int_{\partial\Omega}T_l(\psi(u_\lambda))u_\lambda dx.\end{aligned}\tag{3.10}$$

Since $b, T_l, \beta_\lambda(.,.)$ and ψ are nondecreasing and as $b(0) = \beta_\lambda(0) = \psi(0) = 0$, then $b(T_l(u_\lambda))u_\lambda \geq 0, \beta_\lambda(.,T_l(u_\lambda))u_\lambda \geq 0$ and $T_l(\psi(u_\lambda))u_\lambda \geq 0$. Using the assumptions $(H_2),(H_3)$ and Hölder inequality, we deduce that

$$\begin{aligned}\int_\Omega a\big(T_l(u_\lambda),Du_\lambda\big).Du_\lambda &\geq \lambda_0||Du_\lambda||_p^p+\int_\Omega a(T_l(u_\lambda),0).Du_\lambda dx\\ &\geq \lambda_0||Du_\lambda||_p^p-\Big(\int_\Omega\big(\Lambda(l)\big)^{p'}dx\Big)^{\frac{1}{p'}}\Big(\int_\Omega|Du_\lambda|^p dx\Big)^{\frac{1}{p}}\\ &\geq \lambda_0||Du_\lambda||_p^p-C||u_\lambda||_{1,p}.\end{aligned}$$

Therefore, we get from the relation (3.10)

$$\begin{aligned}\langle A_{\delta,\lambda,b}\ u_\lambda,u_\lambda\rangle &\geq \lambda||u_\lambda||_p^p+\lambda_0||Du_\lambda||_p^p-C||u_\lambda||_{1,p}\\ &\geq C'||u_\lambda||_{1,p}^p-C||u_\lambda||_{1,p},\end{aligned}$$

with $C' = \min(\lambda,\lambda_0)$.
Then

$$\frac{\langle A_{\delta,\lambda,b}\ u_\lambda,u_\lambda\rangle}{||u_\lambda||_{1,p}} \geq C'||u_\lambda||_{1,p}^{p-1}-C \longrightarrow +\infty \text{ as } ||u_\lambda||_{1,p}\longrightarrow +\infty.$$

- **The operator $A_{\delta,\lambda,b}$ verify the (M)-property.**

For the proof, we need the following lemmas:

Lemma 3.2. *(cf. [28]) Let $\mathcal{A}$ and $\mathcal{B}$ be two operators. If $\mathcal{A}$ is of type (M) and $\mathcal{B}$ is monotone, weakly continuous, then $\mathcal{A}+\mathcal{B}$ is of type (M).*

Lemma 3.3. *(cf. [30]) Let $(f_k)_{k>0}$ and $(g_k)_{k>0}$ be two sequences of functions. If $f_k, f : \Omega \longrightarrow \mathbb{R}$ are measurable, $g_k, g \in L^p(\Omega)$, $1 \leq p < +\infty$ such that $g_k \longrightarrow g$ a.e. in Ω, $f_k \longrightarrow f$ a.e. in Ω, $g_k \longrightarrow g$ in $L^p(\Omega)$ and $\forall k > 0$, $|f_k| \leq g_k$ in Ω, then $f_k \longrightarrow f$ in $L^p(\Omega)$.*

We have

$$\begin{aligned}\langle A_{\delta,\lambda,b}\ u_\lambda,u_\lambda\rangle &= \int_\Omega b(T_l(u_\lambda))u_\lambda+\lambda\int_\Omega|u_\lambda|^p+\int_\Omega a(T_l(u_\lambda),Du_\lambda).Du_\lambda\\ &+\int_{\partial\Omega}\beta_\lambda(.,u_\lambda)u_\lambda+\delta\int_{\partial\Omega}T_l(\psi(u_\lambda))u_\lambda\\ &= \langle a(T_l(u_\lambda),Du_\lambda),u_\lambda\rangle+\langle\mathcal{B}u_\lambda,u_\lambda\rangle.\end{aligned}$$

We now have to show that $\mathcal{B}$ is monotone and weakly continuous.

$$\begin{aligned}\langle\mathcal{B}u_\lambda,u_\lambda\rangle = &\int_\Omega b(T_l(u_\lambda))u_\lambda+\lambda\int_\Omega|T_l(u_\lambda)|^{p-2}T_l(u_\lambda)u_\lambda\\ &+\int_{\partial\Omega}\beta_\lambda(.,T_l(u_\lambda))u_\lambda+\delta\int_{\partial\Omega}T_l(\psi(u_\lambda))u_\lambda.\end{aligned}$$

For the monotonicity of $\mathcal{B}$, we have to show that $\langle \mathcal{B}u - \mathcal{B}v, u - v\rangle \geq 0$, for all u and v in $W^{1,p}(\Omega)$. We have:

$$\langle \mathcal{B}u - \mathcal{B}v, u - v\rangle = \langle \mathcal{B}u, u - v\rangle - \langle \mathcal{B}v, u - v\rangle = \int_\Omega \Big[b\big(T_l(u)\big) - b\big(T_l(v)\big)\Big](u - v) +$$
$$\lambda \int_\Omega \Big(|T_l(u)|^{p-2}T_l(u) - |T_l(v)|^{p-2}T_l(v)\Big)(u - v) + \int_{\partial\Omega} \Big[\beta_\lambda(., T_l(u)) - \beta_\lambda(., T_l(v))\Big](u - v)$$
$$+\delta \int_{\partial\Omega} \Big[T_l\big(\psi(u)\big) - T_l\big(\psi(v)\big)\Big](u - v).$$

From the monotonicity of $b, T_l, \psi, \beta_\lambda$ and the map $u \longmapsto |u|^{p-2}u$, we conclude that

$$\langle \mathcal{B}u - \mathcal{B}v, u - v\rangle \geq 0. \tag{3.11}$$

We now show that the operator $\mathcal{B}$ is weakly continuous, i.e. for all sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$, we have $\mathcal{B}u_n \rightharpoonup \mathcal{B}u$.
For all $\phi \in W^{1,p}(\Omega)$, we have

$$\langle \mathcal{B}u_n, \phi\rangle = \int_\Omega b(T_l(u_n))\phi + \lambda \int_\Omega |T_l(u_n)|^{p-2}T_l(u_n)\phi + \int_{\partial\Omega} \beta_\lambda(., T_l(u_n))\phi + \delta \int_{\partial\Omega} T_l(\psi(u_n))\phi. \tag{3.12}$$

We also have that $|b\big(T_l(u_n)\big)\phi| \leq \max\big(|b(l)|, |b(-l)|\big)|\phi| \in L^p(\Omega)$, $|T_l\big(\psi(u_n)\big)\phi| \leq l|\phi| \in L^p(\Omega)$ and $|T_l(u_n)|^{p-1}|\phi| \leq l^{p-1}|\phi| \in L^p(\Omega)$.
As β_λ is nondecreasing, then $-l \leq T_l(u_n) \leq l \Longrightarrow \beta_\lambda(., -l) \leq \beta_\lambda(., T_l(u_\lambda)) \leq \beta_\lambda(., l)$ $\Longrightarrow |\beta_\lambda(., T_l(u_\lambda))| \leq \max\big(|\beta_\lambda(., l)|, |\beta_\lambda(., -l)|\big) = C_1$.
Therefore,

$$|\beta_\lambda(., T_l(u_n))\phi| \leq C_1|\phi| \in L^p(\Omega).$$

Passing to the limit when n goes to $+\infty$ in (3.12), we obtain thanks to Lemma 3.3

$$\lim_{n\longrightarrow+\infty} \langle \mathcal{B}u_n, \phi\rangle = \langle \mathcal{B}u, \phi\rangle, i.e.\ \mathcal{B}u_n \rightharpoonup \mathcal{B}u.$$

The operator $\mathcal{A} : W^{1,p}(\Omega) \longrightarrow \mathbb{R}, u \longmapsto \langle a(T_k(u), Du), Du\rangle$ is of the type (M) and as $\mathcal{B}$ is monotone and weakly continuous, thanks to Lemma 3.2, we conclude that the operator $A_{\delta,\lambda,b}$ is of the type (M). That concludes the proof of Lemma 3.1. □

Lemma 3.4. *(cf. [28]) Let X be a reflexive Banach space and $A : X \longrightarrow X'$ an operator such that*
(i) A is bounded,
(ii) A is coercive,
(iii) A is of the type (M),
then A is surjective.

By Lemma 3.4, the operator $A_{\delta,\lambda,b}$ is surjective. So, for all $f \in (W^{1,p}(\Omega))^*$, there exists $u_\lambda \in W^{1,p}(\Omega)$ such that for all $\phi \in W^{1,p}(\Omega)$,

$$\langle A_{\delta,\lambda,b}\ b(u_\lambda) - f, u_\lambda - \phi\rangle \leq 0. \tag{3.13}$$

Taking $\phi = u_\lambda - p^+_\varepsilon(u_\lambda - k)$ as a test function in (3.13), where $p^+_\varepsilon(.)$ is an approximation of $sign^+_0(.)$ defined as follow

$$p^+_\varepsilon(r) = \begin{cases} 1 \text{ if } r > \varepsilon \\ \frac{1}{\varepsilon} r \text{ if } 0 < r < \varepsilon \\ 0 \text{ if } r < 0 \end{cases}$$

and using hypothesis (H_2), we obtain

$$\int_\Omega b(T_l(u_\lambda))p_\varepsilon^+(u_\lambda-k)+\lambda\int_\Omega |u_\lambda|^{p-2}u_\lambda p_\varepsilon^+(u_\lambda-k)+\frac{1}{\varepsilon}\int_{\{k<u_\lambda<k+\varepsilon\}} a(T_l(u_\lambda),0).Du_\lambda$$
$$\leq \int_\Omega fp_\varepsilon^+(u_\lambda-k)-\delta\int_{\partial\Omega} T_l(\psi(u_\lambda))p_\varepsilon^+(u_\lambda-k)-\int_{\partial\Omega}\beta_\lambda(.,T_l(u_\lambda))p_\varepsilon^+(u_\lambda-k). \tag{3.14}$$

Note that since $l>k$,

$$\begin{aligned}
\Big|\frac{1}{\varepsilon}\int_{\{k<u_\lambda<k+\varepsilon\}} a(T_l(u_\lambda),0).Du_\lambda\Big| &\leq \Big|\int_\Omega div\ (\int_0^{\frac{(u_\lambda-k)^+}{\varepsilon}\wedge 1} a(T_l(\varepsilon r+k),0)\,dr)\Big| \\
&= \Big|\int_{\partial\Omega}\Big(\int_0^{\frac{(u_\lambda-k)^+}{\varepsilon}\wedge 1} a(T_l(\varepsilon r+k),0)\,dr\Big).\eta d\sigma\Big| \\
\text{(Green formula)} \qquad & \longrightarrow \Big|\int_{\partial\Omega} sign_0^+(u_\lambda-k)a(k,0)d\sigma\Big| \text{ as } \varepsilon\longrightarrow 0 \\
&\leq \int_{\partial\Omega}\big|sign_0^+(u_\lambda-k)\big|\big|a(k,0)|d\sigma \\
&\leq \int_{\partial\Omega\cap\{u_\lambda>k\}}\big|a(k,0)\big|d\sigma.
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
\liminf_{\varepsilon\longrightarrow 0}\frac{1}{\varepsilon}\int_{\{k<u_\lambda<k+\varepsilon\}} a(T_l(u_\lambda),0).Du_\lambda &\geq -|a(k,0)|\int_{\partial\Omega\bigcap\{u_\lambda>k\}} d\sigma \\
&\geq -|a(k,0)|\int_{\partial\Omega\bigcap\{u_\lambda>k\}}\frac{T_l(\psi(u_\lambda))}{T_l(\psi(k))}d\sigma \\
&\geq \frac{-|a(k,0)|}{T_l(\psi(k))}\int_{\partial\Omega\bigcap\{u_\lambda>k\}} T_l(\psi(u_\lambda))d\sigma \\
&\geq -\delta\int_{\partial\Omega\cap\{u_\lambda>k\}} T_l(\psi(u_\lambda))d\sigma.
\end{aligned}$$

Passing to the limit in (3.14) with $\varepsilon\longrightarrow 0$ and regarding that $\beta_\lambda(.,T_l(u_\lambda))$ and $|u_\lambda|^{p-2}u_\lambda$ are nonnegative in $\{u_\lambda>k\}$, we get

$$\begin{aligned}
\int_{\{u_\lambda>k\}} b(T_l(u_\lambda))dx &\leq \int_{\{u_\lambda>k\}} fdx+\delta\int_{\partial\Omega\cap\{u_\lambda>k\}} T_l(\psi(u_\lambda))d\sigma- \\
\delta\int_{\partial\Omega\cap\{u_\lambda>k\}} T_l(\psi(u_\lambda))d\sigma &\leq \int_{\{u_\lambda>k\}} fdx.
\end{aligned}$$

Then $\int_{\{u_\lambda>k\}}\Big(b\big(T_l(u_\lambda)\big)-b\big(T_l(k)\big)\Big)dx\leq\int_{\{u_\lambda>k\}}\Big(f-b\big(T_l(k)\big)\Big)dx.$

As $l>k$ then $T_l(k)=k$. Thus, we have $\int_{\{u_\lambda>k\}}\big(f-b\big(T_l(k)\big)\big)=\int_{\{u_\lambda>k\}}\big(f-b(k)\big)\leq 0$

since $k \geq (b^{-1})_0\big(||f||_\infty + 1\big)$. From inequality above, we get

$$\int_{\{u_\lambda > k\}} \Big(\Big[b\big(T_l(u_\lambda)\big) - b\big(T_l(k)\big)\Big]\Big)^+ dx \leq 0, \quad \forall l > k$$

and then $b(T_l(u_\lambda)) \leq b(k)$ a.e. in $\{u_\lambda > k\}$.

We conclude that $b(u_\lambda) \leq b(k)$ a.e. in Ω.

Similarly, we prove that $b(u_\lambda) \geq b(-k)$ a.e. in Ω. Consequently $|b(u_\lambda)| \leq b(k) = C$.

We deduce that $|u_\lambda| \leq C$ (since b is continuous and surjective) and then

$$||u_\lambda||_\infty \leq C, \tag{3.15}$$

where C is a constant depending only on $||f||_\infty$ and b.

Taking $\phi = 0$ as a test function in (3.13), we get, according to (H_2),

$$\begin{aligned} &\int_\Omega b(T_l(u_\lambda))u_\lambda dx + \lambda \int_\Omega |u_\lambda|^p dx + \lambda_0 \int_\Omega |Du_\lambda|^p dx + \int_\Omega a(T_l(u_\lambda), 0).Du_\lambda dx \\ &\quad + \int_{\partial\Omega} \beta_\lambda(., T_l(u_\lambda))u_\lambda d\sigma + \delta \int_{\partial\Omega} T_l(\psi(u_\lambda))u_\lambda d\sigma \leq \int_\Omega f u_\lambda dx. \end{aligned} \tag{3.16}$$

By Gauss-Green formula, according to the hypothesis (H_3) and (3.15), we deduce that

$$\begin{aligned} \Big|\int_\Omega a(T_l(u_\lambda), 0).Du_\lambda\Big| &\leq \Big|\int_{\partial\Omega} \Big(\int_0^{u_\lambda} a(T_l(r), 0)dr\Big).\eta d\sigma\Big| \\ &\leq \int_{\partial\Omega} \Big(\int_0^{u_\lambda} \Lambda(|r|)dr\Big).\eta d\sigma \\ &\leq C. \end{aligned}$$

As b, β_λ and $T_l \circ \psi$ are nondecreasing then, according to Young inequality, we get from (3.16):

$$\lambda_0 \int_\Omega |Du_\lambda|^p dx - \Big|\int_\Omega a(T_l(u_\lambda), 0).Du_\lambda\Big| \leq \int_\Omega f u_\lambda dx \Longrightarrow \lambda_0 \int_\Omega |Du_\lambda|^p dx - C \leq \int_\Omega f u_\lambda dx.$$

We deduce from inequality above that

$$\begin{aligned} \lambda_0 \int_\Omega |Du_\lambda|^p &\leq C + ||f||_1 ||u_\lambda||_\infty \\ &\leq C'. \end{aligned} \tag{3.17}$$

From (3.15) and (3.17), it follows that $(u_\lambda)_\lambda$ is uniformly bounded in $W^{1,p}(\Omega)$. Hence, there exists a subsequence still denoted (u_λ), such that $u_\lambda \longrightarrow u$ weakly in $W^{1,p}(\Omega)$ as $\lambda \longrightarrow 0$. By Rellich-Kondrachov theorem, $u_\lambda \longrightarrow u$ in $L^p(\Omega)$ and $\tau(u_\lambda) \longrightarrow \tau(u)$ in $L^p(\partial\Omega)$ as $\lambda \longrightarrow 0$. Then $T_l(\psi(u_\lambda)) \longrightarrow \psi(u)$ on $\partial\Omega$. We may also assume that $u_\lambda \longrightarrow u$ a.e. in Ω. Therefore, by (3.15), $||u||_\infty \leq C(||f||_\infty, b)$.

We have $|\beta_\lambda(., T_l(u_\lambda))| \leq \beta_\lambda(., l)$, so

$$\int_{\partial\Omega} |\beta_\lambda(., T_l(u_\lambda))| \leq C. \tag{3.18}$$

Thus, passing to a subsequence if necessary, we have $\beta_\lambda(., T_l(u_\lambda)) \rightharpoonup \mu$ weakly in $\mathcal{M}_b(\partial\Omega)$ as $\lambda \longrightarrow 0$.

Note that for all $\nu > \lambda > 0$, we have for a.e. $x \in \partial\Omega$, $|\beta_\lambda(x,r)| \geq |\beta_\nu(x,r)|\ \forall r \in \mathbb{R}$. Thus, from (3.18), $\int_{\partial\Omega} |\beta_\nu(., T_l(u_\lambda))| \leq C$. Passing to the limit as $\lambda \longrightarrow 0$, we get

$$\int_{\partial\Omega} |\beta_\nu(., T_l(u))| \leq C.$$

As $\nu \longrightarrow 0$, we obtain $\int_{\partial\Omega} |\beta^0(., T_l(u))| \leq C$. Here $\beta^0(., r)$ is the main section of $\beta(., r)$.
Next, thanks to (3.15), (3.17) and hypothesis (H_3), we have

$$\begin{aligned}\int_\Omega |a(u_\lambda, Du_\lambda)|^{p'} dx &\leq \int_\Omega \Big[\Lambda(|u_\lambda|)(1+|Du_\lambda|^{p-1})\Big]^{p'} dx \\ &\leq \int_\Omega (\Lambda(|u_\lambda|))^{p'} 2^{p'} (\frac{1}{2} + \frac{1}{2}|Du_\lambda|^p) dx \\ &\leq C.\end{aligned} \tag{3.19}$$

From (3.19), it follows that $(a(u_\lambda, Du_\lambda))_\lambda$ is uniformly bounded in $(L^{p'}(\Omega))^N$. After passing to a suitable subsequence, we can assume that $a(u_\lambda, Du_\lambda) \rightharpoonup \chi$ weakly in $(L^{p'}(\Omega))^N$ as $\lambda \longrightarrow 0$. The aim is to show, via a pseudo-monotonicity argument that $div\ a(u, Du) = div\ \chi$. To this end, we must show that

$$\limsup_{\lambda \longrightarrow 0} \int_\Omega a(u_\lambda, Du_\lambda).D(u_\lambda - u) = 0. \tag{3.20}$$

Taking $\phi = u_\lambda - (u_\lambda - u)^+$ as a test function in (3.13), we get

$$\begin{aligned}\int_\Omega a(u_\lambda, Du_\lambda).D(u_\lambda - u)^+ &\leq \int_\Omega f(u_\lambda - u)^+ - \int_\Omega b(u_\lambda)(u_\lambda - u)^+ \\ - \lambda \int_\Omega |u_\lambda|^{p-2} u_\lambda (u_\lambda - u)^+ &- \int_{\partial\Omega} \beta_\lambda(., u_\lambda)(u_\lambda - u)^+ \\ - \delta \int_{\partial\Omega} T_l(\psi(u_\lambda))(u_\lambda - u)^+.\end{aligned} \tag{3.21}$$

We have $\beta_\lambda(., u_\lambda) = \beta_\lambda(., u_\lambda^+) + \beta_\lambda(., -u_\lambda^-)$ and $\beta_\lambda(., u_\lambda^+)(u_\lambda - u)^+ \geq 0$. Then, from inequality (3.21) we deduce that

$$\begin{aligned}\int_\Omega a(u_\lambda, Du_\lambda).D(u_\lambda - u)^+ &\leq \int_\Omega f(u_\lambda - u)^+ - \int_\Omega b(u_\lambda)(u_\lambda - u)^+ \\ - \lambda \int_\Omega |u_\lambda|^{p-2} u_\lambda (u_\lambda - u)^+ &- \int_{\partial\Omega} \beta_\lambda(., -u_\lambda^-)(u_\lambda - u)^+ - \delta \int_{\partial\Omega} T_l(\psi(u_\lambda))(u_\lambda - u)^+.\end{aligned} \tag{3.22}$$

Having in mind that $(u_\lambda)_\lambda$ is uniformly bounded in $L^\infty(\partial\Omega)$, we have

$$||(u_\lambda - u)^+||_\infty \leq C$$

and $(u_\lambda - u)^+ \longrightarrow 0$ a.e. as $\lambda \longrightarrow 0$.
Next, observe that $\beta_\lambda(., -u_\lambda^-) \geq \beta_\lambda(., -u^-) \geq \beta^0(., -u^-)$ on $\{u_\lambda \geq u\}$.

As $|\beta^0(., -u^-)| \in L^1(\partial\Omega)$, by Lebesgue dominated convergence theorem, it follows that $\int_{\partial\Omega} \beta_\lambda(., -u_\lambda^-)(u_\lambda - u)^+ \longrightarrow 0$, as $\lambda \longrightarrow 0$. Consequently, passing to the limit in (3.22) with $\lambda \longrightarrow 0$, we get

$$\limsup_{\lambda \longrightarrow 0} \int_\Omega a(u_\lambda, Du_\lambda).D(u_\lambda - u)^+ \leq 0.$$

$\limsup_{\lambda \longrightarrow 0} \int_\Omega a(u_\lambda, Du_\lambda).D(-(u_\lambda - u)^-) \leq 0$ follows similarly.

Hence $\limsup_{\lambda \longrightarrow 0} \int_\Omega a(u_\lambda, Du_\lambda).D(u_\lambda - u) \leq 0$ and (3.20) follows from the monotonicity of a.

Now, let $\phi \in \mathcal{C}_c(\mathbb{R}^N)$ and $\alpha \in \mathbb{R}^*$. Using the hypothesis (H_1), the Lebesgue dominated convergence theorem and relation (3.20), we get

$$\begin{aligned} &\alpha \lim_{\lambda \longrightarrow 0} \int_\Omega \Big[a(u_\lambda, Du_\lambda) - a(u, D(u - \alpha\phi))\Big].D\phi dx \\ &\geq \limsup_{\lambda \longrightarrow 0} \int_\Omega \Big[a(u_\lambda, Du_\lambda) - a(u, D(u - \alpha\phi))\Big].\Big[D(u_\lambda - u + \alpha\phi)\Big]dx \\ &\quad + \limsup_{\lambda \longrightarrow 0} \int_\Omega \Big[a(u, D(u - \alpha\phi)\Big].D(u_\lambda - u)dx \\ &\geq \limsup_{\lambda \longrightarrow 0} \int_\Omega \Big[a(u, D(u - \alpha\phi)\Big].D(u_\lambda - u)dx \\ &= 0. \end{aligned}$$

Dividing the quantity $\alpha \lim_{\lambda \longrightarrow 0} \int_\Omega \Big[a(u_\lambda, Du_\lambda) - a(u, D(u - \alpha\phi))\Big].D\phi dx$ by $\alpha > 0$ and by $\alpha < 0$ successively, and passing to the limit with $\alpha \longrightarrow 0$, we get

$$\lim_{\lambda \longrightarrow 0} \int_\Omega a(u_\lambda, Du_\lambda).D\phi dx = \lim_{\alpha \longrightarrow 0} \int_\Omega a(u, D(u - \alpha\phi)).D\phi dx = \int_\Omega a(u, Du).D\phi dx,$$

i.e. $a(u_\lambda, Du_\lambda) \rightharpoonup a(u, Du)$ weakly in $\big(L^{p'}(\Omega)\big)^N$.
Hence $div\ a(u, Du) = div\ \chi$.

Up to now, we have shown that for all $\phi \in \mathcal{C}_c(\mathbb{R}^N)$ (after passing to the limit in (3.13) with $\lambda \longrightarrow 0$),

$$\int_\Omega a(u, Du).D(u - \phi) + \delta \int_{\partial\Omega} \psi(u)(u - \phi) \leq \int_\Omega (f - b(u))(u - \phi) - \int_{\partial\Omega} (\tilde{u} - \tilde{\phi})\, d\mu.$$

By density, inequality above remains true for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.
Then, we can conclude that

$$\int_\Omega a(u, Du).D\phi + \delta \int_{\partial\Omega} \psi(u)\phi = \int_\Omega [f - b(u)]\phi - \int_{\partial\Omega} \tilde{\phi}\, d\mu, \tag{3.23}$$

for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.
Finally, we must characterize the measure μ. First, according to equation (3.23), we can say that $\mu \in \mathcal{M}_b(\partial\Omega) \cap \big(W^{-\frac{1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^*\big)$ and $|\mu|$ dœs not charge the sets of $0-$capacity. Let us show now that $\mu \in \partial J(u)$. For this, we proceed as in [27]. Note that $\beta_\lambda = \partial j_\lambda$, where $j_\lambda \in \mathcal{J}_0(\partial\Omega)$, $j_\lambda(x, r) = \inf_{s\in\mathbb{R}} \big\{\frac{1}{2\lambda}|r - s|^2 + j(x, s)\big\}$.
Recall that, for a.e. $x \in \partial\Omega$ and for all $r \in \mathbb{R}$, $j_\lambda(x, r) \uparrow j(x, r)$ as $\lambda \downarrow 0$. Thus, by definition of the subdifferential, for all $\nu > \lambda > 0$ and a.e. $x \in \partial\Omega$,

$$\begin{aligned} j(x, r) &\geq j_\lambda(x, r) \\ &\geq j_\lambda(x, u_\lambda(x)) + \partial j_\lambda(x, u_\lambda(x))(r - u_\lambda(x)) \\ &\geq j_\nu(x, u_\lambda(x)) + \partial j_\lambda(x, u_\lambda(x))(r - u_\lambda(x)),\ \forall r \in \mathbb{R}. \end{aligned}$$

Therefore,

$$\int_{\partial\Omega} j(.,\xi) \geq \int_{\partial\Omega} j_\nu(.,u_\lambda) + \int_{\partial\Omega} \partial j_\lambda(.,u_\lambda)(\xi - u_\lambda) \ \forall\ \xi \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega).$$

Having in mind that $u_\lambda \longrightarrow u$ a.e. in Ω as $\lambda \longrightarrow 0$ then, according to Fatou's lemma and Lebesgue monotone convergence theorem, passing first to the limit with $\lambda \longrightarrow 0$ and after with $\nu \longrightarrow 0$, we get for all $\xi \in \mathcal{C}(\partial\Omega)$(the set of continuous functions on $\partial\Omega$)

$$\begin{aligned}\int_{\partial\Omega} j(.,\xi) &\geq \int_{\partial\Omega} j(.,u) + \liminf_{\lambda\longrightarrow 0} \int_{\partial\Omega} \beta_\lambda(.,u_\lambda)(\xi - u_\lambda)\\ &\geq \int_{\partial\Omega} j(.,u) + \liminf_{\lambda\longrightarrow 0} \int_{\partial\Omega} \beta_\lambda(.,u_\lambda)(\xi - u) + \liminf_{\lambda\longrightarrow 0} \int_{\partial\Omega} \beta_\lambda(.,u_\lambda)(u - u_\lambda)\\ &\geq \int_{\partial\Omega} j(.,u) + \int_{\partial\Omega} (\xi - u)\, d\mu + \liminf_{\lambda\longrightarrow 0} \int_{\partial\Omega} \beta_\lambda(.,u_\lambda)(u - u_\lambda).\end{aligned} \tag{3.24}$$

Now using (3.20), the monotonicity of ψ, the uniform $L^\infty-$estimate on u_λ and the a.e. convergence of u_λ to u, we get from (3.13),

$$\begin{aligned}&\liminf_{\lambda\longrightarrow 0} \int_{\partial\Omega} \beta_\lambda(.,u_\lambda)(u - u_\lambda)\\ &\geq \lim_{\lambda\longrightarrow 0} \int_{\Omega} (f - b(u_\lambda))(u - u_\lambda) + \limsup_{\lambda\longrightarrow 0} \int_{\Omega} a(u_\lambda, Du_\lambda).D(u_\lambda - u)\\ &\quad +\delta \lim_{\lambda\longrightarrow 0} \int_{\partial\Omega} (\psi(u_\lambda) - \psi(u))(u_\lambda - u) + \delta \lim_{\lambda\longrightarrow 0} \int_{\Omega} \psi(u)(u_\lambda - u)\\ &\geq 0.\end{aligned}$$

Consequently, we conclude from (3.24) that

$$\int_{\partial\Omega} j(.,\xi) \geq \int_{\partial\Omega} j(.,u) + \int_{\partial\Omega} (\xi - u)\, d\mu;$$

i.e.

$$J(\xi) \geq J(u) + \langle \mu, \xi - u \rangle,\ \forall \xi \in \mathcal{C}(\partial\Omega). \tag{3.25}$$

Since $\mu \in \mathcal{M}_b^p(\partial\Omega)$, one can say that inequality (3.25) holds for $\xi \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and thus we deduce that $\mu \in \partial J(u)$.

To conclude the proof of $ii)$, we prove, using the fact that $\mu \in \partial\mathcal{J}(u)$ and same technics as in the proof of Proposition 20 in [18], that the measure μ satisfies

$$\mu_r(x) \in \partial j(x, u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x))\ a.e.\ x \in \partial\Omega$$

$$\tilde{u} = \gamma_-\ \mu_s^- - a.e. \text{ on } \partial\Omega,\ \tilde{u} = \gamma_+\ \mu_s^+ - a.e. \text{ on } \partial\Omega.$$

$iii)$ We show that $D(A_{\delta,b})$ is dense in $L^1(\Omega)$ i.e. $\overline{D(A_{\delta,b})}^{\|\cdot\|_1} = L^1(\Omega)$.

We have $D(A_{\delta,b}) \subset L^\infty(\Omega) \subset L^1(\Omega)$ (since Ω is bounded). Therefore $\overline{D(A_{\delta,b})}^{\|\cdot\|_1} \subset L^1(\Omega)$. Mutually, let's show that $L^1(\Omega) \subset \overline{D(A_{\delta,b})}^{\|\cdot\|_1}$. To this end, it suffices to prove that $L^\infty(\Omega) \subset \overline{D(A_{\delta,b})}^{\|\cdot\|_1}$ (since $L^\infty(\Omega)$ is dense in $L^1(\Omega)$).

Let $\alpha > 0$. Given $f \in L^\infty(\Omega)$, if we set $b(u_\alpha) := (I + \alpha A_{\delta,b})^{-1} f$, then $(b(u_\alpha), \dfrac{1}{\alpha}(f - b(u_\alpha))) \in A_{\delta,b}$. So, taking $\phi = 0$ as a test function in the definition of the operator $A_{\delta,b}$, we get

$$\int_\Omega a(u_\alpha, Du_\alpha).Du_\alpha + \delta \int_{\partial\Omega} \psi(u_\alpha)(u_\alpha) \leq \frac{1}{\alpha}\int_\Omega (f - b(u_\alpha))u_\alpha - \int_{\partial\Omega} \tilde{u}_\alpha \, d\mu_\alpha. \tag{3.26}$$

Using hypothesis (H_2), we have $\int_\Omega [a(u_\alpha, Du_\alpha) - a(u_\alpha, 0)].Du_\alpha \geq \lambda_0 \|Du_\alpha\|_p^p$.
Then, we deduce from inequality (3.26) that

$$\lambda_0 ||Du_\alpha||_p^p \leq \frac{1}{\alpha}\int_\Omega (f - b(u_\alpha))u_\alpha - \delta \int_{\partial\Omega} \psi(u_\alpha)(u_\alpha) - \int_{\partial\Omega} \tilde{u}_\alpha \, d\mu_\alpha - \int_\Omega a(u_\alpha, 0).Du_\alpha. \tag{3.27}$$

Using the hypothesis (H_3), the monotonicity of ψ, properties of μ and the $L^\infty-$estimate on u_α, we get from (3.27)

$$\lambda_0 ||Du_\alpha||_p^p \leq \frac{1}{\alpha}C' + C. \tag{3.28}$$

Using the hypothesis (H_3), Hölder inequality and (3.28), we get

$$\begin{aligned}
\alpha \int_\Omega |a(u_\alpha, Du_\alpha)| & \leq \alpha \int_\Omega \Lambda(|u_\alpha|)(1 + |Du_\alpha|^{p-1}) \\
& \leq \alpha C_1 + \alpha \Big(\int_\Omega \big(\Lambda(|u_\alpha|)\big)^p \Big)^{\frac{1}{p}} \Big(\int_\Omega |Du_\alpha|^p \Big)^{\frac{1}{p'}} \\
& \leq \alpha C_1 + \alpha C_2 \big(\frac{1}{\alpha}C' + C \big)^{\frac{1}{p'}} \\
& \leq \alpha C_1 + \alpha 2^{\frac{1}{p'}} C_2 \Big(\frac{1}{2} \big(\frac{C'}{\alpha} \big)^{\frac{1}{p'}} + \frac{1}{2} C^{\frac{1}{p'}} \Big) \\
& \leq \alpha C_1 + \alpha^{\frac{1}{p}} C_3 + \alpha C_4 \\
& \longrightarrow 0 \text{ as } \alpha \longrightarrow 0.
\end{aligned}$$

On the other hand, if $\phi \in D(\Omega)$, taking $u_\alpha + \phi$ and $u_\alpha - \phi$ as test functions in the definition of the operator $A_{\delta,b}$, we get after adding both inequalities

$$\alpha \int_\Omega a(u_\alpha, Du_\alpha).D\phi + \alpha\delta \int_{\partial\Omega} \psi(u_\lambda)\phi = \int_\Omega (f - b(u_\alpha))\phi - \alpha \int_{\partial\Omega} \tilde{\phi} \, d\mu_\alpha. \tag{3.29}$$

Passing to the limit as $\alpha \longrightarrow 0$ in inequality (3.29), we get

$$\lim_{\alpha \longrightarrow 0} \int_\Omega b(u_\alpha)\phi = \int_\Omega f\phi, \ \forall \phi \in \mathcal{D}(\Omega). \tag{3.30}$$

Since $(u_\alpha)_\alpha$ is bounded in $L^\infty(\Omega)$, there exists a subsequence $(u_{\alpha_n})_n$ such that $u_{\alpha_n} \rightharpoonup u$ weakly in $L^p(\Omega)$; so $b(u_{\alpha_n}) \rightharpoonup b(u)$. Therefore, using (3.30), we get $b(u) = f$.
As $(u_\alpha)_\alpha$ is bounded in $L^\infty(\Omega)$ and b is continuous, we have

$$||b(u_\alpha)||_p^p \ = \int_\Omega |b(u_\alpha)|^p \ \leq \ \int_\Omega ||b(u_\alpha)||_\infty^p \ \leq \ C.$$

By Lebesgue dominated convergence theorem, $b(u_\alpha) \longrightarrow f$ in $L^p(\Omega)$. Consequently, $f \in \overline{D(A_{\delta,b})}^{||.||_1}$. □

4. Entropy solution

Before introducing the notion of entropy solutions for the problem $(E_b)(f)$, we define the following spaces which are similar to that introduced in [7, 11]. We note

$$\mathcal{T}^{1,p}(\Omega) := \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable; } T_k(u) \in W^{1,p}(\Omega) \text{ for all } k > 0\}.$$

In [11], the author proved that for $u \in \mathcal{T}^{1,p}(\Omega)$, there exists a unique measurable function $w : \Omega \longrightarrow \mathbb{R}$ such that $DT_k(u) = w\chi_{\{|w|<k\}}, \ \forall k > 0$. This function w will be denoted by Du.

Denote by $\mathcal{T}^{1,p}_{tr}(\Omega)$ the subset of $\mathcal{T}^{1,p}(\Omega)$ consisting of the function that can be approximated by functions of $W^{1,p}(\Omega)$ in the following sense: a function $u \in \mathcal{T}^{1,p}(\Omega)$ belongs to $\mathcal{T}^{1,p}_{tr}(\Omega)$ if there exists a sequence $(u_\delta)_\delta \in W^{1,p}(\Omega)$ such that:
(i) $u_\delta \longrightarrow u$ a.e. in Ω;
(ii) $DT_k(u_\delta) \rightharpoonup DT_k(u)$ weakly in $L^1(\Omega)$ for any $k > 0$;
(iii) there exists a measurable function $v : \partial\Omega \longrightarrow \mathbb{R}$ such that $(\tau(u_\delta))_\delta$ converges a.e. on $\partial\Omega$ to v.
The function v is called the trace of u, denoted $\tau(u)$ or u.

The concept of entropy solution for a problem with boundary conditions was introduced in [7] for the problem

$$\begin{cases} -div\ a(x, Du) = f \text{ in } \Omega \\ -a(x, Du).\eta \in \beta(u) \text{ on } \partial\Omega \end{cases} \tag{4.1}$$

and adapted by Sbihi and Wittbold [27] for the problem

$$\begin{cases} u - div\ a(u, Du) = f \text{ in } \Omega \\ -a(u, Du).\eta \in \beta(x, u) \text{ on } \partial\Omega. \end{cases} \tag{4.2}$$

Following [27], we define an entropy solution of $(E_b)(f)$.

Definition 4.1. A function $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ is an *entropy solution* of problem $(E_b)(f)$ if $b(u) \in L^1(\Omega)$ and there exists a measure $\mu \in \mathcal{M}^p_b(\partial\Omega)$ with

$$\mu_r(x) \in \partial j(x, u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x)) \ a.e.\ x \in \partial\Omega \tag{4.3}$$

such that for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(u, Du).DT_k(u - \phi) \le \int_\Omega (f - b(u))T_k(u - \phi) - \int_{\partial\Omega} T_k(\tilde{u} - \tilde{\phi})\, d\mu,$$

$$\tilde{u} = \gamma_+ \ \mu_s^+ - \ a.e.\ \partial\Omega,\ \tilde{u} = \gamma_- \ \mu_s^- - \ a.e.\ \partial\Omega. \tag{4.4}$$

Remark 4.1. *Note that each integral in the preceding definition is well defined. Indeed, the first term can be understood as* $\int_\Omega a(T_l(u), DT_l(u)).DT_k(u - \phi)$ *where* $l \ge k + ||\phi||_\infty$. *The second is well defined according to Hölder inequality. Since* $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, *we have* $u - \phi \in \mathcal{T}^{1,p}_{tr}(\Omega)$ *(see [7], Theorem 3.1). Hence,* $T_k(u - \phi) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and admits a trace which has a quasi-continuous representative, according to the remarks made in the preliminary. Thus, the last integral in the above definition is well defined.*

We define an operator $\mathcal{A}$ by the rule

$$(b(u), f - b(u)) \in \mathcal{A} \text{ if and only if } \begin{cases} f \in L^1(\Omega) \text{ and} \\ \\ u \text{ is an entropy solution of problem } (E_b)(f). \end{cases}$$

In the following, we use the notation $A_{m,n}$ (resp. $\psi_{m,n}$) instead of A_δ (resp. $\delta\psi$), where $\psi_{m,n}(u) = \frac{1}{m}\psi(u^+) - \frac{1}{n}\psi(u^-),\ m,n \in \mathbb{N}^*$.

Theorem 4.1. *The operator $\mathcal{A}$ is $m-$accretive with dense domain in $L^1(\Omega)$ and $\mathcal{A} = \liminf_{m,n\longrightarrow+\infty} A_{m,n}$ where $\liminf_{m,n\longrightarrow+\infty} A_{m,n}$ is the operator defined by $(x,y) \in \liminf_{m,n\longrightarrow+\infty} A_{m,n}$, if for all $m>0, n>0$, there are $(x_{m,n}, y_{m,n}) \in A_{m,n}$ such that*

$$(x,y) = \liminf_{m,n\longrightarrow+\infty}(x_{m,n}, y_{m,n})$$

in $X \times X$.

Proof. We divide the proof into six steps.

Step 1. A priori estimates.

Let $f \in L^1(\Omega)$. We approximate f and b respectively by $f_{m,n} = (f \wedge m) \vee (-n) \in L^\infty(\Omega)$ nondecreasing in m, nonincreasing in n and $b_{m,n}(\sigma) = b(\sigma) + \dfrac{1}{m}\sigma^+ - \dfrac{1}{n}\sigma^-$, $\forall\ \sigma \in \mathbb{R}$.
Note that $||f_{m,n}||_1 \ \leq\ ||f||_1$.
Then by Theorem 3.1, $f_{m,n} \in R(I + A_{m,n})$ and there exists $u_{m,n} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and a measure $\mu_{m,n} \in \mathcal{M}_b^p(\partial\Omega)$ satisfying

$$(\mu_{m,n})_r(x) \in \partial j(x, u_{m,n}(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u_{m,n}(x)) \text{ a.e. } x \in \partial\Omega,$$

such that for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\begin{aligned} &\int_\Omega a(u_{m,n}, Du_{m,n}).D(u_{m,n} - \phi) + \int_{\partial\Omega} \psi_{m,n}(u_{m,n})(u_{m,n} - \phi) \\ &\quad \leq \int_\Omega (f_{m,n} - b_{m,n}(u_{m,n}))(u_{m,n} - \phi) - \int_{\partial\Omega} (\tilde{u}_{m,n} - \tilde{\phi})\, d\mu_{m,n} \end{aligned} \tag{4.5}$$

with $\tilde{u}_{m,n}^{+/-} = \gamma_{+/-}\ (\mu_{m,n})_s^{+/-}$ a.e. on $\partial\Omega$.
Now, let $k > 0$ be fixed. Using $\phi = u_{m,n} - T_k(u_{m,n})$ as a test function in (4.5) and applying hypothesis (H_2), we obtain

$$\begin{aligned} &\lambda_0 \int_\Omega |DT_k(u_{m,n})|^p + \frac{1}{m}\int_{\partial\Omega} T_k(u_{m,n})\psi(u_{m,n}^+) - \frac{1}{n}\int_{\partial\Omega} T_k(u_{m,n})\psi(u_{m,n}^-) \\ &\leq \int_\Omega T_k(u_{m,n})(f_{m,n} - b_{m,n}(u_{m,n})) - \int_{\partial\Omega} T_k(\tilde{u}_{m,n})d\mu_{m,n} - \int_\Omega a(u_{m,n},0).DT_k(u_{m,n}). \end{aligned} \tag{4.6}$$

By Gauss-Green Formula and hypothesis (H_3), we have

$$\begin{aligned} \Big|\int_\Omega a(u_{m,n},0).DT_k(u_{m,n})\Big| &\leq \Big|\int_{\partial\Omega}\Big(\int_0^{T_k(u_{m,n})} a(r,0)dr\Big).\eta d\sigma\Big| \\ &\leq \int_{\partial\Omega}\Big|\int_0^{T_k(u_{m,n})} \Lambda(|r|)dr\Big|d\sigma \\ &\leq C, \end{aligned} \tag{4.7}$$

where C is a constant depending on k. Then, from inequality (4.6), according to the monotonicity of ψ, we conclude that

$$\lambda_0 \int_\Omega |DT_k(u_{m,n})|^p \ \leq\ C. \tag{4.8}$$

Thus, $(T_k(u_{m,n}))_{m,n}$ is a bounded sequence of $W^{1,p}(\Omega)$. Hence, after passing to a suitable subsequence if necessary, $(T_k(u_{m,n}))_{m,n}$ converges weakly in $W^{1,p}(\Omega)$. Then, $T_k(u_{m,n}) \longrightarrow v_k$ in $L^p(\Omega)$ as $m, n \longrightarrow \infty$. We may also assume that $DT_k(u_{m,n}) \rightharpoonup g_k$ in $(L^p(\Omega))^N$ as $m, n \longrightarrow \infty$.

Now, we must prove the almost everywhere convergence of $u_{m,n}$. As $A_{m,n}$ is $T-$accretive in $L^1(\Omega)$, we have for all $m \geq m'$,

$$\int_\Omega \big(b_{m',n}(u_{m',n}) - b_{m,n}(u_{m,n})\big)^+ \;\leq\; \int_\Omega (f_{m',n} - f_{m,n})^+.$$

As $f_{m,n}$ is nondecreasing in m, we have: $m \geq m' \Longrightarrow f_{m',n} - f_{m,n} \;\leq\; 0 \Longrightarrow (f_{m',n} - f_{m,n})^+ = 0$. Then $m \geq m' \Longrightarrow \big(b_{m',n}(u_{m',n}) - b_{m,n}(u_{m,n})\big)^+ = 0$, i.e. $b_{m',n}(u_{m',n}) - b_{m,n}(u_{m,n}) \leq 0$ a.e. on Ω. Thus,

$$\Big(b(u_{m',n}) - b(u_{m,n})\Big) + \frac{1}{m'}\big((u_{m',n})^+ - (u_{m,n})^+\big) + \frac{1}{n}\big((u_{m,n})^- - (u_{m',n})^-\big) \leq 0. \quad (4.9)$$

It is easy to see that the three terms of the inequality (4.9) have the same sign, then they are negatives which implies that $u_{m',n} - u_{m,n} \leq 0$ for $m \geq m'$ and n fixed. Then, $(u_{m,n})_m$ is nondecreasing. By the same method, we show that $(u_{m,n})_n$ is nonincreasing. Since $(u_{m,n})_m$ is uniformly bounded then we deduce that

$$u_{m,n} \uparrow u_n \text{ when } m \longrightarrow +\infty \text{ and } u_n \downarrow u \text{ when } n \longrightarrow +\infty.$$

By applying Lebesgue dominated convergence theorem, we get

$$u_{m,n} \uparrow_m u_n \downarrow_n u, \; u_{m,n} \downarrow_n u_m \uparrow_m u \text{ in } L^1(\Omega). \quad (4.10)$$

Therefore, from (4.10) we get the convergence of $(u_{m,n})_{m,n}$ to u in $L^1(\Omega)$ and also the convergence almost everywhere on Ω.

Then, we conclude that $v_k = T_k(u)$ and $g_k = DT_k(u)$. Therefore, $T_k(u) \in W^{1,p}(\Omega)$ for all $k > 0$. Consequently, $u \in \mathcal{T}^{1,p}(\Omega)$.

Finally, we show following [7], that $(\tau(u_{m,n}))_{m,n}$ converges a.e. on $\partial\Omega$.

For every $k > 0$, let $A_k := \{x \in \partial\Omega : |T_k(u(x))| < k\}$ and $C := \partial\Omega \setminus \bigcup_{k>0} A_k$. Then,

$$\begin{aligned}
\text{meas}(C) &= \frac{1}{k}\int_C |T_k(u(x))|dx \\
&\leq \frac{1}{k}\int_{\partial\Omega} |T_k(u(x))|dx \\
&\leq \frac{C_1}{k}||T_k(u)||_{W^{1,1}(\Omega)} \\
&= \frac{C_1}{k}||T_k(u)||_{L^1(\Omega)} + \frac{C_1}{k}||DT_k(u)||_{L^1(\Omega)} \\
&\leq \frac{C_1}{k}||T_k(u)||_{L^1(\Omega)} + \frac{C_1'}{k}||DT_k(u)||_{L^p(\Omega)}.
\end{aligned}$$

According to (4.8) and the boundedness of $\Big\{\|T_k(u)\|_{L^1(\Omega)} \,;\; k > 0\Big\}$, we deduce by letting $k \longrightarrow +\infty$ that meas$(C) = 0$.

Let us define in $\partial\Omega$ the function v by

$$v(x) := T_k(u(x)) \text{ if } x \in A_k.$$

As $T_k(u_{m,n})$ converges to $T_k(u)$ a.e. on $\partial\Omega$, there exists $C' \subset \partial\Omega$ such that $T_k(u_{m,n})$ converges to $T_k(u)$ on $\partial\Omega\backslash C'$ with meas$(C') = 0$.
We take $x \in \partial\Omega\backslash(C \cup C')$, then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_{m,n}(x) - v(x) = (u_{m,n}(x) - T_k(u_{m,n}(x))) + (T_k(u_{m,n}(x)) - T_k(u(x))).$$

Since $x \in A_k$, we have $|T_k(u(x))| < k$ and so $|T_k(u_{m,n}(x))| < k$, from which we deduce that $|u_{m,n}| < k$ and $T_k(u_{m,n}(x)) = u_{m,n}(x)$.
Therefore

$$u_{m,n}(x) - v(x) = T_k(u_{m,n}(x)) - T_k(u(x)) \longrightarrow 0, \text{ as } n \longrightarrow +\infty.$$

This means that $u_{m,n}$ converges to v a.e. on $\partial\Omega$ and then, $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$.

Step 2. Existence of the measure.

It remains to show the existence of a measure $\mu \in \mathcal{M}^p_b(\partial\Omega)$ such that $\mu_{m,n} \longrightarrow \mu$ strongly in $\mathcal{M}^p_b(\partial\Omega)$.
Let $u^\lambda_{m,n}$ be a solution to the problem

$$\begin{aligned} \int_\Omega a(u^\lambda_{m,n}, Du^\lambda_{m,n}).D\varphi + \frac{1}{m}\int_{\partial\Omega} \psi(u^{\lambda,+}_{m,n})\varphi - \frac{1}{n}\int_{\partial\Omega} \psi(u^{\lambda,-}_{m,n})\varphi \\ = \int_\Omega (f_{m,n} - b_{m,n}(u^\lambda_{m,n}))\varphi - \int_{\partial\Omega} \beta_\lambda(.,u^\lambda_{m,n})\varphi, \end{aligned} \tag{4.11}$$

for all $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We know from Theorem 3.1 (part ii)) that $||\beta_\lambda(.,u^\lambda_{m,n})||_1$ is uniformly bounded by a constant C independent of λ, thus $\beta_\lambda(.,u^\lambda_{m,n}) \rightharpoonup \mu_{m,n}$ in $\mathcal{M}_b(\partial\Omega)$ as $\lambda \longrightarrow 0$. Therefore

$$||\mu_{m,n}||_{\mathcal{M}_b(\partial\Omega)} \ \leq \ \liminf_{\lambda\longrightarrow 0} ||\beta_\lambda(.,u^\lambda_{m,n})||_{\mathcal{M}_b(\partial\Omega)} \ \leq C$$

and we deduce, after extracting a subsequence if necessary that $\mu_{m,n} \rightharpoonup \mu$ weakly in $\mathcal{M}_b(\partial\Omega)$ as $m,n \longrightarrow \infty$.

In order to prove the strong convergence of $\mu_{m,n}$, we need the following comparison result.

Lemma 4.1. *Assume that $\tilde{m} \geq m$, $\tilde{n} \geq n$ and $f_{m,n}, f_{\tilde{m},n} \in L^\infty(\Omega)$. Let $u^\lambda_{m,n}, u^\lambda_{\tilde{m},n}$ be the weak solutions which verify (4.11). Then*

$$u^\lambda_{m,\tilde{n}} \leq u^\lambda_{m,n} \leq u^\lambda_{\tilde{m},n} \text{ a.e. in } \Omega$$

and

$$\beta_\lambda(.,u^\lambda_{m,\tilde{n}}) \leq \beta_\lambda(.,u^\lambda_{m,n}) \leq \beta_\lambda(.,u^\lambda_{\tilde{m},n}) \text{ a.e. on } \partial\Omega.$$

Proof. Of Lemma 4.1. As $A_{m,n}$ is $T-$accretive in $L^1(\Omega)$, we have for all $\tilde{m} \geq m$,

$$\int_\Omega \left(b_{m,n}(u^\lambda_{m,n}) - b_{\tilde{m},n}(u^\lambda_{\tilde{m},n})\right)^+ \ \leq \ \int_\Omega (f_{m,n} - f_{\tilde{m},n})^+.$$

As $f_{m,n}$ is nondecreasing in m then, $\tilde{m} \geq m \Longrightarrow f_{m,n} - f_{\tilde{m},n} \ \leq \ 0 \Longrightarrow (f_{m,n} - f_{\tilde{m},n})^+ = 0$.
Therefore

$$\tilde{m} \geq m \Longrightarrow \left(b_{m,n}(u^\lambda_{m,n}) - b_{\tilde{m},n}(u^\lambda_{\tilde{m},n})\right)^+ = 0, \text{ i.e. } b_{m,n}(u^\lambda_{m,n}) - b_{\tilde{m},n}(u^\lambda_{\tilde{m},n}) \leq 0 \text{ a.e. on } \Omega.$$

Thus, $\Big(b(u^{\lambda}_{m,n})-b(u^{\lambda}_{\tilde m,n})\Big)+\dfrac{1}{m}\big(u^{\lambda}_{m,n}\big)^{+}-\dfrac{1}{\tilde m}\big(u_{m,n}\big)^{+}+\dfrac{1}{n}\Big((u^{\lambda}_{\tilde m,n})^{-}-(u^{\lambda}_{m,n})^{-}\Big)\le 0$ and then

$$\Big(b(u^{\lambda}_{m,n})-b(u^{\lambda}_{\tilde m,n})\Big)+\frac{1}{m}\Big(\big(u^{\lambda}_{m,n}\big)^{+}-\big(u_{m,n}\big)^{+}\Big)+\frac{1}{n}\Big((u^{\lambda}_{\tilde m,n})^{-}-(u^{\lambda}_{m,n})^{-}\Big)\le 0. \quad (4.12)$$

It is easy to see that the three terms of the inequality (4.12) have the same sign, then they are negative, which implies that $u^{\lambda}_{m,n}-u^{\lambda}_{\tilde m,n}\le 0$ a.e. on Ω and as β_λ is monotone then $\beta_\lambda(.,u^{\lambda}_{m,n})\le\beta_\lambda(.,u^{\lambda}_{\tilde m,n})$ a.e. on $\partial\Omega$. By the same methods, we show the other inequalities. Thus the result of Lemma 4.1 follows. □

Note that the result of Lemma 4.1 remains true for the positive and negative parts, i.e. $\pm\beta_\lambda(.,u^{\lambda}_{m,\tilde n})^{\pm}\le\pm\beta_\lambda(.,u^{\lambda}_{m,n})^{\pm}\le\pm\beta_\lambda(.,u^{\lambda}_{\tilde m,n})^{\pm}$.

Thus, by the previous result of convergence, we deduce that $\pm\mu^{\pm}_{m,\tilde n}\le\pm\mu^{\pm}_{m,n}\le\pm\mu^{\pm}_{\tilde m,n}$, which is equivalent to say that the regular and the singular parts verify this comparison result. From this, it follows that $\mu^{+}_{m,n}\uparrow_m\mu^{+}_{n}$ in $\mathcal{M}_b(\partial\Omega)$ as $m\longrightarrow+\infty$. Indeed, let $\mu^{+}_{n}:\mathcal{B}(\partial\Omega)\longrightarrow[0,+\infty]$ defined by $\mu^{+}_{n}(A)=\lim\limits_{m\longrightarrow+\infty}\mu^{+}_{m,n}(A)<+\infty$. Here, $\mathcal{B}(\partial\Omega)$ denotes the set of Borel sets of $\partial\Omega$. Note that μ^{+}_{n} is a Radon measure. We have

$$\begin{aligned}\big\|\mu^{+}_{m,n}-\mu^{+}_{n}\big\| &= \sup_{(E_i)_{i=1,n}\in\mathcal{B}(\partial\Omega)}\Big[\sum_{i=1}^{n}\big(\mu^{+}_{m,n}-\mu^{+}_{n}\big)(E_i)\Big]\\ &= \sum_{i=1}^{n}\Big(\mu^{+}_{m,n}(E_i)-\mu^{+}_{n}(E_i)\Big)\\ &= \mu^{+}_{m,n}(\partial\Omega)-\mu^{+}_{n}(\partial\Omega)\\ &\longrightarrow 0 \text{ as } m\longrightarrow+\infty,\end{aligned}$$

where $(E_i)_i$ denotes a finite partition of $\partial\Omega$. We applied the same methods to show that $\mu^{+}_{n}\downarrow\mu^{+}$ as $n\longrightarrow+\infty$. Note that we get the same results for the negative parts and this concludes the proof of Step 2.

Step 3. The pseudo-monotonicity argument.

We recall that $u_{m,n}$ satisfies, for all $\varphi\in W^{1,p}(\Omega)\cap L^{\infty}(\Omega)$

$$\begin{aligned}&\int_\Omega a(u_{m,n},Du_{m,n}).D\varphi+\frac{1}{m}\int_{\partial\Omega}\psi(u^{+}_{m,n})\varphi-\frac{1}{n}\int_{\partial\Omega}\psi(u^{-}_{m,n})\varphi\\ &=\int_\Omega(f_{m,n}-b_{m,n}(u_{m,n}))\varphi-\int_{\partial\Omega}\beta_\lambda(.,u_{m,n})\varphi.\end{aligned}\quad (4.13)$$

Since $T_k(u_{m,n})$ is bounded in $W^{1,p}(\Omega)$ then, thanks to the growth assumption (H_3), there exists a vector fields $\chi_k\in(L^{p'}(\Omega))^N$ such that $a(T_k(u_{m,n}),DT_k(u_{m,n}))\rightharpoonup\chi_k$ weakly in $(L^{p'}(\Omega))^N$ as $m,n\longrightarrow+\infty$, for all $k\in\mathbb{N}^*$. The aim is to prove, via a pseudo-monotonicity argument, that $div\ \chi_k=div\ a(T_k(u),DT_k(u))$ in $\mathcal{D}'(\Omega)$. To this end, we define for $l<k$, the following integral

$$I=\int_\Omega[a(T_k(u_{m,n}),DT_k(u_{m,n}))-a(T_k(u_{m',n'}),DT_k(u_{m',n'}))].DT_l(T_k(u_{m,n})-T_k(u_{m',n'})), \quad (4.14)$$

which can be written as

$$\int_{\{|u_{m,n}|<k,|u_{m',n'}|<k\}}[a(u_{m,n},D(u_{m,n}))-a(u_{m',n'},D(u_{m',n'}))].DT_l(u_{m,n}-u_{m',n'})$$
$$+\int_{\{|u_{m,n}|<k,|u_{m',n'}|\geq k\}}[a(u_{m,n},D(u_{m,n}))-a(T_k(u_{m',n'}),0)].DT_l(u_{m,n}-T_k(u_{m',n'}))$$
$$+\int_{\{|u_{m,n}|\geq k,|u_{m',n'}|<k\}}[a(T_k(u_{m,n}),0)-a(u_{m',n'},D(u_{m',n'}))].DT_l(T_k(u_{m,n})-u_{m',n'})$$
$$:= \quad I_1+I_2+I_3.$$

We want to pass to the limit in I, in the following order, with $m',n'\longrightarrow+\infty, m,n\longrightarrow+\infty$ and then $l\longrightarrow 0$. Note that the term I_1 can be written as

$$I_1=\int_\Omega[a(u_{m,n},Du_{m,n})-a(u_{m',n'},D(u_{m',n'}))].DT_l(u_{m,n}-u_{m',n'})$$
$$-\int_{\{|u_{m,n}|<k,|u_{m',n'}|\geq k\}}[a(u_{m,n},D(u_{m,n}))-a(u_{m',n'},D(u_{m',n'}))].DT_l(u_{m,n}-u_{m',n'})$$
$$-\int_{\{|u_{m,n}|\geq k,|u_{m',n'}|<k\}}[a(u_{m,n},D(u_{m,n}))-a(u_{m',n'},D(u_{m',n'}))].DT_l(u_{m,n}-u_{m',n'})$$
$$-\int_{\{|u_{m,n}|\geq k,|u_{m',n'}|\geq k\}}[a(u_{m,n},D(u_{m,n}))-a(u_{m',n'},D(u_{m',n'}))].DT_l(u_{m,n}-u_{m',n'})$$
$$:= \quad I_1^1-I_1^2-I_1^3-I_1^4.$$

Choosing $T_l(u_{m,n}-u_{m',n'})$ and $T_l(u_{m',n'}-u_{m,n})$ corresponding to solutions $u_{m,n}$ and $u_{m',n'}$ respectively in the equation (4.13), adding both equalities, we get

$$\begin{aligned}&\int_\Omega\big[a(u_{m,n},Du_{m,n})-a(u_{m',n'},Du_{m',n'})\big].DT_l(u_{m,n}-u_{m',n'})\\&\qquad+\int_{\partial\Omega}\big(\psi_{m,n}(u_{m,n})-\psi_{m',n'}(u_{m',n'})\big)T_l(u_{m,n}-u_{m',n'})\\&\quad=\int_\Omega\big(f_{m,n}-f_{m',n'}+b_{m',n'}(u_{m',n'})-b_{m,n}(u_{m,n})\big)T_l(u_{m,n}-u_{m',n'})\\&\qquad-\int_{\partial\Omega}\big(\beta_\lambda(.,u_{m,n})-\beta_\lambda(.,u_{m',n'})\big)T_l(u_{m,n}-u_{m',n'}).\end{aligned}\tag{4.15}$$

Using the fact that $u_{m,n},b_{m,n},f_{m,n},\psi_{m,n}$ are uniformly bounded, $u_{m,n},u_{m',n'}\longrightarrow u$ a.e. in Ω, $b_{m,n},b_{m',n'}\longrightarrow b$ which is continuous in $\mathbb{R}$, $f_{m,n},f_{m',n'}\longrightarrow f$ in $L^1(\Omega)$, $\mu_{m,n},\mu_{m',n'}\longrightarrow\mu$ strongly in $\mathcal{M}_b(\partial\Omega)$, by Lebesgue dominated convergence theorem, passing to the limit in equation (4.15) we obtain

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty}\int_\Omega\big[a(u_{m,n},Du_{m,n})-a(u_{m',n'},Du_{m',n'})\big].DT_l(u_{m,n}-u_{m',n'})=0,$$

i.e.

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} I_1^1 = 0.$$

By hypothesis (H_1), we have

$$\begin{aligned} I_1^2 &= \int_{\{|u_{m,n}|<k,|u_{m',n'}|\geq k\}} [a(u_{m,n}, D(u_{m,n})) - a(u_{m',n'}, D(u_{m',n'}))].DT_l(u_{m,n} - u_{m',n'}) \\ &\geq \int_{\{|u_{m,n}|<k,|u_{m',n'}|\geq k\}\cap\{|u_{m,n}-u_{m',n'}|<l\}} |a(u_{m,n}, Du_{m',n'}) - a(u_{m',n'}, Du_{m',n'})| \\ &\quad \times |D(u_{m,n} - u_{m',n'})|. \end{aligned}$$

Note that
$|u_{m,n} - u_{m',n'}| < l \Longrightarrow |u_{m,n}| - |u_{m',n'}| < l \Longrightarrow |u_{m',n'}| < |u_{m,n}| + l \Longrightarrow |u_{m',n'}| < k + l$ (since $|u_{m,n}| < k$) $\Longrightarrow |u_{m',n'}| < 2k$ (since $l < k$). Therefore, using hypothesis (H_4), Hölder inequality, coerciveness of the power application and (4.8) we obtain

$$\begin{aligned} I_1^2 &\geq -\int_{\mathcal{F}_1} |a(u_{m,n}, Du_{m',n'}) - a(u_{m',n'}, Du_{m',n'})||D(u_{m,n} - u_{m',n'})| \\ &\geq -\Big[\int_{\mathcal{F}_1} 2^{p'} C\big(u_{m,n}, u_{m',n'}\big)^{p'} |u_{m,n} - u_{m',n'}|^{p'}\big(1 + |Du_{m',n'}|^p\big)\Big]^{\frac{1}{p'}} \\ &\quad \times\Big[\int_{\mathcal{F}_1} |D(u_{m,n} - u_{m',n'})|^p\Big]^{\frac{1}{p}} \\ &\geq -Cl, \end{aligned}$$

where $\mathcal{F}_1 := \{|u_{m,n}| < k, |u_{m',n'}| < 2k, |u_{m,n} - u_{m',n'}| < l\}$ and C is a constant depending on f, b, p and k. Then

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} I_1^2 \geq 0.$$

By the same methods, we show that

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} I_1^3 \geq 0.$$

For the term I_1^4, define the function h_k by

$$h_k(r) = \begin{cases} 0 & \text{if } |r| < k \\ r - ksign_0(r) & \text{if } |r| \geq k. \end{cases}$$

Then, I_1^4 is equal to

$$\begin{aligned} I_1^4 = &\int_\Omega [a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})].DT_l(h_k(u_{m,n}) - h_k(u_{m',n'})) \\ &- \int_{\{|u_{m,n}|<k,|u_{m',n'}|\geq k\}} [a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})].DT_l(-h_k(u_{m',n'})) \\ &- \int_{\{|u_{m,n}|\geq k,|u_{m',n'}|<k\}} [a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})].DT_l(h_k(u_{m,n}) \\ := \quad & K_1 - K_2 - K_3. \end{aligned} \tag{4.16}$$

Using $T_l(h_k(u_{m,n})-h_k(u_{m',n'}))$ as a test function in the equalities corresponding to both solutions $u_{m,n}$ and $u_{m',n'}$, we show as for I_1^1, that

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} K_1 = 0.$$

Note that, by using $T_l(h_k(u_{m,n}))$ as a test function in (4.13), by the same technics as for (4.8), it follows

$$\int_\Omega |DT_l(h_k(u_{m,n}))|^p \leq lC, \tag{4.17}$$

where C is a constant depending only on f, b and k.
Now, by Hölder inequality

$$\begin{aligned} |K_2| &\leq \int_{\mathcal{F}_2} |a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})||DT_l(h_k(u_{m',n'}))| \\ &\leq \Big[\int_{\{|u_{m,n}|<k, |u_{m',n'}|<2k\}} |a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})|^{p'}\Big]^{\frac{1}{p'}} \\ &\quad\times\Big[\int_\Omega |DT_l(h_k(u_{m',n'}))|^p\Big]^{\frac{1}{p}}, \end{aligned}$$

where $\mathcal{F}_2 = \{|u_{m,n}| < k, |u_{m',n'}| < 2k, |h_k(u_{m',n'})| < l\}$.
Then, the hypothesis (H_3) and the estimations (4.8) and (4.17) imply

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} K_2 = 0.$$

Similarly, we have $\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} K_3 = 0$.
Consequently, combining all limits in (4.16), we get $\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} I_1^4 = 0$ and we conclude that

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} I_1 = 0.$$

Now, consider the term I_2. Let's remark that

$$\begin{aligned} I_2 &= \int_{\{|u_{m,n}|<k, |u_{m',n'}|\geq k\}} [a(u_{m,n}, Du_{m,n}) - a(u_{m,n}, 0)].DT_l(u_{m,n} - T_k(u_{m',n'})) \\ &\quad + \int_{\{|u_{m,n}|<k, |u_{m',n'}|\geq k\}} [a(u_{m,n}, 0) - a(T_k(u_{m',n'}), 0)].DT_l(u_{m,n} - T_k(u_{m',n'})) \\ &:= I_2^1 + I_2^2. \end{aligned}$$

Hypothesis (H_4), Hölder's inequality and (4.8) yield

$$\begin{aligned} |I_2^2| &\leq \int_{\mathcal{F}_3} C\big(u_{m,n}, u_{m',n'}\big)|T_k(u_{m,n}) - T_k(u_{m',n'})||DT_k(u_{m,n})| \\ &\leq C\Big[\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} |T_k(u_{m,n}) - T_k(u_{m',n'})|^{p'}\Big]^{\frac{1}{p'}} \end{aligned}$$

where $\mathcal{F}_3 = \{|u_{m,n}| < k, |u_{m',n'}| < 2k, |T_k(u_{m,n}) - T_k(u_{m',n'})| < l\}$, and then

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} I_2^2 = 0.$$

Hypothesis (H_2) ensures that $I_2^1 \geq 0$. On the other hand

$$I_2^1 \leq \int_{\{k-l<|u_{m,n}|<k\}} [a(u_{m,n}, D(u_{m,n})) - a(u_{m,n}, 0)].D(u_{m,n}).$$

Now, taking $T_k(u_{m,n}) - T_{k-l}(u_{m,n})$ as a test function in (4.13), we have

$$\begin{aligned}
&\int_{\{k-l<|u_{m,n}|<k\}} a(u_{m,n}, Du_{m,n}).D(u_{m,n}) \\
&\quad + \frac{1}{m}\int_{\partial\Omega\cap\{k-l<|u_{m,n}|<k\}} \psi(u_{m,n}^+)(T_k(u_{m,n}) - T_{k-l}(u_{m,n}) \\
&\quad - \frac{1}{n}\int_{\partial\Omega\cap\{k-l<|u_{m,n}|<k\}} \psi(u_{m,n}^-)(T_k(u_{m,n}) - T_{k-l}(u_{m,n})) \\
&= \int_{\{k-l<|u_{m,n}|<k\}} (f_{m,n} - b_{m,n}(u_{m,n}))(T_k(u_{m,n}) - T_{k-l}(u_{m,n})) \\
&\quad - \int_{\partial\Omega\cap\{k-l<|u_{m,n}|<k\}} \beta_\lambda(., u_{m,n})(T_k(u_{m,n}) - T_{k-l}(u_{m,n})).
\end{aligned} \tag{4.18}$$

As $l \longrightarrow 0$, we have $T_{k-l}(u_{m,n}) \longrightarrow T_k(u_{m,n})$. Then, passing to the limit in (4.18) with $l \longrightarrow 0$ we obtain

$$\lim_{l\longrightarrow 0}\int_{\{k-l<|u_{m,n}|<k\}} a(u_{m,n}, Du_{m,n}).D(u_{m,n}) = 0.$$

We have

$$\begin{aligned}
&\int_{\{k-l<|u_{m,n}|<k\}} a(u_{m,n}, Du_{m,n}).D(u_{m,n}) \\
&\quad = \int_{\{k-l<|u_{m,n}|<k\}} (a(u_{m,n}, Du_{m,n}) - a(u_{m,n}, 0)).D(u_{m,n}) + \int_{\{k-l<|u_{m,n}|<k\}} a(u_{m,n}, 0).
\end{aligned}$$

Using the hypothesis (H_3), we deduce that

$$\int_{\{k-l<|u_{m,n}|<k\}} |a(u_{m,n}, 0)| \leq \int_{\{k-l<|u_{m,n}|<k\}} \Lambda(|u_{m,n}|) \longrightarrow 0 \text{ as } l \longrightarrow 0.$$

Then,

$$\lim_{l\longrightarrow 0}\int_{\{k-l<|u_{m,n}|<k\}} (a(u_{m,n}, Du_{m,n}) - a(u_{m,n}, 0)).D(u_{m,n}) \leq 0.$$

We conclude that

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} I_2^1 = 0.$$

An analogous decomposition and estimates can be applied to I_3. Thus, combining all limits yields

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} I \leq 0. \tag{4.19}$$

Now, let $\varphi \in W^{1,p}(\Omega)$. Then, I can be written as

$$I = -\int_\Omega a(T_k(u_{m,n}), DT_k(u_{m,n})).D\varphi - \int_\Omega a(T_k(u_{m',n'}), DT_k(u_{m',n'})).D\varphi + J_1 + J_2 + J_3 + J_4$$

where

$$J_1 := \int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} a(T_k(u_{m,n}), DT_k(u_{m,n})).D(T_k(u_{m,n}) - T_k(u_{m',n'}) + \varphi),$$

$$J_2 := \int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} a(T_k(u_{m',n'}), DT_k(u_{m',n'})).D(T_k(u_{m',n'}) - T_k(u_{m,n}) + \varphi),$$

$$J_3 := \int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|\geq l\}} a(T_k(u_{m,n}), DT_k(u_{m,n})).D\varphi,$$

$$J_4 := \int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|\geq l\}} a(T_k(u_{m',n'}), DT_k(u_{m',n'})).D\varphi \ .$$

Passing to the limit in this last equality and using the relation (4.19), we obtain

$$2\int_\Omega \chi_k.D\varphi \ \geq \lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty}(J_1+J_2+J_3+J_4). \tag{4.20}$$

Consider the term J_1. Using hypothesis (H_1) we have $\Big[a(T_k(u_{m,n}), DT_k(u_{m,n})) - a\Big(T_k(u_{m,n}), D\Big(T_k(u_{m',n'}) - \varphi\Big)\Big)\Big].D(T_k(u_{m,n}) - T_k(u_{m',n'}) + \varphi) \geq 0$, which implies that

$$\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} a(T_k(u_{m,n}), DT_k(u_{m,n})).D(T_k(u_{m,n}) - T_k(u_{m',n'}) + \varphi) \geq$$

$$\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} a\Big(T_k(u_{m,n}), D\Big(T_k(u_{m',n'}) - \varphi\Big)\Big).D(T_k(u_{m,n}) - T_k(u_{m',n'}) + \varphi).$$

As $T_k(u_{m,n}), DT_k(u_{m,n})$ are uniformly bounded, $DT_k(u_{m,n}), DT_k(u_{m',n'}) \rightharpoonup DT_k(u)$ weakly in $(L^p(\Omega))^N$ and $T_k(u_{m,n}), T_k(u_{m',n'}) \longrightarrow T_k(u)$ a.e. in Ω as $m, n, m', n' \longrightarrow \infty$. Then, applying Lebesgue dominated convergence theorem to above inequality, we obtain

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} J_1 \geq$$

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\int_{\{|T_k(u_{m,n})-T_k(u)|<l\}} a(T_k(u_{m,n}), D(T_k(u)-\varphi)).D(T_k(u_{m,n}) - T_k(u) + \varphi)$$

$$\geq \int_\Omega a(T_k(u), D(T_k(u)-\varphi)).D\varphi.$$

Now, we treat the term J_3. As $a(T_k(u_{m,n}), DT_k(u_{m,n}))$ is bounded in $(L^{p'}(\Omega))^N$, Hölder inequality applied to J_3 gives

$$|J_3| \ \leq C\Big[\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})| \ \geq l\}} |D\varphi|^p\Big]^{\frac{1}{p}}.$$

As $T_k(u_{m,n}) \longrightarrow T_k(u)$ a.e. in Ω then, by Lebesgue dominated convergence theorem, we get

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} J_3 = 0.$$

Analogously, we also have

$$\lim_{l\longrightarrow 0}\lim_{m,n\longrightarrow\infty}\lim_{m',n'\longrightarrow\infty} J_4 = 0.$$

For the term J_2, it can be written as:

$$\begin{aligned} J_2 = & \int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} a(T_k(u_{m',n'}), DT_k(u_{m',n'})).D(T_k(u_{m',n'}) - T_k(u) + \varphi) \\ & + \int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} a(T_k(u_{m',n'}), DT_k(u_{m',n'})).D(T_k(u) - T_k(u_{m,n})) \\ := & \quad J_2^1 + J_2^2. \end{aligned}$$

Using hypothesis (H_1) and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} & \lim_{l \longrightarrow 0} \lim_{m,n \longrightarrow \infty} \lim_{m',n' \longrightarrow \infty} J_2^1 \geq \\ & \quad \lim_{l \longrightarrow 0} \lim_{m,n \longrightarrow \infty} \lim_{m',n' \longrightarrow \infty} \int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} a(T_k(u_{m',n'}), D(T_k(u) - \varphi)) \\ & \quad\quad .D(T_k(u_{m',n'}) - T_k(u) + \varphi) \geq \int_\Omega a(T_k(u), DT_k(u) - \varphi).D\varphi. \end{aligned}$$

On the other hand, since $a(T_k(u_{m',n'}), DT_k(u_{m',n'})) \rightharpoonup \chi_k$ weakly in $(L^{p'}(\Omega))^N$ and $DT_k(u_{m,n}) \rightharpoonup DT_k(u)$ weakly in $(L^p(\Omega))^N$ as $m, n, m', n' \longrightarrow \infty$, we deduce that

$$\lim_{l \longrightarrow 0} \lim_{m,n \longrightarrow \infty} \lim_{m',n' \longrightarrow \infty} J_2^2 = 0.$$

Combining together all limits in (4.20), we obtain

$$2\int_\Omega \chi_k.D\varphi \geq 2\int_\Omega a(T_k(u), DT_k(u) - \varphi).D\varphi, \tag{4.21}$$

for all $\varphi \in W^{1,p}(\Omega)$.
Now, taking $\varphi = \alpha\zeta$ in (4.21), where $\zeta \in D(\Omega)$ and $\alpha \in \mathbb{R}$. Dividing the inequality (4.21) by $\alpha > 0$, respectively $\alpha < 0$, we get

$$2\int_\Omega \chi_k.D\zeta \ \geq \ 2\int_\Omega a(T_k(u), DT_k(u) - \alpha\zeta).D\zeta$$

and

$$2\int_\Omega \chi_k.D\zeta \ \leq \ 2\int_\Omega a(T_k(u), DT_k(u) - \alpha\zeta).D\zeta.$$

Passing to the limit in the last two inequalities with $\alpha \downarrow 0$, respectively $\alpha \uparrow 0$, it follows that
$\int_\Omega \chi_k.D\zeta = \int_\Omega a(T_k(u), DT_k(u)).D\zeta$ for all $\zeta \in D(\Omega)$, i.e $div\ \chi_k = div\ a(T_k(u), DT_k(u))$.

Step 4. Passage to the limit in equation (4.13).

Taking $\varphi = S(u_{m,n} - \phi)$ as a test function in (4.13), where $S \in \mathcal{P} = \{p \in C^1(\mathbb{R}); p(0) = 0,\ 0 \leq p' \leq 1,\ \operatorname{supp}(p') \text{ is compact}\}$, $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and define $l = ||\phi||_\infty + \max\{|z|, z \in \operatorname{supp}(S')\}$.

Considering the first integral, we obtain

$$\int_\Omega a(u_{m,n}, Du_{m,n}).DS(u_{m,n} - \phi)$$

$$= \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})).DS(u_{m,n} - \phi)$$

$$= \int_\Omega (a(T_l(u_{m,n}), DT_l(u_{m,n})) - a(T_l(u_{m,n}), DT_l(u))).D(T_l(u_{m,n}) - T_l(u))S'(u_{m,n} - \phi)$$

$$+ \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})).DT_l(u)S'(u_{m,n} - \phi)$$

$$+ \int_\Omega a(T_l(u_{m,n}), DT_l(u)).D(T_l(u_{m,n}) - T_l(u))S'(u_{m,n} - \phi)$$

$$- \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})).D\phi S'(u_{m,n} - \phi).$$

Using hypothesis (H_1) and the fact that $0 \le S'(u_{m,n} - \phi) \le 1$, we deduce that

$$\int_\Omega (a(T_l(u_{m,n}), DT_l(u_{m,n})) - a(T_l(u_{m,n}), DT_l(u))).D(T_l(u_{m,n}) - T_l(u))S'(u_{m,n} - \phi) \ge 0.$$

Therefore, we have

$$\begin{aligned} &\int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})).DS(u_{m,n} - \phi) \ge \\ &\quad \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})).DT_l(u)S'(u_{m,n} - \phi) \\ &\quad + \int_\Omega a(T_l(u_{m,n}), DT_l(u)).D(T_l(u_{m,n}) - T_l(u))S'(u_{m,n} - \phi) \\ &\quad - \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})).D\phi S'(u_{m,n} - \phi). \end{aligned} \tag{4.22}$$

Since $S'(u_{m,n} - \phi) \longrightarrow S'(u - \phi)$ a.e. in Ω, $DT_l(u_{m,n}) \rightharpoonup DT_l(u)$ weakly in $(L^p(\Omega))^N$, $T_l(u_{m,n}) \longrightarrow T_l(u)$ a.e. in Ω and $a(T_l(u_{m,n}), DT_l(u_{m,n})) \rightharpoonup \chi_l$ weakly in $(L^{p'}(\Omega))^N$ as $m, n \longrightarrow \infty$, we obtain after passing to the limit in (4.22) the following:

$$\begin{aligned} \lim_{m,n\longrightarrow\infty} \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})).DS(u_{m,n} - \phi) &\ge \int_\Omega \chi_l.DT_l(u)S'(u - \phi) - \\ \int_\Omega \chi_l.D\phi S'(u - \phi) &= \int_\Omega \chi_l.DS(u - \phi). \end{aligned}$$

Consequently,

$$\lim_{m,n\longrightarrow\infty} \int_\Omega a(u_{m,n}, Du_{m,n}).DS(u_{m,n} - \phi) \ge \int_\Omega a(u, Du).DS(u - \phi). \tag{4.23}$$

By Lebesgue dominated convergence theorem, we get

$$\lim_{m,n\longrightarrow\infty} \int_\Omega (f_{m,n} - b_{m,n}(u_{m,n})).DS(u_{m,n} - \phi) = \int_\Omega (f - b(u)).DS(u - \phi). \tag{4.24}$$

Now, note that

$$\int_{\partial\Omega} \psi_{m,n}(u_{m,n}) S(u_{m,n}-\phi)$$

$$= \int_{\partial\Omega} \Big[\psi_{m,n}(u_{m,n}) - \psi_{m,n}(\phi)\Big] S(u_{m,n}-\phi) + \int_{\partial\Omega} \psi_{m,n}(\phi) S(u_{m,n}-\phi)$$

$$= \int_{\partial\Omega} \Big[\psi_{m,n}(u_{m,n}) - \psi_{m,n}(\phi)\Big] S(u_{m,n}-\phi) + \frac{1}{m}\int_{\partial\Omega} \psi(\phi^+) S(u_{m,n}-\phi)$$

$$-\frac{1}{n}\int_{\partial\Omega} \psi(\phi^-) S(u_{m,n}-\phi).$$

As the functions $\psi_{m,n}$ and S are nondecreasing, we get

$$\int_{\partial\Omega} \Big[\psi_{m,n}(u_{m,n}) - \psi_{m,n}(\phi)\Big] S(u_{m,n}-\phi) \geq 0.$$

On the other hand, as $\psi, u_{m,n}$ and S are bounded, then

$$\lim_{m,n\longrightarrow\infty} \frac{1}{m}\int_{\partial\Omega} \psi(\phi^+) S(u_{m,n}-\phi) = 0 \quad \text{and} \quad \lim_{m,n\longrightarrow\infty} \frac{1}{n}\int_{\partial\Omega} \psi(\phi^-) S(u_{m,n}-\phi) = 0.$$

Therefore,

$$\lim_{m,n\longrightarrow\infty} \int_{\partial\Omega} \psi_{m,n}(u_{m,n}) S(u_{m,n}-\phi) \ \geq \ 0. \tag{4.25}$$

To complete the proof, it remains to show that μ verifies $\mu_r \in \partial j(.,u) + \partial I_{[\gamma_-,\gamma_+]}(u)$ a.e. in $\partial\Omega$, $\tilde{u} = \gamma_+ \ \mu_s^+$ a.e. on $\partial\Omega$, $\tilde{u} = \gamma_- \ \mu_s^-$ a.e. on $\partial\Omega$ and

$$\lim_{m,n\longrightarrow\infty} \int_{\partial\Omega} S(\tilde{u}_{m,n} - \tilde{\phi}) d\mu_{m,n} = \int_{\partial\Omega} S(\tilde{u} - \tilde{\phi}) d\mu. \tag{4.26}$$

We know from the proof of Theorem 3.1 (part ii)) that $\mu_{m,n} \in \partial \mathcal{J}(u_{m,n})$, thus

$$(\mu_{m,n})_r \in \partial j(.,u_{m,n}) + \partial I_{[\gamma_-,\gamma_+]}(u_{m,n}) \text{ a.e. on } \partial\Omega.$$

As $u_{m,n} \longrightarrow u$ a.e. on $\partial\Omega$ and $||(\mu_{m,n})_r - \mu_r||_{L^1(\partial\Omega)} \ \leq \ ||\mu_{m,n} - \mu||_{\mathcal{M}_b(\partial\Omega)} \longrightarrow 0$ as $m,n \longrightarrow \infty$ then,

$$\mu_r \in \partial j(.,u) + \partial I_{[\gamma_-,\gamma_+]}(u) \text{ a.e. on } \partial\Omega.$$

On the other hand, we have $\tilde{u}_{m,n} = \gamma_+ \ (\mu_{m,n})_s^+$ a.e. on $\partial\Omega$, $\tilde{u}_{m,n} = \gamma_- \ (\mu_{m,n})_s^-$ a.e. on $\partial\Omega$, which are equivalent to say

$$\int_{\partial\Omega} (\gamma_+ - \tilde{u}_{m,n}) d(\mu_{m,n})_s^+ = 0 \quad \text{and} \quad \int_{\partial\Omega} (\gamma_- - \tilde{u}_{m,n}) d(\mu_{m,n})_s^- = 0.$$

As u is bounded on $\partial\Omega$ and $(\mu_{m,n})_s \longrightarrow \mu_s$ strongly in $\mathcal{M}_b(\partial\Omega)$ as $m,n \longrightarrow \infty$ then, after passing to the limit in the last both integrals according to the Lebesgue dominated convergence theorem, we obtain

$$\int_{\partial\Omega} (\gamma_+ - \tilde{u}) d\mu_s^+ = 0 \quad \text{and} \quad \int_{\partial\Omega} (\gamma_- - \tilde{u}) d\mu_s^- = 0;$$

which are equivalent to say $\tilde{u} = \gamma_\pm \ \mu_s^\pm$ a.e. on $\partial\Omega$.

As $u_{m,n} \longrightarrow u$ a.e. on $\partial\Omega$ and $\mu_{m,n} \longrightarrow \mu$ weakly in $\mathcal{M}_b(\partial\Omega)$ then, using Lebesgue dominated convergence theorem, we get (4.26).

Finally, putting together all the limits (4.23)-(4.26), we conclude that:

$$\int_{\Omega} a(u,Du).DS(u-\phi) + \int_{\partial\Omega} S(\tilde{u}-\tilde{\phi}) d\mu \ \leq \ \int_{\Omega} (f - b(u)) S(u-\phi),$$

for all $\phi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$.

Taking S as an approximation of T_k, we get the desired entropy inequality. Therefore, we have shown that, for all $f \in L^{\infty}(\Omega)$, $(I + A_{m,n})^{-1} f$ converges in $L^1(\Omega)$ to an entropy solution of the problem $(E_b)(f)$, hence $\liminf_{m,n\longrightarrow\infty} A_{m,n} \subset \mathcal{A}$. For the inverse inclusion, we refer to the step below.

Step 5. The accretivity of $\mathcal{A}$.

To prove the accretivity of $\mathcal{A}$, we must show that

$$\int_{\Omega} |b(w) - b(v)| \leq \int_{\Omega} |f - g|, \tag{4.27}$$

where $f \in b(w) + \mathcal{A}w$ and $g \in b(v) + \mathcal{A}(v)$.
Let $w_{m,n}$ and $v_{m,n}$ verifying $f \in b_{m,n}(w_{m,n}) + A_{m,n}w_{m,n}$ and $g \in b_{m,n}(v_{m,n}) + A_{m,n}v_{m,n}$. Observe that

$$b(w) = \lim_{m,n\longrightarrow\infty} b(w_{m,n}) \text{ and } b(v) = \lim_{m,n\longrightarrow\infty} b(v_{m,n}).$$

Indeed, taking $\phi_1 = w_{m,n}$ and $\phi_2 = w_{m,n} - T_h(w_{m,n} - w)$ as test functions in inequalities corresponding to solutions w and $w_{m,n}$ respectively, we obtain:

$$\int_{\Omega} a(w, Dw).DT_h(w - w_{m,n}) \leq \int_{\Omega} (f - b(w))T_h(w - w_{m,n}) - \int_{\partial\Omega} T_h(\tilde{w} - \tilde{w}_{m,n})\, d\mu$$

and

$$\int_{\Omega} a(w_{m,n}, Dw_{m,n}).DT_h(w_{m,n} - w) + \int_{\partial\Omega} \psi_{m,n}(w_{m,n})T_h(w_{m,n} - w)$$

$$\leq \int_{\Omega} \big(f_{m,n} - b_{m,n}(w_{m,n})\big)T_h(w_{m,n} - w) - \frac{1}{h}\int_{\partial\Omega} T_h(\tilde{w}_{m,n} - \tilde{w})\, d\mu_{m,n}.$$

Adding the two inequalities above and dividing their sum by $h > 0$, we get

$$\begin{aligned} &\frac{1}{h}\int_{\Omega} \big(a(w_{m,n}, Dw_{m,n}) - a(w, Dw)\big).DT_h(w_{m,n} - w) \leq \\ &-\frac{1}{h}\int_{\Omega} \Big(f - f_{m,n} + b_{m,n}(w_{m,n}) - b(w)\Big)T_h(w_{m,n} - w) - \frac{1}{h}\int_{\partial\Omega} \psi_{m,n}(w_{m,n})T_h(w_{m,n} - w) \\ &-\frac{1}{h}\int_{\partial\Omega} T_h(\tilde{w} - \tilde{w}_{m,n})\, d\mu - \frac{1}{h}\int_{\partial\Omega} T_h(\tilde{w} - \tilde{w}_{m,n})\, d\mu_{m,n}. \end{aligned} \tag{4.28}$$

Assumptions (H_1) and (H_4) imply that

$$\begin{aligned} &\frac{1}{h}\int_{\Omega} \big(a(w_{m,n}, Dw_{m,n}) - a(w, Dw)\big).DT_h(w_{m,n} - w) \\ &\quad \geq \frac{1}{h}\int_{\Omega} \big(a(w_{m,n}, Dw_{m,n}) - a(w, Dw_{m,n})\big).DT_h(w_{m,n} - w) \\ &\quad \geq -\frac{1}{h}\int_{\mathcal{F}} C\big(w_{m,n}, w\big)|w_{m,n} - w|\big(1 + |Dw_{m,n}|^{p-1}\big)|D(w_{m,n} - w)| \\ &\quad \longrightarrow 0 \text{ as } h \longrightarrow 0, \end{aligned}$$

where $\mathcal{F} := \{|w| \leq ||w_{m,n}||_{\infty} + 1\} \cap \{|w_{m,n} - w| < h\}$.

On the other hand, the last two integrals in the right hand side of inequality (4.28) are negative thanks to properties of functions γ_+ and γ_-.

As $\lim_{h\longrightarrow 0}\frac{1}{h}T_h(w_{m,n}-w) = sign_0(w_{m,n}-w)$ then, passing to the limit as h tend to zero in (4.28), we obtain

$$\int_\Omega \Big(f - f_{m,n} + b_{m,n}(w_{m,n}) - b(w)\Big) sign_0(w_{m,n}-w) \ \le\ \int_{\partial\Omega} (\psi_{m,n}(w_{m,n})) sign_0(w_{m,n}-w)$$

which imply

$$\begin{aligned}
&\int_\Omega (b_{m,n}(w_{m,n}) - b(w)) sign_0(w_{m,n}-w)\\
&\quad\le\ -\int_\Omega (f - f_{m,n}) sign_0(w_{m,n}-w) + \frac{1}{m}\int_{\partial\Omega} \psi(w^+_{m,n}) sign_0(w_{m,n}-w)\\
&\qquad\qquad -\frac{1}{n}\int_{\partial\Omega} \psi(w^-_{m,n}) sign_0(w_{m,n}-w)\\
&\quad\le\ \int_\Omega \big|f - f_{m,n}\big| + \frac{1}{m}\int_{\partial\Omega} \big|\psi(w^+_{m,n})\big| + \frac{1}{n}\int_{\partial\Omega} \big|\psi(w^-_{m,n})\big| \longrightarrow 0
\end{aligned} \tag{4.29}$$

as $m,n \longrightarrow +\infty$ (since b is bounded and $f_{m,n} \longrightarrow f$ as $m,n \longrightarrow +\infty$).

Note also that

$$\begin{aligned}
&\int_\Omega (b_{m,n}(w_{m,n}) - b(w)) sign_0(w_{m,n}-w)\\
&\quad=\ \int_\Omega (b(w_{m,n}) - b(w)) sign_0(w_{m,n}-w) + \frac{1}{m}\int_{\partial\Omega} (w^+_{m,n}) sign_0(w_{m,n}-w)\\
&\qquad\qquad -\frac{1}{n}\int_{\partial\Omega} (w^-_{m,n}) sign_0(w_{m,n}-w)\\
&\quad\ge\ \int_\Omega (b(w_{m,n}) - b(w)) sign_0(w_{m,n}-w) - \frac{1}{m}\int_{\partial\Omega} |w^+_{m,n}| - \frac{1}{n}\int_{\partial\Omega} |w^-_{m,n}|.
\end{aligned}$$

This imply that

$$\begin{aligned}
&\int_\Omega (b(w_{m,n}) - b(w)) sign_0(w_{m,n}-w)\\
&\quad\le\ \int_\Omega (b_{m,n}(w_{m,n}) - b(w)) sign_0(w_{m,n}-w) + \frac{1}{m}\int_{\partial\Omega} |w^+_{m,n}| + \frac{1}{n}\int_{\partial\Omega} |w^-_{m,n}|
\end{aligned}$$

$\longrightarrow 0$ as $m,n \longrightarrow +\infty$ (according to inequality (4.29)).

Therefore,

$$\lim_{m,n\longrightarrow +\infty} \int_\Omega \big|b(w_{m,n}) - b(w)\big| = 0$$

i.e.

$$\|b(w_{m,n}) - b(w)\|_1 \longrightarrow 0 \text{ as } m,n \longrightarrow +\infty.$$

By the same technics, we show that

$$\|b(v_{m,n}) - b(v)\|_1 \longrightarrow 0 \text{ as } m,n \longrightarrow +\infty.$$

As the operator $A_{m,n}$ is T-accretive, we can write

$$\int_\Omega \big|b(w_{m,n}) - b(v_{m,n})\big| \ \le\ \int_\Omega |f-g|.$$

Now

$$\begin{aligned}\int_\Omega |b(w)-b(v)| &\le \int_\Omega \big|b(w)-b(w_{m,n})\big| + \int_\Omega \big|b(w_{m,n})-b(v_{m,n})\big| + \int_\Omega \big|b(v_{m,n})-b(v)\big| \\ &\le \int_\Omega \big|b(w)-b(w_{m,n})\big| + \int_\Omega |f-g| + \int_\Omega \big|b(v)-b(v_{m,n})\big|. \end{aligned} \tag{4.30}$$

After passing to the limit in (4.30) with $m,n \longrightarrow +\infty$, we obtain

$$\int_\Omega \big|b(w)-b(v)\big| \le \int_\Omega |f-g|. \tag{4.31}$$

Step 6. $D(\mathcal{A})$ is dense in $L^1(\Omega)$.

For this, we show that $L^\infty(\Omega) \subset \overline{D(\mathcal{A})}^{||.||_1}$.

Let $u \in L^\infty(\Omega)$ and consider $u^\alpha_{m,n}$ and u_α, $\alpha > 0$ such that

$$b_{m,n}(u^\alpha_{m,n}) + \alpha A_{m,n} u^\alpha_{m,n} \ni b(u) \text{ and } b(u_\alpha) + \alpha \mathcal{A} u_\alpha \ni b(u). \tag{4.32}$$

We know from Theorem 3.1 that $D(A_{m,n})$ is dense in $L^1(\Omega)$; then, for all $m,n \in \mathbb{N}^*$, we deduce that

$$b(u^\alpha_{m,n}) \longrightarrow b(u) \text{ in } L^1(\Omega) \text{ as } \alpha \longrightarrow 0.$$

We show now that $b(u^\alpha_{m,n}) \longrightarrow b(u_\alpha)$ in $L^1(\Omega)$ as $m,n \longrightarrow \infty$.

To this end, taking $u^\alpha_{m,n} - T_l(u^\alpha_{m,n} - u_\alpha)$, respectively $u^\alpha_{m,n}$ as test functions in the entropy formulation of the problems defined by (4.32), we obtain

$$\begin{aligned}&\int_\Omega a(u^\alpha_{m,n}, Du^\alpha_{m,n}).DT_l(u^\alpha_{m,n}-u_\alpha) + \int_{\partial\Omega} \psi_{m,n}(u^\alpha_{m,n}) T_l(u^\alpha_{m,n}-u_\alpha) \\ &\quad \le \int_\Omega (b(u) - b_{m,n}(u^\alpha_{m,n})) T_l(u^\alpha_{m,n}-u_\alpha) - \int_{\partial\Omega} T_l(\tilde{u}^\alpha_{m,n} - \tilde{u}_\alpha) d\mu^\alpha_{m,n}.\end{aligned}$$

and

$$\int_\Omega a(u_\alpha, Du_\alpha).DT_l(-u^\alpha_{m,n}+u_\alpha) \le \frac{1}{\alpha}\int_\Omega (b(u_\alpha)-b(u)) T_l(-u^\alpha_{m,n}+u_\alpha) - \int_{\partial\Omega} T_l(\tilde{u}_\alpha - \tilde{u}^\alpha_{m,n}) d\mu_\alpha.$$

Adding the two inequalities above and dividing their sum by $l > 0$, we get

$$\begin{aligned}&\frac{1}{l}\int_\Omega [a(u^\alpha_{m,n}, Du^\alpha_{m,n}) - a(u_\alpha, Du_\alpha)].DT_l(u^\alpha_{m,n}-u_\alpha) + \frac{1}{l}\int_{\partial\Omega} \psi_{m,n}(u^\alpha_{m,n}) T_l(u^\alpha_{m,n}-u_\alpha) \\ &\quad \le -\int_\Omega (b_{m,n}(u^\alpha_{m,n}) - b(u_\alpha)) \frac{1}{l} T_l(u^\alpha_{m,n}-u_\alpha) \\ &\qquad - \frac{1}{l}\int_{\partial\Omega} T_l(\tilde{u}^\alpha_{m,n}-\tilde{u}_\alpha) d\mu^\alpha_{m,n} - \frac{1}{l}\int_{\partial\Omega} T_l(\tilde{u}_\alpha - \tilde{u}^\alpha_{m,n}) d\mu_\alpha.\end{aligned} \tag{4.33}$$

Using assumptions (H_1) and (H_4), we deduce that

$$\begin{aligned}
&\frac{1}{l}\int_\Omega [a(u^\alpha_{m,n}, Du^\alpha_{m,n}) - a(u_\alpha, Du_\alpha)].DT_l(u^\alpha_{m,n} - u_\alpha) \\
&\quad \geq \frac{1}{l}\int_\Omega [a(u^\alpha_{m,n}, Du^\alpha_{m,n}) - a(u_\alpha, Du^\alpha_{m,n})].DT_l(u^\alpha_{m,n} - u_\alpha) \\
&\quad \geq -\frac{1}{l}\int_{\mathcal{F}} C(u^\alpha_{m,n}, u_\alpha)|u^\alpha_{m,n} - u_\alpha|(1 + |Du^\alpha_{m,n}|^{p-1})|D(u^\alpha_{m,n} - u_\alpha)| \\
&\quad \longrightarrow 0 \text{ as } l \longrightarrow 0,
\end{aligned}$$

where $\mathcal{F} = \{|u_\alpha| \ \leq \ ||u^\alpha_{m,n}||_\infty + l\} \cap \{|u^\alpha_{m,n} - u_\alpha| \ < \ l\}$.

Noticing that the last two integrals in the right hand side of inequality (4.33) are nonnegative. Indeed these integrals can be written as

$$\int_{\partial\Omega} T_l(\tilde{u}^\alpha_{m,n} - \tilde{u}_\alpha)((u^\alpha_{m,n})_r - (u_\alpha)_r) + \int_{\partial\Omega} T_l(\gamma_+ - \tilde{u}_\alpha)d(\mu^\alpha_{m,n})^+_s$$

$$-\int_{\partial\Omega} T_l(\gamma_- - \tilde{u}_\alpha)d(\mu^\alpha_{m,n})^-_s - \int_{\partial\Omega} T_l(-\gamma_+ + \tilde{u}^\alpha_{m,n})d(\mu_\alpha)^+_s + \int_{\partial\Omega} T_l(-\gamma_- + \tilde{u}^\alpha_{m,n})d(\mu_\alpha)^-_s,$$

which are clearly nonnegative by properties of the measures and $\gamma_\pm$.

As $\lim\limits_{l\longrightarrow 0} \frac{1}{l} T_l(u^\alpha_{m,n} - u_\alpha) = sign_0(u^\alpha_{m,n} - u_\alpha)$, we get after passing to the limit in (4.33) as $l \longrightarrow 0$

$$\begin{aligned}
&\int_\Omega (b_{m,n}(u^\alpha_{m,n}) - b(u_\alpha))sign_0(u^\alpha_{m,n} - u_\alpha) \\
&\quad \leq -\frac{1}{m}\int_{\partial\Omega} \psi(u^{\alpha,+}_{m,n})sign_0(u^\alpha_{m,n} - u_\alpha) + \frac{1}{n}\int_{\partial\Omega} \psi(u^{\alpha,-}_{m,n})sign_0(u^\alpha_{m,n} - u_\alpha) \\
&\quad \leq \frac{1}{m}\int_{\partial\Omega} |\psi(u^{\alpha,+}_{m,n})| + \frac{1}{n}\int_{\partial\Omega} |\psi(u^{\alpha,-}_{m,n})| \\
&\quad \longrightarrow 0 \text{ as } m, n \longrightarrow +\infty.
\end{aligned} \tag{4.34}$$

Note also that

$$\begin{aligned}
&\int_\Omega (b_{m,n}(u^\alpha_{m,n}) - b(u_\alpha))sign_0(u^\alpha_{m,n} - u_\alpha) \\
&\quad = \int_\Omega (b(u^\alpha_{m,n}) - b(u_\alpha))sign_0(u^\alpha_{m,n} - u_\alpha) + \frac{1}{m}\int_{\partial\Omega} (u^{\alpha,+}_{m,n})sign_0(u^\alpha_{m,n} - u_\alpha) \\
&\qquad - \frac{1}{n}\int_{\partial\Omega} (w^{\alpha,-}_{m,n})sign_0(u^\alpha_{m,n} - u_\alpha) \\
&\quad \geq \int_\Omega (b(u^\alpha_{m,n}) - b(u_\alpha))sign_0(u^\alpha_{m,n} - u_\alpha) - \frac{1}{m}\int_{\partial\Omega} |u^{\alpha,+}_{m,n}| - \frac{1}{n}\int_{\partial\Omega} |u^{\alpha,-}_{m,n}|,
\end{aligned}$$

which imply that

$$\begin{aligned}
&\int_\Omega (b(u^\alpha_{m,n}) - b(u_\alpha))sign_0(u^\alpha_{m,n} - u_\alpha) \\
&\quad \leq \int_\Omega (b_{m,n}(u^\alpha_{m,n}) - b(u_\alpha))sign_0(u^\alpha_{m,n} - u_\alpha) + \frac{1}{m}\int_{\partial\Omega} |u^{\alpha,+}_{m,n}| + \frac{1}{n}\int_{\partial\Omega} |w^{\alpha,-}_{m,n}| \\
&\quad \longrightarrow 0 \text{ as } m, n \longrightarrow +\infty \text{ (according to inequality (4.34))}.
\end{aligned}$$

Then, it follows that

$$\lim_{m,n\longrightarrow+\infty}\int_{\Omega}\left|b(u^{\alpha}_{m,n})-b(u_{\alpha})\right|=0$$

i.e.

$$\left|\left|b(u^{\alpha}_{m,n})-b(u_{\alpha})\right|\right|_1\longrightarrow 0 \text{ as } m,n\longrightarrow+\infty.$$

As $\left|\left|b(u_{\alpha})-b(u)\right|\right|_1 \le \left|\left|b(u^{\alpha}_{m,n})-b(u_{\alpha})\right|\right|_1+\left|\left|b(u^{\alpha}_{m,n})-b(u)\right|\right|_1\longrightarrow 0$ then

$$\left|\left|b(u_{\alpha})-b(u)\right|\right|_1\longrightarrow 0 \text{ as } m,n\longrightarrow+\infty.$$

We deduce that $b(u)\in\overline{D(\mathcal{A})}^{||\cdot||_1}$.

The proof of Theorem 4.1 is now complete. □

Corollary 4.1. *Under the assumptions of Theorem 4.1, $(E_b)(f)$ admits a unique entropy solution.*

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