Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 39(2), 2012, Pages 226–236
ISSN: 1223-6934

# Statistical Convergence of Triple Sequences on Probabilistic Normed Space

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Abstract. The concept of statistical convergence was presented by Fast [7]. This concept was extended to the double sequences by Mursaleen and Edely [21]. In this paper, we define statistical analogues of convergence and Cauchy for triple sequences on probabilistic normed space.

*2010 Mathematics Subject Classification.* 46S50; 42B15; 40C05.
*Key words and phrases.* Double sequence, double statistical convergence, t-norm, probabilistic normed space.

## 1. Introduction

A probabilistic normed space (PN space) is a natural generalization of an ordinary normed linear space. In PN space, the norms of vectors are represented by probability distribution functions rather than a positive number. Such spaces were first introduced by Serstnev in 1963, (see, [27]). In [3] Alsina et al. gave a new definition of PN-spaces which includes Serstnev's a special case and leads naturally to the identification of the principle class of PN-spaces, the Menger spaces. This definition becomes the standard one and has been adopted by many authors (for instance, [4], [15], [16], [17]) who have investigated the properties of PN spaces. The detailed history and the development of the subject up to 2006 can be found in [19].

On the other hand, statistical convergence was first introduced by Fast [16] as a generalization of ordinary convergence for real number sequences. Since then it has been studied by many authors (for instance, [24], [8], [5], [9]). Statistical convergence has also been discussed in more general abstract spaces such as the fuzzy number space [1], locally convex spaces [18] and Banach spaces [14]. Karakus [12] introduced and studied statistical convergence on PN spaces and followed by Karakus and Demirci [13] studied statistical convergence of double sequences on PN spaces. Recently Esi and Özdemir [11] introduced generalized $\Delta^m-$statistical convergence in probabilistic normed space for single generalized difference sequences and Esi [10] has introduced lacunary statistical convergence of double sequences in probabilistic normed space.

It seems therefore reasonable to think if the concept of statistical convergence can be extended to probabilistic normed spaces and in that case enquire how the basic properties are affected. But basic properties do not hold on probabilistic normed spaces. The problem is that the triangle function in such spaces.

In this paper we extend the concept of statistical convergence of triple sequences to probabilistic normed spaces and observe that some basic properties are also preserved

Received October 31, 2011.
This paper is in final form and no version of it will be submitted for publication elsewhere.

on probabilistic normed spaces. Since the study of convergence in PN-spaces is fundamental to probabilistic functional analysis, we feel that the concepts of statistical convergence and statistical Cauchy for triple sequences in a PN-space would provide a more general framework for the subject.

## 2. Preliminaries

Now we recall some notations and definitions used in paper.

**Definition 2.1.** [30]. *A function $x : N \times N \times N \to R(C)$ is called a real (complex) triple sequence.*

**Definition 2.2.** [3]. *A function $f : \mathbb{R} \to \mathbb{R}_o^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$. We will denote the set of all distribution functions by $D$.*

**Definition 2.3.** [3]. *A triangular norm, briefly t-norm, is a binary operation on $[0,1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, that is, it is the continuous mapping $\circledast : [0,1] \times [0,1] \to [0,1]$ such that for all $a, b, c \in [0,1]$ :*

*(1)* $a \circledast 1 = a$,
*(2)* $a \circledast b = b \circledast a$,
*(3)* $c \circledast d \geq a \circledast b$ *if* $c \geq a$ *and* $d \geq b$,
*(4)* $(a \circledast b) \circledast c = a \circledast (b \circledast c)$.

**Example 2.1.** *The $\circledast$ operations $a \circledast b = \max\{a + b - 1, 0\}$, $a \circledast b = a.b$ and $a \circledast b = \min\{a, b\}$ on $[0,1]$ are t-norms.*

**Definition 2.4.** [26, 25]. *A triple $(X, N, \circledast)$ is called a probabilistic normed space or shortly PN-space if $X$ is a real vector space, $N$ is a mapping from $X$ into $D$ (for $x \in X$, the distribution function $N(x)$ is denoted by $N_x$ and $N_x(t)$ is the value of $N_x$ at $t \in \mathbb{R}$) and $\circledast$ is a t-norm satisfying the following conditions:*

*(PN-1)* $N_x(0) = 0$,
*(PN-2)* $N_x(t) = 1$ *for all* $t > 0$ *if and only if* $x = 0$,
*(PN-3)* $N_{\alpha x}(t) = N_x\left(\frac{t}{|\alpha|}\right)$ *for all* $\alpha \in R\backslash\{0\}$,
*(PN-4)* $N_{x+y}(s+t) \geq N_x(s) \circledast N_y(t)$ *for all* $x, y \in X$ *and* $s, t \in \mathbb{R}_o^+$.

**Example 2.2.** *Suppose that $(X, \|.\|)$ is a normed space $\mu \in D$ with $\mu(0) = 0$ and $\mu \neq h$, where*

$$h(t) = \begin{cases} 0 & , \quad t \leq 0 \\ 1 & , \quad t > 0 \end{cases}.$$

*Define*

$$N_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ \mu\left(\frac{t}{\|x\|}\right) & , \quad x \neq 0 \end{cases},$$

*where $x \in X$, $t \in \mathbb{R}$. Then $(X, N, \circledast)$ is a PN-space. For example if we define the functions $\mu$ and $\nu$ on $\mathbb{R}$ by*

$$\mu(x) = \begin{cases} 0 & , \quad x \leq 0 \\ \frac{x}{1+x} & , \quad x > 0 \end{cases}, \quad \nu(x) = \begin{cases} 0 & , \quad x \leq 0 \\ e^{\frac{-1}{x}} & , \quad x > 0 \end{cases}$$

*then we obtain the following well-known $\circledast$ norms:*

$$N_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ \frac{t}{t+\|x\|} & , \quad x \neq 0 \end{cases}, \quad M_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ e^{\left(\frac{-\|x\|}{t}\right)} & , \quad x \neq 0 \end{cases}.$$

We recall the concepts of convergence and Cauchy sequences for single sequences in a probabilistic normed space.

**Definition 2.5.** [2]. *Let $(X, N, \ast)$ is a PN-space. Then a sequence $x = (x_k)$ is said to be convergent to $l \in X$ with respect to the probabilistic norm $N$ if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer $k_o$ such that $N_{x_k - l}(\varepsilon) > 1 - \lambda$ whenever $k \geq k_o$. It is denoted by $N - \lim x = L$ or $x_k \xrightarrow{N} L$ as $k \to \infty$.*

**Definition 2.6.** [2]. *Let $(X, N, \ast)$ is a PN-space. Then a sequence $x = (x_k)$ is called a Cauchy sequence with respect to the probabilistic norm $N$ if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer $k_o$ such that $N_{x_k - x_l}(\varepsilon) > 1 - \lambda$ for all $k, l \geq k_o$.*

**Definition 2.7.** [2]. *Let $(X, N, \ast)$ is a PN-space. Then a sequence $x = (x_k)$ is said to be bounded in $X$, if there is a $r \in \mathbb{R}$ such that $N_{x_k}(r) > 1 - \lambda$, where $\lambda \in (0,1)$. We denote by $l_\infty^N$ the space of all bounded sequences in PN space.*

**Remark 2.1.** [1] *Let $(X, \|.\|)$ be a real normed space and $N_x(t) = \frac{t}{t + |x|}$, where $x \in X$ and $t \geq 0$ (standard $\ast$ norm induced by $|.|$). Then it is not hard to see that $x_n \xrightarrow{\|.\|} x$ iff $x_n \xrightarrow{N} x$.*

The idea of statistical convergence for single sequences was introduced by Fast [7] and then studied by various authors, e.g., Salat [24], Fridy [8], Connor [5], Esi [9], Mohiuddine and Savas [20], Savas and Mohiuddine [29], Savas [28], Patterson and Savas [22] and many others and in normed space by Kolk [14]. Recently Karakus [12] and Alotaibi [2] have studied the concept of statistical convergence in probabilistic normed spaces.

Firstly, we recall some definitions.

In 1900 Pringsheim presented the following definition for the convergence of double sequences.

**Definition 2.8.** [23]. *A double sequence $x = (x_{jk})$ has Pringsheim limit $L$ (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. We shall describe such an $x = (x_{jk})$ more briefly as "P-convergent".*

We shall denote the space of all P-convergent sequences by $c^2$. By a bounded double sequence we shall mean there exists a positive number $K$ such that $|x_{jk}| < K$ for all $(j,k)$ and denote such bounded by $\|x\|_{(\infty,2)} = \sup_{j,k} |x_{jk}| < \infty$.We shall also denote the set of all bounded double sequences by $l_\infty^2$.We also note in contrast to the case for single sequence, a P-convergent double sequence need not be bounded.

Definitions 2.4 and 2.5 for double sequences on probabilistic normed space are as follows:

**Definition 2.9.** [12]. *Let $(X, N, \ast)$ is a PN-space. Then a double sequence $x = (x_{jk})$ is said to be convergent to $L \in X$ with respect to the probabilistic norm $N$, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer $k_o$ such that $N_{x_{jk} - L}(\varepsilon) > 1 - \lambda$ whenever $j, k \geq k_o$. It is denoted by $N_2 - \lim x = L$ or $x_{jk} \xrightarrow{N} L$ as $j,k \to \infty$.*

**Definition 2.10.** [12]. *Let $(X, N, \ast)$ is a PN-space. Then a double sequence $x = (x_{jk})$ is said to be Cauchy sequence with respect to the probabilistic norm $N$ if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$,there exist $M = M(\epsilon)$ and $T = T(\varepsilon)$ such that $N_{x_{jk} - x_{pq}}(\varepsilon) > 1 - \lambda$ for all $j, p \geq M$ and $k, q \geq T$.*

Let $K \subset \mathbb{N} \times \mathbb{N}$ be two-dimensional set of positive integers and let $K(n,m)$ be the numbers of $(i,j)$ in $K$ such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of a set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = P - \liminf_{n,m} \frac{K(n,m)}{nm}.$$

In this case $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense then we say that $K$ has a double natural density and is defined as

$$\delta_2(K) = P - \lim_{n,m} \frac{K(n,m)}{nm}.$$

For example, let $K = \left\{\left(i^2, j^2\right) : \ i,j \in \mathbb{N}\right\}$. Then

$$\delta_2(K) = P - \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

i.e., the set $K$ has double natural density zero, while the set $L = \{(i,2j) : \ i,j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$.

**Definition 2.11.** [21]. *A real double sequence $x = (x_{jk})$ is said to be statistically convergent to a number $L$ provided that, for each $\varepsilon > 0$, the set*

$$\{(j,k) : \ |x_{jk} - L| \geq \varepsilon\}$$

*has double natural density zero. In this case, one writes $st_2 - \lim x = L$.*

**Definition 2.12.** [21]. *A real double sequence $x = (x_{jk})$ is said to be statistically Cauchy provided that, for every $\varepsilon > 0$ there exist $M = M(\epsilon)$ and $T = T(\varepsilon)$ such that for all $j,p \geq M$, $k,q \geq T$, the set*

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \ |x_{jk} - x_{pq}| \geq \varepsilon\}$$

*has double natural density zero.*

**Definition 2.13.** [30]. *A subset $K$ of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to be natural density $\delta_3$ if*

$$\delta_3(K) = \lim_{p,q,r \to \infty} \frac{K(p,q,r)}{pqr} \text{ exists}$$

*where $K(p,q,r)$ denote the number of $(j,k,l)$ in $K$ such that $j \leq p, k \leq q$ and $l \leq r$. For example, let $K = \left\{\left(j^3, k^3, l^3\right) : \ j,k,l \in \mathbb{N}\right\}$, then $\delta_3(K) = \lim_{p,q,r \to \infty} \frac{K(p,q,r)}{pqr} \leq \lim_{p,q,r \to \infty} \frac{\sqrt[3]{p}\sqrt[3]{q}\sqrt[3]{r}}{pqr} = 0$, i.e., the set $K$ has triple natural density zero, while the set $L = \{(j,3k,5l) : j,k,l \in \mathbb{N}\}$ has triple natural density $\frac{1}{15}$.*

**Definition 2.14.** [30]. *A real triple sequence $x = (x_{jkl})$ is said to be statistically convergent to the number $L$ if for each $\varepsilon > 0$*

$$\delta_3(\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \ |x_{jkl} - L| \geq \varepsilon\}) = 0.$$

*In this case, one writes $st_3 - \lim x = L$.*

## 3. Main Results

Now we give the analogues of these definitions with respect to the probabilistic norm $N$.

**Definition 3.1.** *Let $(X, N, \circledast)$ be a PN-space. Then, a triple sequence $x = (x_{jkl})$ is said to be convergent to $L \in X$ with respect to the probabilistic norm $N$ provided that, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $k_o$ such that $N_{x_{jkl}-L}(\varepsilon) > 1 - \lambda$ whenever $j, k, l \geq k_o$. It is denoted by $N_3 - \lim x = L$ or or $x_{jkl} \xrightarrow{N} L$ as $j,k,l \to \infty$.*

**Definition 3.2.** *Let $(X, N, \circledast)$ be a PN-space. Then, a triple sequence $x = (x_{jkl})$ is said to be a Cauchy sequence with respect to the probabilistic norm $N$ provided that, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exist $M(\varepsilon), T(\varepsilon), P(\varepsilon) \in \mathbb{N}$ such that $N_{x_{jkl}-x_{pqu}}(\varepsilon) > 1 - \lambda$ for all $j, p \geq M, k, q \geq T$ and $l, u \geq P$.*

**Definition 3.3.** *Let $(X, N, \circledast)$ be a PN-space. A triple sequence $x = (x_{jkl})$ is statistically convergent to $L \in X$ with respect to the probabilistic norm $N$ provided that, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$*

$$K = \left\{(j, k, l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ N_{x_{jkl}-L}(\varepsilon) \leq 1 - \lambda\right\}$$

*has triple natural density zero, that is if $K(n, m, s)$ become the numbers of $(j, k, l)$ in $K$*

$$\lim_{n,m,s} \frac{K(n, m, s)}{nms} = 0.$$

In this case, one writes $st_{N_3} - \lim_{j,k,l} x_{j,k,l} = L$, where $L$ is said to be $st_{N_3}-$limit. Also one denotes the set of all statistically convergent triple sequences with respect to the probabilistic norm $N$ by $st_{N_3}$.

Now we give a useful lemma as follows.

**Lemma 3.1.** *Let $(X, N, \circledast)$ be a PN-space. Then, for $\varepsilon > 0$ and $\lambda \in (0, 1)$ the following statements are equivalent:*

***(i)*** $st_{N_3} - \lim_{j,k,l} x_{j,k,l} = L$,
***(ii)*** $\delta_3\left(\left\{(j, k, l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ N_{x_{jkl}-L}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0$,
***(iii)*** $\delta_3\left(\left\{(j, k, l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ N_{x_{jkl}-L}(\varepsilon) > 1 - \lambda\right\}\right) = 1$,
***(iv)*** $st_3 - \lim N_{x_{jkl}-L}(\varepsilon) = 1$.

*Proof.* The first three parts are equivalent is trivial from Definition 3.3. It follows from Definition 2.14 that

$$\begin{aligned} &\left\{(j, k, l) : \ j \leq n, k \leq m \text{ and } l \leq s, \left| \ N_{x_{jkl}-L}(\varepsilon) - 1\right| \geq \lambda\right\} \\ &\quad = \left\{(j, k, l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ N_{x_{jkl}-L}(\varepsilon) \geq 1 + \lambda\right\} \\ &\qquad \cup \left\{(j, k, l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ N_{x_{jkl}-L}(\varepsilon) \leq 1 - \lambda\right\}. \end{aligned}$$

□

Also, Definition 2.14 implies that (ii) and (iv) are equivalent.

**Theorem 3.1.** *Let $(X, N, \circledast)$ be a PN-space. If a triple sequence $x = (x_{jkl})$ is statistically convergent with respect to the probabilistic norm $N$, then $st_{N_3} - \lim x$ is unique.*

*Proof.* Let $x = (x_{jkl})$ be a triple sequence. Suppose that $st_{N_3} - \lim x = L_1$ and $st_{N_3} - \lim x = L_2$. Let $\varepsilon > 0$ and $\lambda \in (0,1)$. Choose $\gamma \in (0,1)$ such that $(1-\gamma) \circledast (1-\gamma) \geq 1-\lambda$. Then, we define the following sets:

$$K_1(\gamma,\varepsilon) = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \ N_{x_{jkl}-L_1}(\varepsilon) \leq 1-\gamma\right\},$$
$$K_2(\gamma,\varepsilon) = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \ N_{x_{jkl}-L_2}(\varepsilon) \leq 1-\gamma\right\}.$$

Since $st_{N_3} - \lim x = L_1$, we have $\delta_3(\{K_1(\gamma,\varepsilon)\}) = 0$ for all $\varepsilon > 0$. Furthermore, using $st_{N_3} - \lim x = L_2$, we get $\delta_3(\{K_2(\gamma,\varepsilon)\}) = 0$ for all $\varepsilon > 0$. Now let $K(\gamma,\varepsilon) = K_1(\gamma,\varepsilon) \cap K_2(\gamma,\varepsilon)$. Then observe that $\delta_3(\{K(\gamma,\varepsilon)\}) = 0$ for all $\varepsilon > 0$ which implies $\delta_3(\{\mathbb{N}\times\mathbb{N}\times\mathbb{N}/K(\gamma,\varepsilon)\}) = 1$. If $(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}/K(\gamma,\varepsilon)$, then we have

$$\begin{aligned} N_{L_1-L_2}(\varepsilon) &\geq N_{x_{jkl}-L_1}\left(\frac{\varepsilon}{2}\right) \circledast N_{x_{jkl}-L_2}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\gamma) \circledast (1-\gamma) \geq 1-\lambda. \end{aligned}$$

Since $\lambda > 0$ was arbitrary, we get $N_{L_1-L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which yields $L_1 = L_2$. Therefore, we conclude that $st_{N_3} - \lim x$ is unique. □

**Theorem 3.2.** *Let $(X,N,\circledast)$ be a PN-space. If $N_3 - \lim x = L$ for a triple sequence $x = (x_{jkl})$, then $st_{N_3} - \lim x = L$.*

*Proof.* By hypothesis, for every $\lambda > 0$ and $\varepsilon > 0$, there is a number $k_o \in \mathbb{N}$ such that $N_{x_{jkl}-L}(\varepsilon) > 1-\lambda$ for all $j \geq k_o, k \geq k_o$ and $l \geq k_o$. This guarantees that the set $\left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \ N_{x_{jkl}-L}(\varepsilon) \leq 1-\lambda\right\}$ has at most finitely many terms. Since every finite subset of the $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ has triple density zero, we immediately see that $\delta_3\left(\left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \ N_{x_{jkl}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0$, whence the result. □

The following example shows that the converse of this theorem does not hold in general.

**Example 3.1.** *Let $(\mathbb{R}, |.|)$ be a normed space and $N_x(t) = \frac{t}{t+|x|}$, where $x \in X$ and $t \geq 0$ (standard $\circledast$ norm induced by $|.|$). In this case, observe that $(X,N,\circledast)$ be a PN-space. Now we define a sequence $x = (x_{jkl})$ whose terms are given by*

$$x_{jkl} = \begin{cases} jkl & , \ j,k \text{ and } l \text{ are cubes} \\ 0 & , \ \text{otherwise} \end{cases}. \tag{1}$$

*Then, for every $\lambda \in (0,1)$ and for any $\varepsilon > 0$, let*

$$K_{(\lambda,\varepsilon)}(n,m,s) = \left\{(j,k,l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ N_{x_{jkl}}(\varepsilon) \leq 1-\lambda\right\}.$$

*Since*

$$\begin{aligned} K_{(\lambda,\varepsilon)}(n,m,s) &= \left\{(j,k,l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ \frac{t}{t+|x_{jkl}|} \leq 1-\lambda\right\} \\ &= \left\{(j,k,l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ |x_{jkl}| \geq \frac{\lambda t}{1-\lambda} > 0\right\} \\ &= \{(j,k,l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ |x_{jkl}| = jkl\} \\ &= \{(j,k,l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ j,k \text{ and } l \text{ are cubes}\}, \end{aligned}$$

*then we get*

$$\lim_{n,m,s} \frac{1}{nms} \left| K_{(\lambda,\varepsilon)}(n,m,s) \right| \leq \lim_{n,m,s} \frac{1}{nms} \left| \{(j,k,l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ j,k \text{ and } l \text{ are cubes}\} \right|$$

$$\leq \lim_{n,m,s} \frac{\sqrt[3]{n}\sqrt[3]{m}\sqrt[3]{s}}{nms} = 0$$

*which implies*

$$\delta_3 \left( \left\{ K_{(\lambda,\varepsilon)}(n,m,s) \right\} \right) = 0.$$

*Hence, by Definition 3.3, we get* $st_{N_3} - \lim x = 0$. *However, since the sequence* $x = (x_{jkl})$ *given by (1) is not convergent in the space* $(\mathbb{R}, |.|)$, *by Remark 2.1, we also see that* $x = (x_{jkl})$ *is not convergent with respect to the probabilistic norm* $N$.

**Theorem 3.3.** *Let* $(X, N, \circledast)$ *be a PN-space and* $x = (x_{jkl})$ *be a triple sequence. Then* $st_{N_3} - \lim x = L$ *iff there exists a subset* $K = \{(j,k,l) : \ j,k,l = 1,2,3,\ldots\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ *such that* $\delta_3(\{K\}) = 1$ *and* $N_3 - \lim_{\substack{j,k,l\to\infty \\ (j,k,l)\in K}} x_{jkl} = L$.

*Proof.* We first assume that $st_{N_3} - \lim x = L$. Now, for any $\varepsilon > 0$ and $r \in \mathbb{N}$, let

$$K(r,\varepsilon) = \left\{ (j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \ N_{x_{jkl}-L}(\varepsilon) \leq 1 - \frac{1}{r} \right\},$$

$$M(r,\varepsilon) = \left\{ (j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \ N_{x_{jkl}-L}(\varepsilon) > 1 - \frac{1}{r} \right\}.$$

Then $\delta_3(\{K(r,\varepsilon)\}) = 0$ and

$i)$ $M(1,\varepsilon) \supset M(2,\varepsilon) \supset \cdots \supset M(i,\varepsilon) \supset M(i+1,\varepsilon) \supset \ldots$

$ii)$ $\delta_3(\{M(r,\varepsilon)\}) = 1, \ r = 1,2,3,\ldots.$

Now we have to show that for $(j,k,l) \in M(r,\varepsilon)$, $x = (x_{jkl})$ is $N_3 -$ *convergent* to $L$. Suppose that $x = (x_{jkl})$ is not $N_3 -$ *convergent* to $L$. Therefore there is $\lambda > 0$ such that

$$\left\{ (j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \ N_{x_{jkl}-L}(\varepsilon) \leq 1 - \lambda \right\}$$

for finitely many terms. Let

$$M(\lambda,\varepsilon) = \left\{ (j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \ N_{x_{jkl}-L}(\varepsilon) \geq 1 - \lambda \right\},$$
$$\lambda > \frac{1}{r} \ (r = 1,2,3,\ldots).$$

Then

$iii)$ $\delta_3(\{M(\lambda,\varepsilon)\}) = 0$ and by (i), $M(r,\varepsilon) \subset M(\lambda,\varepsilon)$.

Hence $\delta_3(\{M(r,\varepsilon)\}) = 0$ which contradicts (ii). Therefore $x = (x_{jkl})$ is $N_3 -$ *convergent* to $L$.

Conversely, suppose that there exists a subset $K = \{(j,k,l) : \ j,k,l = 1,2,3,\ldots\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $\delta_3(\{K\}) = 1$ and $N_3 - \lim_{\substack{j,k,l\to\infty \\ (j,k,l)\in K}} x_{jkl} = L$ that is there exists $k_o \in \mathbb{N}$ such that for every $\lambda \in (0,1)$ and for any $\varepsilon > 0$

$$N_{x_{jkl}-L}(\varepsilon) > 1 - \lambda, \forall j,k,l \geq k_o.$$

Now

$$M(\lambda,\varepsilon) = \left\{ (j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} \ : \ N_{x_{jkl}-L}(\varepsilon) \leq 1 - \lambda \right\}$$
$$\subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} / \left\{ (j_{k_o+1}, k_{k_o+1}, l_{k_o+1}), (j_{k_o+2}, k_{k_o+2}, l_{k_o+2}), \ldots \right\}.$$

Therefore, $\delta_3(\{M(\lambda,\varepsilon)\}) \le 1-1=0$. Hence, we conclude that $st_{N_3}-\lim x = L$. □

**Lemma 3.2.** *Let $(X,N,\circledast)$ be a PN-space.*

***(i)*** *If $st_{N_3}-\lim x = L_1$ and $st_{N_3}-\lim y = L_2$, then $st_{N_3}-\lim(x+y) = L_1+L_2$.*

***(ii)*** *If $st_{N_3}-\lim x = L$ and $\alpha \in \mathbb{R}$, then $st_{N_3}-\lim \alpha x = \alpha L$.*

***(iii)*** *If $st_{N_3}-\lim x = L_1$ and $st_{N_3}-\lim y = L_2$, then $st_{N_3}-\lim(x-y) = L_1-L_2$.*

*Proof.* **(i)**. Let $st_{N_3}-\lim x = L_1$ and $st_{N_3}-\lim y = L_2$, $\varepsilon>0$ and $\lambda\in(0,1)$. Choose $\gamma\in(0,1)$ such that $(1-\gamma)\circledast(1-\gamma)\ge 1-\lambda$. Then, we define the following sets:

$$K_1(\gamma,\varepsilon) = \left\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ N_{x_{jkl}-L_1}(\varepsilon)\le 1-\gamma\right\},$$
$$K_2(\gamma,\varepsilon) = \left\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ N_{x_{jkl}-L_2}(\varepsilon)\le 1-\gamma\right\}.$$

Since $st_{N_3}-\lim x = L_1$, we have $\delta_3(\{K_1(\gamma,\varepsilon)\})=0$ for all $\varepsilon>0$ and since $st_{N_3}-\lim x = L_2$, we get $\delta_3(\{K_2(\gamma,\varepsilon)\})=0$ for all $\varepsilon>0$. Now let $K(\gamma,\varepsilon) = K_1(\gamma,\varepsilon)\cap K_2(\gamma,\varepsilon)$. Then observe that $\delta_3(\{K(\gamma,\varepsilon)\})=0$ for all $\varepsilon>0$ which implies $\delta_3(\{\mathbb{N}\times\mathbb{N}\times\mathbb{N}/K(\gamma,\varepsilon)\})=1$. If $(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}/K(\gamma,\varepsilon)$, then we have

$$\begin{aligned}N_{(x_{jkl}-L_1)+(y_{jkl}-L_2)}(\varepsilon) &\ge N_{x_{jkl}-L_1}\left(\frac{\varepsilon}{2}\right)\circledast N_{y_{jkl}-L_2}\left(\frac{\varepsilon}{2}\right)\\ &> (1-\gamma)\circledast(1-\gamma)\ge 1-\lambda.\end{aligned}$$

This shows that

$$\delta_3\left(\left\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ N_{(x_{jkl}-L_1)+(y_{jkl}-L_2)}(\varepsilon)\le 1-\lambda\right\}\right)=0$$

so, $st_{N_3}-\lim(x+y) = L_1+L_2$.

**(ii)** Let $st_{N_3}-\lim x = L$, $\varepsilon>0$ and $\lambda\in(0,1)$. First of all, we consider the case of $\alpha=0$. In this case

$$N_{0x_{jkl}-0L}(\varepsilon) = N_0(\varepsilon) = 1 > 1-\lambda.$$

So we obtain $N_3-\lim 0x = 0$. Then from Theorem 3.3, we have $st_{N_3}-\lim 0x = 0$. Now we consider the case $\alpha\ne 0$. Since $st_{N_3}-\lim x = L$, if we define the set

$$K(\lambda,\varepsilon) = \left\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ N_{x_{jkl}-L}(\varepsilon)\le 1-\lambda\right\}$$

then, we can say that

$$\delta_3(\{K(\lambda,\varepsilon)\}) = 0 \text{ for all } \varepsilon>0.$$

In this case

$$\delta_3(\{\mathbb{N}\times\mathbb{N}\times\mathbb{N}/K(\lambda,\varepsilon)\}) = 1.$$

If $(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}/K(\lambda,\varepsilon)$, then

$$\begin{aligned}N_{\alpha x_{jkl}-\alpha L}(\varepsilon) &= N_{x_{jkl}-L}\left(\frac{\varepsilon}{|\alpha|}\right)\circledast N_0\left(\frac{\varepsilon}{|\alpha|}-\varepsilon\right)\\ &= N_{x_{jkl}-L}(\varepsilon)\circledast 1 = N_{x_{jkl}-L}(\varepsilon) > 1-\lambda\end{aligned}$$

for $\alpha\in\mathbb{R}$ $(\alpha\ne 0)$. This shows that

$$\delta_3\left(\left\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ N_{\alpha x_{jkl}-\alpha L}(\varepsilon)\le 1-\lambda\right\}\right)=0$$

so, $st_{N_3}-\lim \alpha x = \alpha L$.

**(iii)** The proof is clear from (i) and (ii). □

**Definition 3.4.** *Let $(X,N,\circledast)$ be a PN-space. For $x=(x_{jkl})\in X$, $t>0$ and $0<r<1$, the sphere centered at $x$ with radius $r$ is defined by*

$$B(x,r,t) = \{y\in X: \ N_{x-y}(t) > 1-r\}.$$

**Definition 3.5.** *A subset $Y$ of PN-space $(X, N, \circledast)$ is called bounded on $X$ if for every $r \in (0,1)$, there exists $t_o > 0$ such that $N_{x_{jkl}}(t_o) > 1 - r$ for all $x \in Y$.*

It follows from Lemma 3.2 that the set of all bounded statistically convergent triple sequences on PN-space is linear subspace of the linear normed space $l_\infty^{N_3}(X)$ of all bounded sequences on PN-space.

**Theorem 3.4.** *Let $(X, N, \circledast)$ be a PN-space and the set $st_{N_3}(X) \cap l_\infty^{N_3}(X)$ is closed linear subspace of the set $l_\infty^{N_3}(X)$.*

*Proof.* Let $y \in \overline{st_{N_3}(X) \cap l_\infty^{N_3}(X)}$. Since $B(y,r,t) \cap st_{N_3}(X) \cap l_\infty^{N_3}(X) \neq \varnothing$, there is an $x \in B(y,r,t) \cap st_{N_3}(X) \cap l_\infty^{N_3}(X)$. Let $t > 0$ and $0 < \varepsilon < 1$. Choose $r \in (0,1)$ such that $(1-r) \circledast (1-r) \geq 1 - \varepsilon$. Since $x \in B(y,r,t) \cap st_{N_3}(X) \cap l_\infty^{N_3}(X)$, there is a set $K \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $\delta_3(K) = 1$ such that

$$N_{y_{jkl} - x_{jkl}}\left(\frac{t}{2}\right) > 1 - r, \; N_{x_{jkl}}\left(\frac{t}{2}\right) > 1 - r, \text{ for all } (j,k,l) \in K.$$

Then we have

$$\begin{aligned} N_{y_{jkl}}(t) = N_{y_{jkl} - x_{jkl} + x_{jkl}}(t) &\geq N_{y_{jkl} - x_{jkl}}\left(\frac{t}{2}\right) \circledast N_{x_{jkl}}\left(\frac{t}{2}\right) \\ &> (1-r) \circledast (1-r) \geq 1 - \varepsilon, \text{ for all } (j,k,l) \in K. \end{aligned}$$

Hence

$$\delta_3\left(\left\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \; N_{y_{jkl}}(t) > 1 - \varepsilon\right\}\right) = 1$$

and thus $y \in st_{N_3}(X) \cap l_\infty^{N_3}(X)$. This completes the proof. □

**Definition 3.6.** *Let $(X, N, \circledast)$ be a PN-space. A triple sequence $x = (x_{jkl}) \in X$ is said to be statistically Cauchy with respect to the probabilistic norm **N** provided that, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exist $M(\varepsilon), T(\varepsilon), P(\varepsilon) \in \mathbb{N}$ such that for all $j, p \geq M, k, q \geq T$ and $l, u \geq P$, the set*

$$\left\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \; j \leq n, k \leq m \text{ and } l \leq s, N_{x_{jkl} - x_{pqu}}(\varepsilon) \leq 1 - \lambda\right\}$$

*has triple natural density zero.*

Now using a similar technique in the proof of Theorem 3.3, one can get the following result at once.

**Theorem 3.5.** *Let $(X, N, \circledast)$ be a PN-space and $x = (x_{jkl}) \in X$. Then, the following conditions are equivalent:*

***(i)*** *The triple sequence $x = (x_{jkl})$ is a statistically Cauchy sequence with respect to the probabilistic norm $N$;*

***(ii)*** *There exists an increasing index sequence $K = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : j,k,l = 1,2,3,\ldots\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3(K) = 1$ and the subsequence $\{x_{jkl}\}_{(j,k,l) \in K}$ is a Cauchy sequence with respect to the probabilistic norm $N$.*

## 4. Conclusion

In this paper we obtained some results on statistical convergence in probabilistic normed space. As every ordinary norm induces a probabilistic norm, the results obtained here are more general then the corresponding of normed spaces.

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