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Solving nonlinear fractional differential equation using a multi-step Laplace Adomian decomposition method

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Abstract. This paper presents a numerical technique for solving fractional differential equations by employing the multi-step Laplace Adomian decomposition method (MLADM). The proposed scheme is only a simple modification of the Adomian decomposition method, in which it is treated as an algorithm in a sequence of small intervals (i.e. time step) for finding accurate approximate solutions to the corresponding problems. This method was applied in four examples to solve non-linear fractional differential equations which were presented as fractional initial value problems. The fractional derivatives are described in the Caputo sense. Figurative comparisons between the MLADM and the classical fourth-order Runge–Kutta method (RK4) reveal that this modified method is more effective and convenient.

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1. Introduction

Fractional order ordinary differential equations, as generalizations of classical integer order ordinary differential equations, are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics, engineering and other applications [1, 2, 3]. The solutions of fractional differential equations are much involved. In general, there exists no method that yields an exact solution for fractional differential equations. Only approximate solutions can be derived. Several methods have been used to solve fractional differential equations, such as Laplace transform method [4, 5], Fourier transform method [6], Adomain decomposition method [7, 8, 9, 10, 11], homotopy perturbation method [12, 13, 14] and homotopy analysis method [15, 16, 17, 18, 19]. The objective of the present paper is to modify the LADM [20, 21] to provide symbolic approximate solutions for nonlinear fractional initial value problems by the MLADM. It can be found that the corresponding numerical solutions obtained by using LADM are valid only for a short time. While by using MLADM, they more valid and accurate during a long time, and are highly in agreement with the RK4-5 numerical solutions. Also, the MLADM gave the same results in the case of a multi-step Adomian decomposition method (MADM) without taking the inverse of the operator D_*^α. This paper is organized as follows. A brief review of the fractional calculus theory is given in section 2. In section 3, the MLADM is used to construct our numerical solutions for general nonlinear fractional differential equations. In section 4, some examples are presented to show the efficiency and simplicity of this method. Conclusions are presented in section 5.

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2. Fractional calculus

In this section, we introduce the linear operators of fractional integration and fractional differentiation in the framework of the Riemann-Liouville and Caputo fractional calculus.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$,where $f_1(x) \in C[0, \infty)$.

Clearly $C_\mu \subset C_\beta$ if $\beta \leq \mu$.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space C_μ^m, $m \in \mathbb{N} \cup \{0\}$, if $f^{(m)} \in C_\mu$.

Definition 2.3. The left sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, \quad x > 0, \\ J^0 f(x) &= f(x), \end{aligned} \tag{2.1}$$

Let $f \in C_{-1}^m$, $m \in \mathbb{N} \cup \{0\}$ then the Caputo fractional derivative of $f(x)$ is defined as

$$D_*^\alpha f\,(x) = \begin{cases} J^{m-\alpha} f^{(m)}(x), & m-1 < \alpha < m, \quad m \in \mathbb{N}, \\ \dfrac{d^m f(x)}{dx^m}, & \alpha = m. \end{cases} \tag{2.2}$$

Hence, we have the following properties [1, 2, 3]

$$\begin{aligned} 1.\ J^\alpha J^v f(t) &= J^{\alpha+v} f(t), \quad \alpha, v \geq 0. \\ 2. \qquad J^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \quad \alpha > 0,\ \gamma > -1,\ \ t > 0. \\ 3. J^\alpha D_*^\alpha f(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0, \quad m-1 < \alpha \leq m. \end{aligned} \tag{2.3}$$

Lemma 2.1. *If* $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, *then the Laplace transform of the fractional derivative* $D_*^\alpha f(t)$ *is*

$$\pounds(D_*^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-k-1}, \quad t > 0, \tag{2.4}$$

where $F(s)$ *be the Laplace transform of* $f(t)$ [19].

3. The Algorithm of the Method

The ADM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergent series with easily computable components, it has some drawbacks: the series solution always converges in a very small region and it has slow convergent rate in the wider region [7, 8, 9, 10, 11]. In this section we employ the MLADM to the discussed problem. To show the basic idea, let us consider the following fractional differential equation

$$D_*^\alpha u(t) + a_m u^{(m)}(t) + a_{m-1} u^{(m-1)}(t) + \ldots\ldots + a_1 u'(t) + a_0 u(t) + N(u(t), u'(t)) = f(t),$$

$$t \geq 0, \quad m-1 < \alpha \leq m, \tag{3.1}$$

subject to the initial conditions

$$u^{(i)}(0) = b_i, \quad i = 0, 1, 2, ..., m-1, \tag{3.2}$$

where a_i, b_i are known real constants, N is a nonlinear operator and $f(t)$ is known function. Let $[0, T]$ be the interval over which we want to find the solution of the above initial value problem. Assume that the interval $[0, T]$ is divided into $n-$subintervals of equal length Δt, $[t_0, t_1)$, $[t_1, t_2)$, ..., $[t_{n-1}, t_n]$ with $t_0 = 0$, $t_n = T$. Let $u_j(t)$ be approximate solutions in each subinterval $[t_{j-1}, t_j)$, $j = 1, 2, ..., n$, then equation (3.1) is transformed into the follwing system

$$D_*^\alpha u_j(t) + a_m u_j^{(m)}(t) + a_{m-1} u_j^{(m-1)}(t) + ... + a_1 u_j'(t) + a_0 u_j(t) + N(u_j(t), u_j'(t)) = f(t),$$
$$t \geq 0, \; j = 1, 2, ..., n, \tag{3.3}$$

subject to the initial conditions

$$u_j^{(i)}(t_{j-1}) = c_{j,i}, \quad j = 1, 2, ..., n, \quad i = 0, 1, 2, ..., m-1, \tag{3.4}$$

with $c_{1,i} = b_i$. Applying the Laplace transform to both sides of equation (3.3) and by using linearity of Laplace transforms, the result is

$$\mathcal{L}(D_*^\alpha u_j(t)) + a_m \mathcal{L}(u_j^{(m)}(t)) + a_{m-1} \mathcal{L}(u_j^{(m-1)}(t)) + ... + a_1 \mathcal{L}(u_j'(t))$$
$$+ a_0 \mathcal{L}(u_j(t)) + \mathcal{L}(N(u_j(t), u_j'(t))) = \mathcal{L}(f(t)). \tag{3.5}$$

Using the previous lemma and applying the formulas of Laplace transform, we get

$$\begin{aligned} s^\alpha \mathcal{L}(u_j(t)) &= \sum_{k=0}^{m-1} u_j^{(k)}(t_{j-1}) s^{\alpha-k-1} + \mathcal{L}(f(t)) - a_m \mathcal{L}(u_j^{(m)}(t)) - a_{m-1} \mathcal{L}(u_j^{(m-1)}(t)) \\ &\quad - ...... - a_1 \mathcal{L}(u_j'(t)) - a_0 \mathcal{L}(u_j(t)) - \mathcal{L}(N(u_j(t), u_j'(t))), \end{aligned}$$

and

$$\mathcal{L}(u_j(t)) = \sum_{k=0}^{m-1} u_j^{(k)}(t_{j-1}) s^{-(k+1)} + \frac{1}{s^\alpha} \mathcal{L}(f(t)) - \frac{1}{s^\alpha} [a_m \mathcal{L}(u_j^{(m)}(t)) + a_{m-1} \mathcal{L}(u_j^{(m-1)}(t))$$
$$+ ...... + a_1 \mathcal{L}(u_j'(t)) + a_0 \mathcal{L}(u_j(t))] - \frac{1}{s^\alpha} \mathcal{L}(N(u_j(t), u_j'(t))). \tag{3.6}$$

The MLADM represents the solution as an infinite series

$$u_j(t) = \sum_{r=0}^{\infty} u_{j,r}(t), \quad j = 1, 2, ..., n, \tag{3.7}$$

and the nonlinear term $N(u_j(t), u_j'(t))$ decomposes as

$$N(u_j(t), u_j'(t)) = \sum_{r=0}^{\infty} A_{j,r}(t), \quad j = 1, 2, ..., n, \tag{3.8}$$

where the $A_{j,r}$'s are Adomian polynomials of $u_{j,r}$'s. One can calculate the polynomials by the following formula

$$A_{i,r}(t) = \frac{1}{r!} \frac{d^r}{dp^r} N \left(\sum_{i=0}^{\infty} p^r u_{j,r}(t), \sum_{i=0}^{\infty} p^r u_{j,r}'(t) \right)_{|p=0} \tag{3.9}$$

Substituting (3.7), (3.8) and (3.9) into (3.6), we have

$$\mathcal{L}(\sum_{r=0}^{\infty} u_{j,r}(t)) = \sum_{k=0}^{m-1} u_j^{(k)}(t_{j-1})s^{-(k+1)} + \frac{1}{s^{\alpha}}\mathcal{L}(f(t))$$

$$-\frac{1}{s^{\alpha}}[a_m\mathcal{L}(\sum_{r=0}^{\infty} u_{j,r}^{(m)}(t)) + ... + a_1\mathcal{L}(\sum_{r=0}^{\infty} u'_{j,r}(t)) + a_0\mathcal{L}(\sum_{r=0}^{\infty} u_{j,r}(t))] - \frac{1}{s^{\alpha}}\mathcal{L}(\sum_{r=0}^{\infty} A_{j,r}(t)). \tag{3.10}$$

Hence the iteration are defined by the following recursive algorithm

$$\mathcal{L}(u_{j,0}(t)) = \sum_{k=0}^{m-1} u_j^{(k)}(t_{j-1})s^{-(k+1)} + \frac{1}{s^{\alpha}}\mathcal{L}(f(t))$$

$$\mathcal{L}(u_{j,r}(t)) = -\frac{1}{s^{\alpha}}[a_m\mathcal{L}(u_{j,r-1}^{(m)}(t)) + ... + a_1\mathcal{L}(u'_{j,r-1}(t)) + a_0\mathcal{L}(u_{j,r-1}(t))]$$
$$-\frac{1}{s^{\alpha}}\mathcal{L}(A_{j,r-1}(t)), \qquad r = 1, 2, 3, ... \tag{3.11}$$

Using the initial conditions (3.4) then applying the inverse Laplace transform to (3.11) we obtain the values $u_{j,r}(t)$ recursively. The solution of the initial value problem (3.1) and (3.2) for $[0, T]$ is given by

$$u(t) = \sum_{j=1}^{n} \chi_v u_j(t),$$

where

$$\chi_v = \begin{cases} 1, & t \in [t_{j-1}, t_j), \\ 0, & t \notin [t_{j-1}, t_j). \end{cases}$$

4. Numerical results

To demonstrate the effectiveness of the method for solving nonlinear fractional differential equations, we consider here the following four examples.

Example 4.1. Consider the following fractional Riccati equation

$$D_*^{\alpha}u(t) = 1 - u^2(t), \quad t \geq 0, \ 0 < \alpha \leq 1, \tag{4.1}$$

subject to the initial condition

$$u(0) = 0. \tag{4.2}$$

In this example, we apply the proposed algorithm on the interval $[0, 10]$. We choose to divide the interval $[0, 10]$ to subintervals with time step $\Delta t = 0.1$. Let $u_j(t)$ be approximate solutions of Riccati equation in each subinterval $[t_{j-1}, t_j)$, $j = 1, 2, ..., 100$, then equation (4.1) is transformed into the follwing system

$$D_*^{\alpha}u_j(t) = 1 - u_j^2(t), \quad t \geq 0, \ 0 < \alpha \leq 1, \quad j = 1, 2, ..., 100, \tag{4.3}$$

subject to the initial condition

$$u_j(t_{j-1}) = c_j, \ \text{ with } c_1 = 0. \tag{4.4}$$

To derive the solution, we use the equation (3.5) to get

$$\mathcal{L}(D_*^{\alpha}u_j(t)) = \mathcal{L}(1) - \mathcal{L}(u^2(t)).$$

Use the initial conditions (4.4), then we have

$$\mathcal{L}(u_j(t)) = \frac{c_j}{s} + \frac{1}{s^{\alpha+1}} - \frac{1}{s^{\alpha}}\mathcal{L}(u_j^2(t)).$$

In view of (3.10), we have

$$\mathcal{L}(\sum_{i=0}^{\infty} u_{j,i}(t)) = \frac{c_j}{s} + \frac{1}{s^{\alpha+1}} - \frac{1}{s^{\alpha}}\mathcal{L}(\sum_{i=0}^{\infty} A_{j,i}(t)).$$

The Laplace Adomian decomposition series (3.11) has the form

$$\begin{aligned}
\mathcal{L}(u_{j,0}(t)) &= \frac{c_j}{s} + \frac{1}{s^{\alpha+1}}, \\
\mathcal{L}(u_{j,i}(t)) &= -\frac{1}{s^{\alpha}}\mathcal{L}(A_{j,i-1}(t)), \qquad i = 1, 2, 3, ...
\end{aligned}$$

where

$$A_{j,i}(t) = \frac{1}{i!}\frac{d^i}{dp^i}\left[u_{j,0}^2 + 2pu_{j,0}u_{j,1} + p^2(u_{j,1}^2 + 2u_{j,0}u_{j,2}) + ...\right]_{|p=0}.$$

Now, we can obtain the following algorithm

$$\begin{aligned}
\mathcal{L}(u_{j,0}(t)) &= \frac{c_j}{s} + \frac{1}{s^{\alpha+1}}, \\
\mathcal{L}(u_{j,1}(t)) &= -\frac{1}{s^{\alpha}}\mathcal{L}(u_{j,0}^2(t)), \\
\mathcal{L}(u_{j,2}(t)) &= -\frac{1}{s^{\alpha}}\mathcal{L}(2u_{j,0}(t)u_{j,1}(t)), \\
\mathcal{L}(u_{j,3}(t)) &= -\frac{1}{s^{\alpha}}\mathcal{L}(u_{j,1}^2(t) + 2u_{j,0}(t)u_{j,2}(t)), \\
&\vdots
\end{aligned}$$

and so, the first three terms of the Laplace decomposition series are derived as follows:

$$\begin{aligned}
u_{j,0}(t) &= c_j + \frac{1}{\Gamma(\alpha+1)}t^{\alpha}, \\
u_{j,1}(t) &= -\frac{c_j^2}{\Gamma(\alpha+1)}t^{\alpha} - \frac{2c_j}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \\
u_{j,2}(t) &= \frac{2c_j^3}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{2c_j^2}{\Gamma(3\alpha+1)}(2 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)})t^{3\alpha} \\
&\quad + \frac{2c_j}{\Gamma(\alpha+1)\Gamma(4\alpha+1)}(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} + \frac{2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)})t^{4\alpha} \\
&\quad + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}, \\
&\vdots
\end{aligned}$$

Fig.1. shows the displacement of the MLADM (when $\alpha = 1$, 0.9 and 0.8) and the fourth-order Runge–Kutta method of the Riccati fractional equation. The results from the MLADM when $\alpha = 1$ match the results of the Runge–Kutta method very well. Therefore, the proposed method is a very efficient and accurate method that can be used to provide analytical solutions for nonlinear fractional differential equations. Also the results from the MLADM when $\alpha = 0.9$ and 0.8 have the same trajectories.

Example 4.2. Consider the following fractional nonlinear equation

$$D_*^{\alpha}u(t) = 0.1 - u(t) + 0.8u^2(t), \quad t \geq 0, \ 0 < \alpha \leq 1, \tag{4.5}$$

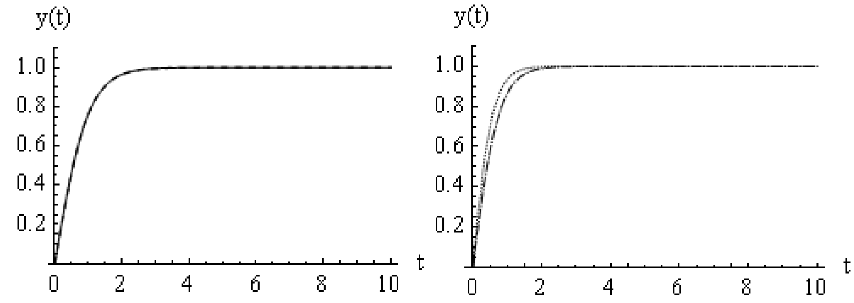

FIGURE 1. The displacement for Example 4.1: solid line: RK4 method solution, dashed line: MLADM solution when $\alpha = 1$, dash-dotted line: MLADM solution when $\alpha = 0.9$, dotted line: MLADM solution when $\alpha = 0.8$.

subject to the initial condition

$$u(0) = 1. \tag{4.6}$$

Apply the proposed algorithm on the interval $[0, 10]$. We choose to divide the interval $[0, 10]$ to subintervals with time step $\Delta t = 0.1$. Let $u_j(t)$ be approximate solutions of equation (4.5) in each subinterval $[t_{j-1}, t_j)$, $j = 1, 2, ..., 100$, then equation (4.5) is transformed into the follwing system

$$D_*^\alpha u_j(t) = 0.1 - u_j(t) + 0.8u_j^2(t), \quad t \geq 0,\ 0 < \alpha \leq 1, \quad j = 1, 2, ..., 100, \tag{4.7}$$

subject to the initial conditions

$$u_j(t_{j-1}) = c_j, \quad \text{with } c_1 = 1. \tag{4.8}$$

Use the initial conditions (4.8), then we have

$$\mathcal{L}(u_j(t)) = \frac{c_j}{s} + \frac{0.1}{s^{\alpha+1}} - \frac{1}{s^\alpha}\mathcal{L}(u_j(t)) + \frac{0.8}{s^\alpha}\mathcal{L}(u_j^2(t)),$$

According to the relation (3.11), we have the following Laplace Adomian decomposition series

$$\begin{aligned} \mathcal{L}(u_{j,0}(t)) &= \frac{c_j}{s} + \frac{0.1}{s^{\alpha+1}}, \\ \mathcal{L}(u_{j,i}(t)) &= -\frac{1}{s^\alpha}\mathcal{L}(u_{j,i-1}(t)) + \frac{0.8}{s^\alpha}\mathcal{L}(A_{j,i-1}(t)), \qquad i = 1, 2, 3, ... \end{aligned}$$

where

$$A_{j,i}(t) = \frac{1}{i!}\frac{d^i}{dp^i}[u_{j,0}^2 + 2pu_{j,0}u_{j,1} + p^2(u_{j,1}^2 + 2u_{j,0}u_{j,2}) + ...]|_{p=0}.$$

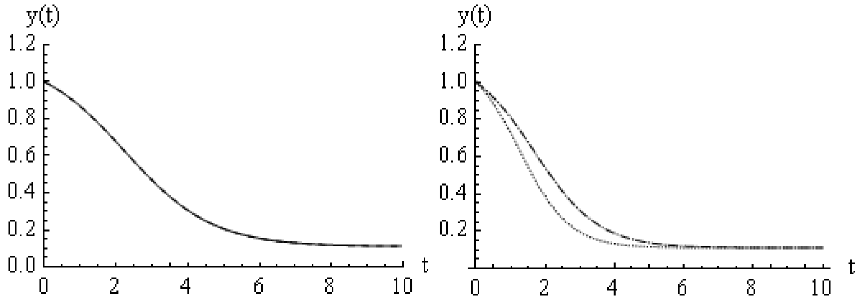

FIGURE 2. The displacement for Example 4.2: solid line: RK4 method solution, dashed line: MLADM solution when $\alpha = 1$, dash-dotted line: MLADM solution when $\alpha = 0.9$, dotted line: MLADM solution when $\alpha = 0.8$.

So, the first three terms of the Laplace decomposition series are derived as follows:

$$\begin{aligned}
u_{j,0}(t) &= c_j + \frac{0.1}{\Gamma(\alpha+1)}t^{\alpha}, \\
u_{j,1}(t) &= \frac{c_j(0.8c_j-1)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{(0.16c_j-0.1)}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{0.008\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \\
u_{j,2}(t) &= \frac{c_j(0.8c_j-1)(1.6c_j-1)}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{1}{\Gamma(3\alpha+1)}[(1.6c_j-1)(0.16c_j-0.1) \\
&\quad + \frac{0.16c_j(0.8c_j-1)\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}]t^{3\alpha} + \frac{1}{\Gamma(\alpha+1)\Gamma(4\alpha+1)}[-\frac{0.008\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} \\
&\quad + \frac{0.0128c_j\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} + \frac{0.16(0.16c_j-0.1)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}]t^{4\alpha} \\
&\quad + \frac{0.00128\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}, \\
&\quad \vdots
\end{aligned}$$

Fig.2. shows the displacement of the MLADM (when $\alpha = 1$, 0.9 and 0.8) and the fourth-order Runge–Kutta method of the fractional equation (4.5). Also the results of our computations when $\alpha = 1$ are in excellent agreement with the results obtained by the Runge–Kutta method and the results from the MLADM when $\alpha = 0.9$ and 0.8 have the same trajectories.

Example 4.3. Consider the following fractional nonlinear equation

$$D_*^{\alpha}u(t) = 1 + u(t) + u'^2(t) - u^2(t), \quad t \geq 0,\ 1 < \alpha \leq 2, \tag{4.9}$$

subject to the initial conditions

$$u(0) = 1,\ u'(0) = 0. \tag{4.10}$$

To demonstrate the effectiveness of the proposed algorithm as an approximate tool for solving the nonlinear fractional differential equations (4.9) for larger t, we apply the proposed algorithm on the interval $[0, 50]$. We choose to divide the interval to subintervals with time step $\Delta t = 0.1$. Let $u_j(t)$ be approximate solutions of equation (4.9) in each subinterval $[t_{j-1}, t_j)$, then equation (4.9) is transformed into the following

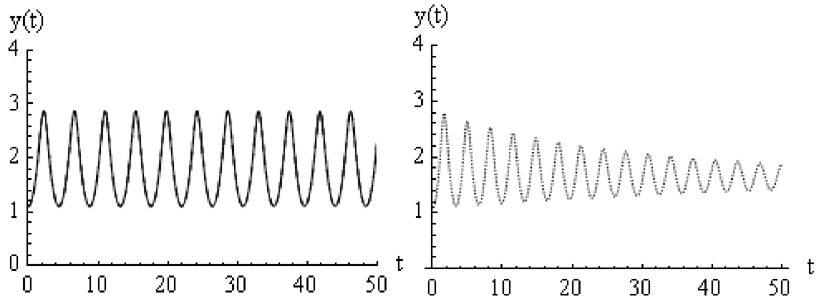

FIGURE 3. The displacement for Example 4.3: solid line: RK4 method solution, dashed line: MLADM solution when $\alpha = 2$, dotted line: MLADM solution when $\alpha = 1.8$.

system

$$D_*^\alpha u_j(t) = 1 + u_j(t) + u_j'^2(t) - u_j^2(t), \quad t \geq 0,\ 1 < \alpha \leq 2, \tag{4.11}$$

subject to the initial conditions

$$u_j(t_{j-1}) = a_j,\ u_j'(t_{j-1}) = b_j \quad \text{with} \ \ a_1 = 1,\ b_1 = 0. \tag{4.12}$$

According to the relation (3.11), we have the following Laplace Adomian decomposition series

$$\begin{aligned} \mathcal{L}(u_{j,0}(t)) &= \frac{a_j}{s} + \frac{b_j}{s^2} + \frac{1}{s^{\alpha+1}}, \\ \mathcal{L}(u_{j,i}(t)) &= \frac{1}{s^\alpha}\mathcal{L}(u_{j,i-1}(t)) - \frac{1}{s^\alpha}\mathcal{L}(A_{j,i-1}(t)), \qquad i = 1, 2, 3, ... \end{aligned}$$

where

$$A_{j,i}(t) = \frac{1}{i!}\frac{d^i}{dp^i}[(u_{j,0}^2 - u_{j,0}'^2) + 2p(u_{j,0}u_{j,1} - u_{j,0}'u_{j,1}') + ...]|_{p=0}.$$

Using mathematica software, the first few components of the Laplace Adomian decomposition solution are derived as follows:

$$\begin{aligned} u_j(t) &= a_j + b_j t + \frac{1 + a_j(1-a_j) + b_j^2}{\Gamma(\alpha+1)}t^\alpha + \frac{b_j(1-2a_j)}{\Gamma(\alpha+2)}t^{\alpha+1} - \frac{2b_j^2}{\Gamma(\alpha+3)}t^{\alpha+2} \\ &\quad + \frac{2b_j}{\Gamma(2\alpha)}t^{2\alpha-1} + \frac{1-2a_j}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{2b_j\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}t^{2\alpha+1}, \\ &\quad + \frac{\Gamma(2\alpha-1)}{\Gamma^2(\alpha)\Gamma(3\alpha-1)}t^{3\alpha-2} + ... \end{aligned}$$

Fig.3. shows the displacement of the MLADM (when $\alpha = 2$ and 1.8) and the fourth-order Runge–Kutta method of the fractional equation (4.9). It can be seen that the results from the MLADM when $\alpha = 2$ match the results of the RK4 solution very well. Therefore, the proposed method is very effcient and accurate method

Example 4.4. Consider the following fractional nonlinear Van der Pol equation

$$D_*^\alpha u(t) = 1 + u'(t) - u(t) - u^2(t)u'(t), \quad t \geq 0,\ 1 < \alpha \leq 2, \tag{4.13}$$

subject to the initial conditions

$$u(0) = 0.5,\ u'(0) = 0. \tag{4.14}$$

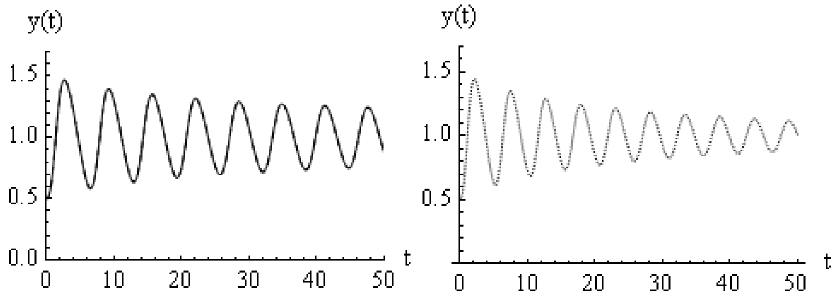

FIGURE 4. The displacement for Example 4.4: solid line: RK4 method solution, dashed line: MLADM solution when $\alpha = 2$, dotted line: MLADM solution when $\alpha = 1.8$.

Also, in this example divide the interval $[0, 50]$ to subintervals with time step $\Delta t = 0.1$. Let $u_j(t)$ be approximate solutions of equation (4.13) in the subinterval $[t_{j-1}, t_j)$, then we have

$$D_*^{\alpha} u_j(t) = 1 + u_j'(t) - u_j(t) - u_j^2(t) u_j'(t), \quad t \geq 0,\ 1 < \alpha \leq 2, \tag{4.15}$$

with initial conditions

$$u_j(t_{j-1}) = a_j,\ u_j^{'}(t_{j-1}) = b_j \quad \text{with} \quad a_1 = 0.5,\ b_1 = 0. \tag{4.16}$$

According to the relation (3.11), we have the following Laplace Adomian decomposition series

$$\begin{aligned} \mathcal{L}(u_{j,0}(t)) &= \frac{a_j}{s} + \frac{b_j}{s^2} - \frac{a_j}{s^{\alpha}} + \frac{1}{s^{\alpha+1}}, \\ \mathcal{L}(u_{j,i}(t)) &= \frac{1}{s^{\alpha-1}} \mathcal{L}(u_{j,i-1}(t)) - \frac{1}{s^{\alpha}} \mathcal{L}(u_{j,i-1}(t)) - \frac{1}{s^{\alpha}} \mathcal{L}(A_{j,i-1}(t)), \qquad i = 1, 2, 3, ... \end{aligned}$$

where

$$A_{j,i}(t) = \frac{1}{i!} \frac{d^i}{dp^i} [u_{j,0}^2 u_{j,0}' + p(2u_{j,0} u_{j,0}' u_{j,1} + u_{j,0}^2 u_{j,1}^{'}) + ...]|_{p=0}.$$

So, the first few components of the Laplace Adomian decomposition solution are derived as follows:

$$\begin{aligned} u_j(t) &= a_j + b_j t + \frac{1 - a_j(1 + a_j b_j) + b_j}{\Gamma(\alpha+1)} t^{\alpha} - \frac{b_j(1 + 2a_j b_j)}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{2b_j^3}{\Gamma(\alpha+3)} t^{\alpha+2} \\ &+ \frac{a_j(a_j^2 - 1)}{\Gamma(2\alpha-1)} t^{2\alpha-2} + \frac{1}{\Gamma(2\alpha)} [1 + a_j - a_j^2 + 2a_j^2 b_j + \frac{2a_j^2 b_j \Gamma(\alpha)}{\Gamma(\alpha-1)}] t^{2\alpha-1} \\ &- \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} [-\frac{a_j b_j^2}{\Gamma(\alpha-1)} + \frac{2a_j b_j(1 - b_j)}{\Gamma(\alpha)} + \frac{1 + 2a_j b_j}{\Gamma(\alpha+1)}] t^{2\alpha} \\ &- \frac{\Gamma(\alpha+2)}{\Gamma(2\alpha+2)} [\frac{b_j^2}{\Gamma(\alpha)} + \frac{2b_j^2}{\Gamma(\alpha+1)}] t^{2\alpha+1} - \frac{2a_j^3 \Gamma(2\alpha-2)}{\Gamma(\alpha-1)\Gamma(\alpha)\Gamma(3\alpha-2)} t^{3\alpha-3} \\ &- \frac{\Gamma(2\alpha-1)}{\Gamma(3\alpha-1)} [\frac{a_j^2(b_j - 2)}{\Gamma^2(\alpha)} + \frac{2a_j^2 b_j}{\Gamma(\alpha-1)\Gamma(\alpha)} - \frac{2a_j^2}{\Gamma(\alpha-1)\Gamma(\alpha+1)}] t^{3\alpha-2} + ... \end{aligned}$$

Fig.4. exhibits the comparison between the MLADM solution and the numerical results obtained by RK4 method for the displacement of nonlinear equation (4.13). From Fig.4, it is obvious that the solution obtained by the present method when $\alpha = 2$ is nearly identical with that given by RK4 method.

5. Conclusions

The main goal of this work is to purpose an efficient algorithm for the solution of nonlinear fractional differential equation. The Adomian decomposition method has been known to be a powerful device for solving many functional equations. In this paper, the multi-step Laplace Adomian decomposition method is suggested, as a modification of the ADM. It is found that the results obtained by using the MLADM with that obtained by the fourth-order Runge–Kutta method revealed that the approximate solutions obtained by ADM are only valid for a small time, compared to that the ones obtained by MLADM which are highly accurate and valid for a long time. It is recommended that the MLADM be used to solve other nonlinear problems in fractional calculus field.

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