Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 38(4), 2011, Pages 100–101
ISSN: 1223-6934, Online 2246-9958

A remark on the behavior of integrable functions at infinity

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Abstract. We prove that any continuous function $f : [0, \infty) \to \mathbb{R}$, for which the integral $\int_0^\infty \frac{f(x)}{x} dx$ exists at least as a Riemann improper integral, verifies the condition

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)dt = 0.$$

2000 *Mathematics Subject Classification.* Primary 26A42; Secondary 37A45.
Key words and phrases. Lebesgue integral, mean value, limit in density.

There are many ways to describe the behavior at infinity of integrable functions. The recent paper by Niculescu and Popovici [2] calls the attention to an old result of B. O. Koopman and J. von Neumann [1] relating the convergence of certain arithmetic means to convergence in density. For the convenience of the reader we recall here the basic facts involved in their approach.

The density of a measurable subset $A \subset \mathbb{R}$ is defined by the formula

$$d(A) = \lim_{r\to\infty} \frac{\lambda (A \cap [0, r))}{r}.$$

Here λ denotes the Lebesgue measure on real line.

Clearly, all bounded measurable subsets of $\mathbb{R}$ have density 0. However, one can exhibit easily examples of subsets having density 0 that are not bounded.

Given a real-valued function f defined on the interval $[0, \infty)$, its *limit in density* at infinity,

$$\ell = (d)\text{-} \lim_{x\to\infty} f(x),$$

is defined by the condition that each of the sets $\{t \geq \alpha : |f(t) - \ell| \geq \varepsilon\}$ has zero density, whenever $\varepsilon > 0$.

Lemma 0.1. *(B. O. Koopman and J. von Neumann). Suppose that $f : [0, \infty) \to \mathbb{R}$ is a nonnegative continuous function. Then*

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)dt = 0 \text{ implies } (d)\text{-} \lim_{x\to\infty} f(x) = 0.$$

The following result outlines a class of integrals satisfying the hypotheses of Lemma 1:

Theorem 0.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\int_0^\infty \frac{f(x)}{x} dx$ existsas a Riemann improper integral. Then*

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)dt = 0.$$

Consequently, if $f(x)/x$ is Lebesgue integrable, then $\lim_{x\to\infty} f(x) = 0$.

Received August 30, 2011. Revision received November 18, 2011.

Proof. Given $\varepsilon > 0$, there exists positive numbers y such that

$$\left| \int_y^x \frac{f(t)}{t} dt \right| < \frac{\varepsilon}{3} \text{ for all } x \in (y, \infty).$$

Integrating by parts we get

$$\begin{aligned}
\frac{1}{x} \int_0^x f(t)dt &= \frac{1}{x} \left(\int_0^y f(t)dt + \int_y^x t \, \frac{f(t)}{t} dt \right) \\
&= \frac{1}{x} \left(\int_0^y f(t)dt + \int_y^x t \, \frac{d}{dt} \left(\int_y^t \frac{f(s)}{s} ds \right) dt \right) \\
&= \frac{1}{x} \int_0^y f(t)dt + \int_y^x \frac{f(s)}{s} ds - \frac{1}{x} \int_y^x \left(\int_y^t \frac{f(s)}{s} ds \right) dt.
\end{aligned}$$

For every $x \in (y, \infty)$ we have

$$\left| \frac{1}{x} \int_y^x \left(\int_y^t \frac{f(s)}{s} ds \right) dt \right| < \frac{1}{x} \frac{\varepsilon}{3} (x - y) < \frac{\varepsilon}{3}.$$

Choose $z \in (y, \infty)$ such that

$$\left| \frac{1}{x} \int_0^y f(t)dt \right| < \frac{\varepsilon}{3} \text{ for every } x \in (z, \infty).$$

Then for every $x \in (z, \infty)$ we have

$$\left| \frac{1}{x} \int_0^x f(t)dt \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

and the proof is done. □

According to Theorem 1, if a continuous function $f : [0, \infty) \to \mathbb{R}$ does not verify the condition $\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)dt = 0$, then the improper integral $\int_0^\infty \frac{f(x)}{x} dx$ is divergent. In particular,

$$\int_0^\infty \frac{\sin^2 x}{x} dx = \infty.$$

Stronger results concerning the behavior at infinity of Lebesgue integrable functions will appear in [3].

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