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The Jensen Inequality for (M, N)-Convex Functions

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Abstract. We prove an Jensen type inequality in the context of functions which are convex with respect to a pair of regular means.

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1. Introduction

In what follows by a *mean* on an interval I we will understand a sequence $\mathcal{M} = (M_n)_{n\geq 2}$ of functions $M_n : I^n \to I$ such that

$$\min\{x_k : k = 1, ..., n\} \leq M_n(x_1, \ldots, x_n) \leq \max\{x_k : k = 1, ..., n\}$$

for all families $(x_1, \ldots, x_n) \in I^n$ and all $n \geq 2$.

Given a continuous increasing function $\varphi : I \to \mathbb{R}$ we can attach to it the so called *quasi-arithmetic mean* $\mathcal{M}^{[\varphi]} = (M_n^{[\varphi]})_{n\geq 2}$, defined by the formula

$$M_n^{[\varphi]}(x_1, \ldots, x_n) = \varphi^{-1}\left(\frac{\varphi(x_1) + \cdots + \varphi(x_n)}{n}\right).$$

The special case where $\varphi(x) = x$ corresponds to the arithmetic mean $\mathcal{A} = (A_n)_{n\geq 2}$, where

$$A_n(x_1, \ldots, x_n) = \frac{\varphi(x_1) + \cdots + \varphi(x_n)}{n},$$

while the function $\varphi(x) = \log x$ corresponds to the geometric mean $\mathcal{G} = (G_n)_{n\geq 2}$, where

$$G_n(x_1, \ldots, x_n) = \sqrt[n]{x_1 \ldots x_n}.$$

The quasi-arithmetic mean is an example of a *regular mean* in the sense that the following three conditions are fulfilled:

Symmetry:

$$M_n(x_1, \ldots, x_n) = M_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for every family $(x_1, \ldots, x_n) \in I^n$ every permutation σ and every $n \geq 2$;

Associativity:

$$M_n(x_1, \ldots, x_k, x_{k+1}, ..., x_n) = M_n(M_k(x_1, \ldots, x_k), ..., M_k(x_1, \ldots, x_k)), x_{k+1}, ..., x_n)$$

for every family $(x_1, \ldots, x_n) \in I^n$ and every $n \geq k \geq 2$;

Strict monotonicity:

$$x < y \text{ implies } M_n(x, z_2, ..., z_n) < M_n(y, z_2, ..., z_n)$$

for every $z_2, ..., z_n \in I$ and every $n \geq 2$.

See [1] for details.

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Every regular mean verifies the equalities

$$M_n(x, \ldots, x) = x$$

and

$$M_{2n}(x, ..., x, y, ..., y) = M_2(x, y).$$

In order to simplify the notation we will denote the means by capitol letters such as $M, N, ...$ and the functions $M_n(x_1, \ldots, x_n)$ simply $M(x_1, \ldots, x_n)$, the lower index resulting from the context.

In what follows we will be interested in the following special class of functions extending the concept of midconvexity of Jensen.

Definition 1.1. *Suppose that M and N are means defined on the intervals I and J respectively. A function $F : I \to J$ is called $(M, N)-$*convex *if*

$$F(M(x, y)) \leq N(F(x, y)) \quad \text{for every pair } x, y \text{ of elements of } I. \qquad (MN)$$

In this paper we will prove an Jensen type inequality in the context of functions which are convex with respect to a pair of regular means. A more restrictive theory of convex functions associated to means is described in [2], [3] and [4].

2. Main results

Theorem 2.1. *Let I and J be intervals and let M and N be regular means on I and J respectively. If $f : I \to J$ is a $(M, N)-$convex function, then the following generalization of Jensen's inequality holds*

$$f(M(x_1, x_2, ..., x_n)) \leq N(f(x_1), f(x_2), ..., f(x_n)) \qquad (J_n)$$

for all families $x_1, x_2, ..., x_n \in I$ and all $n \in \mathbb{N}$, $n \geq 2$.

The proof will be done by mathematical induction, based on the following two technical lemmas.

The initial step where $n = 2$, corresponds to the definition of $(M, N)-$convexity,

Lemma 2.1. *Under the assumptions of Theorem 2.1, if (J_n) works for families of length $n \in \mathbb{N}$, $n \geq 2$, then it also works for families of length $2n$.*

Proof. Let $x_1, x_2, ..., x_{2n} \in I$. Then $f(M(x_1, x_2, ..., x_{2n}))$ equals

$$f\Bigg(M(\underbrace{M(x_1, \ldots, x_n), \ldots, M(x_1, \ldots, x_n)}_{n \text{ times}}),$$

$$M(\underbrace{M(x_{n+1}, \ldots, x_n), \ldots, M(x_{n+1}, \ldots, x_n)}_{n \text{ times}})\Bigg)$$

$$= f(M(M(x_1, \ldots, x_n), M(x_{n+1}, \ldots, x_{2n})))$$

and because f is $(M,N)-$convex, this is less than equal to

$$\begin{aligned}
&N\left(f\left(M\left(x_1,\ldots,x_n\right),f(M\left(x_{n+1},\ldots,x_{2n}\right)\right)\right) \\
&\qquad \le N\left(N\left(f(x_1),\ldots,f(x_n)\right),N\left(f(x_{n+1}),\ldots,f(x_{2n})\right)\right) \\
&\qquad = N\Bigg(\underbrace{N(f(x_1),\ldots,f(x_n)),\ldots,N(f(x_1),\ldots,f(x_n))}_{n\text{ times}}, \\
&\qquad\quad \underbrace{N(f(x_{n+1}),\ldots,f(x_{2n})),\ldots,N(f(x_{n+1}),\ldots,f(x_{2n}))}_{n\text{ times}}\Bigg) \\
&\qquad = N\left(f(x_1),f(x_2),...,f(x_n)\right).
\end{aligned}$$

Therefore the inequality J_n holds also for families of length $2n$. □

Lemma 2.2. *Under the assumptions of Theorem 2.1, if (J_n) holds for families of length $m\in\mathbb{N}$, $m\ge 2$ then it holds also for families of any length $k\in\{2,...,m\}$.*

Proof. Let $k\in\{2,...,m-1\}$. Let $x_1,x_2,...,x_k\in I$. We have

$$\begin{aligned}
&N(\underbrace{f(M(x_1,\ldots,x_k)),\ldots,}_{k\text{ times}}\underbrace{f(M(x_1,\ldots,x_k)),\ldots}_{m-k\text{ times}}) \\
&\qquad = f\left(M\left(x_1,x_2,...,x_k\right)\right) \\
&\qquad \le f(M(\underbrace{M(x_1,\ldots,x_k),\ldots,M(x_1,\ldots,x_k)}_{m\text{ times}})) \\
&\qquad = f(M(x_1,\ldots,x_k,\underbrace{M(x_1,\ldots,x_k),\ldots,M(x_1,\ldots,x_k)}_{m-k\text{ times}})) \\
&\qquad \le N(f(x_1),\ldots,f(x_k),\underbrace{f(M(x_1,\ldots,x_k)),\ldots,f(M(x_1,\ldots,x_k))}_{m-k\text{ times}}) \\
&\qquad = N(\underbrace{H,...,}_{k\text{ times}}\underbrace{f(M(x_1,\ldots,x_k)),\ldots}_{m-k\text{ times}}).
\end{aligned}$$

Here $H=N(f(x_1),\ldots,f(x_k))$. Since N is strictly monotonic, we infer

$$f\left(M\left(x_1,...,x_k\right)\right)\le N\left(f(x_1),...,f(x_n)\right)$$

which yields (J_n) for all k. □

According to [2] (see also [3] and [4]), a nonnegative function f defined on a subinterval I of $[0,\infty)$ is called (G,G)-convex if

$$f\left(\sqrt{xy}\right)\le\sqrt{f(x)f(y)}\text{ for all }x,y\in I.$$

By Theorem 2.1,

$$f\left(\sqrt[n]{x_1\cdots x_n}\right)\le\sqrt[n]{f(x_1)\cdots f(x_n)}$$

for all families $x_1,...,x_n\in I$ and all $n\ge 2$. For $f(x)=1+x$, $x\ge 0$, this yields the inequality of Huygens,

$$1+\sqrt[n]{x_1\cdots x_n}\le\sqrt[n]{(1+x_1)\cdots(1+x_n)}$$

for all $x_1,...,x_n\ge 0$ and all $n\ge 2$.

The sum (as well as the product) of two (G,G)-convex functions is (G,G)-convex too. Thus every polynomial with nonnegative coefficients is (G,G)-convex. Other examples of such functions are presented in [2] and [4].

Let us consider now the sequence

$$a_n = n! \quad \text{for } n \in \mathbb{N}.$$

Since

$$\sqrt{a_n a_{n+2}} = \sqrt{n!\,(n+2)!} = n!\sqrt{n^2+3n+1} > (n+1)! = \sqrt{a_{n+1}},$$

it follows that this sequence is log-convex (that is, $(\log a_n)_n$ is a convex sequence).

Then the function g that interpolates linearly the points $(n, \log a_n)$ is convex (and increasing). As a consequence

$$h(x) = \exp g(x), \quad x \geq 0.$$

is log-convex and increasing. Notice that $h(n) = n!$ for every $n \in \mathbb{N}$. From Theorem 2.1 we infer that

$$h\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \sqrt[n]{h(x_1)\cdots h(x_n)}.$$

Consequently, using the floor function $\lfloor x \rfloor$ (the largest integer less than or equal to x) we conclude that

$$\lfloor \frac{x_1 + \cdots + x_n}{n} \rfloor ! \leq \sqrt[n]{(\lfloor x_1 \rfloor + 1)\cdots(\lfloor x_n \rfloor + 1)}$$

for every $x_1, ..., x_n \geq 0$ and every $n \geq 2$.

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