Fuzzy deductive systems in BE-semigroups

A. H. Handam

Abstract. In this paper, we introduce the notion of fuzzy deductive systems and investigate some of their properties. Also we give the construction of quotient self-distributive BE-semigroup $X/\mu$ induced by a fuzzy deductive system $\mu$ and discuss their interesting properties.

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Key words and phrases. BE-semigroup, deductive system, fuzzy deductive systems.

1. Introduction

Imai and Iséki introduced two classes of abstract algebras, namely, BCK-algebras and BCI-algebras [5], [6]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [11], Neggers and Kim introduced the notion of d-algebras which is a generalization of BCK-algebras. Moreover, Jun et al. [7] introduced a new notion, called a BH-algebra, which is a generalization of BCK/BCI-algebras. Recently, as another generalization of BCK-algebras, the notion of a BE-algebra was introduced by Kim and Kim [9]. They provided an equivalent condition of the filters in BE-algebras using the notion of upper sets. In [2], [3], Ahn and So introduced the notion of ideals in BE-algebras and proved several characterizations of such ideals. In [1], Ahn and Kim combined BE-algebras and semigroups and introduced the notion of BE-semigroups. Also, congruences and BE-Relations in BE-Algebras was studied by Yon et al. [15]. Recently, Handam introduced the notion of BE-homomorphisms between BE-semigroups [4].

The theory of fuzzy sets was first developed by Zadeh [16] and has been applied to many branches in mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [13] and he introduced the notion of fuzzy subgroups. In 1975, Zadeh [17] introduced the concept of interval valued fuzzy subset, where the values of the membership functions are intervals of numbers instead of the numbers. Later on, Song et al. [14] introduced the concept of a fuzzy ideals in BE-algebras. Recently, Rezaei and Saeid [12] introduced the concepts of fuzzy BE-subalgebras and fuzzy topological BE-algebras. In this paper, we introduce the concept of fuzzy deductive systems and investigate some of their properties. We give the construction of quotient self-distributive BE-semigroup $X/\mu$ via a fuzzy deductive system $\mu$. In addition, we establish a generalization of fundamental BE-homomorphism theorem in self-distributive BE-semigroups by using fuzzy deductive systems.

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2. Preliminaries

In this section we cite some elementary aspects that will be used in the sequel of this paper.

**Definition 2.1.** [9]. An algebra $(X, *, 1)$ of type $(2,0)$ is called a **BE-algebra** if

- $(BE1)$ $x * x = 1$ for all $x \in X$,
- $(BE2)$ $x * 1 = 1$ for all $x \in X$,
- $(BE3)$ $1 * x = x$ for all $x \in X$,
- $(BE4)$ $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

**Example 2.1.** [9]. Let $X = \{1, a, b, c, d, 0\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>a</td>
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<td>c</td>
<td>d</td>
<td>0</td>
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<td>c</td>
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</tr>
</tbody>
</table>

Then $(X; *, 1)$ is a BE-algebra.

We can define a relation " $\leq$ " on $X$ by $x \leq y$ if and only if $x * y = 1$.

In an BE-algebra, the following identities are true (see [9]):

- $(a1)$ $x * (y * x) = 1$.
- $(a2)$ $x * ((x * y) * y) = 1$.

**Definition 2.2.** [9]. A BE-algebra $(X, *, 1)$ is said to be **self-distributive** if

$x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

**Example 2.2.** [9]. Let $X = \{1, a, b, c, d\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<tbody>
<tr>
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<td>1</td>
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</table>

Then it is easy to see that $X$ is a self-distributive BE-algebra.

**Definition 2.3.** [1]. An algebraic system $(X; \circ, *, 1)$ is called a **BE-semigroup** if it satisfies the following:

- $(i)$ $(X; \circ)$ is a semigroup,
- $(ii)$ $(X; *, 1)$ is a BE-algebra,
- $(iii)$ the operation "$\circ$" is distributive (on both sides) over the operation "$*$", that is, $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$ and $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$ for all $x, y, z \in X$.

**Example 2.3.** [1]. Define two operations "$\circ$" and "$*$" on a set $X = \{1, a, b, c\}$ as follows:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>a</td>
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<tr>
<td>b</td>
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<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$*$</th>
<th>1</th>
<th>a</th>
<th>b</th>
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<td>0</td>
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<td>1</td>
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<td>c</td>
</tr>
</tbody>
</table>
It is easy to see that $(X; \odot, *, 1)$ is a BE-semigroup.

**Definition 2.4.** [4]. A BE-semigroup $(X; \odot, *, 1)$ is said to be self-distributive BE-semigroup if $X$ is self-distributive BE-algebra.

**Proposition 2.1.** [1]. Let $(X; \odot, *, 1)$ be a BE-semigroup. Then
(i) $(\forall x \in X) \ (1 \odot x = x \odot 1 = 1),$ 
(ii) $(x, y, z \in X) \ (x \leq y \Rightarrow x \odot z \leq y \odot z, z \odot x \leq z \odot y).$

**Definition 2.5.** [1]. Let $(X; \odot, *, 1)$ be a BE-semigroup. A nonempty subset $D$ of $X$ is called a left (resp., right) deductive system if it satisfies:
(ds1) $X \odot D \subseteq D$ (resp. $(D \odot X \subseteq D))$
(ds2) $(\forall a \in D) \ ((\forall x \in X) \ (a \ast x \in D \Rightarrow x \in D).$
Both left and right deductive system is a two sided deductive system or simply deductive system.

**Example 2.4.** [1]. Let $X = \{x, y, z, 1\}$ be a set with the following Cayley tables:

<table>
<thead>
<tr>
<th>$\odot$</th>
<th>1</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>$x$</td>
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<td>$x$</td>
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<td>1</td>
<td>$y$</td>
<td>$z$</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>1</td>
<td>$z$</td>
<td>$y$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>1</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>$x$</td>
<td>1</td>
<td>$x$</td>
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<tr>
<td>$y$</td>
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<td>1</td>
<td>$y$</td>
<td>$z$</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>1</td>
<td>$z$</td>
<td>$y$</td>
</tr>
</tbody>
</table>

It is easy to show that $(X; \odot, *, 1)$ is a BE-semigroup and $D = \{1, x\}$ is an left deductive system of $X.$

**Definition 2.6.** [4]. Let $X$ and $Y$ be two BE-semigroups. A mapping $\psi : X \to Y$ is called a BE-homomorphism if for all $a, b \in X,$
$\psi(a \ast b) = \psi(a) \ast \psi(b)$ and $\psi(a \odot b) = \psi(a) \odot \psi(b).$

**Proposition 2.2.** [4]. Suppose that $\psi : X \to Y$ is a BE-homomorphism of BE-semigroups. Then $\psi(1) = 1.$

A BE-homomorphism $\psi$ is called a BE-monomorphism (resp. BE-epimorphism) if it is injective (resp. surjective). A bijective BE-homomorphism is called a BE-isomorphism. For any BE-homomorphism $\psi : X \to Y,$ the set $\{x \in X \mid \psi(x) = 1\}$ is called the kernel of $\psi,$ denoted by $\text{Ker}(\psi)$ and the set $\{\psi(x) \mid x \in X\}$ is called the image of $\psi,$ denoted by $\text{Im}(\psi).$ We denote by $\text{Hom}(X,Y)$ the set of all BE-homomorphisms of BE-semigroups from $X$ to $Y.$

We now review some fuzzy logic concepts. The readers are referred to [10] for some basic definitions and results on fuzzy sets and fuzzy algebras, not given in this paper. Let $X$ be a set. A fuzzy set $A$ in $X$ is characterized by a membership function $\mu_A : X \to [0, 1].$ For any $t \in [0, 1],$ the set $U(\mu, t) = \{x \in A : \mu(x) \geq t\}$ is called level subset of $\mu.$ Let $\xi$ be a mapping from the set $X$ to the set $Y$ and let $B$ be a fuzzy set in $Y$ with membership function $\mu_B.$ The inverse image of $B,$ denoted $\xi^{-1}(B),$ is the fuzzy set in $X$ with membership function $\mu_{\xi^{-1}(B)}$ defined by $\mu_{\xi^{-1}(B)}(x) = \mu_B(\xi(x))$ for all $x \in X.$ Conversely, let $A$ be a fuzzy set in $X$ with membership function $\mu_A.$ Then the image of $A,$ denoted by $\xi(A),$ is the fuzzy set in $Y$ such that:

$$
\mu_{\xi(A)}(y) = \begin{cases} 
\sup_{z \in \xi^{-1}(y)} \mu_A(z), & \text{if } \xi^{-1}(y) = \{x : \xi(x) = y\} \neq \emptyset, \\
0, & \text{otherwise.}
\end{cases}
$$
A fuzzy set \( A \) in a BE-semigroup \( X \) with the membership function \( \mu_A \) is said to have the sup property if for any subset \( T \subseteq X \) there exists \( x_0 \in T \) such that 
\[
\mu_A(x_0) = \sup_{t \in T} \mu_A(t).
\]

3. Fuzzy deductive systems

In what follows, \( X \) denotes a BE-semigroup, \( A \) or \( \mu_A \) denotes a fuzzy set \( A \) in \( X \).

**Definition 3.1.** A fuzzy set \( \mu \) in \( X \) is called a fuzzy deductive system of \( X \) if it satisfies the following conditions:

\[
\begin{align*}
(FD1) & \quad \mu(x \odot y) \geq \mu(y) \text{ for all } x, y \in X, \\
(FD2) & \quad \mu(x \odot y) \geq \mu(x) \text{ for all } x, y \in X, \\
(FD3) & \quad \mu(x) \geq \min \{ \mu(y), \mu(y \ast x) \} \text{ for all } x, y \in X.
\end{align*}
\]

Note that \( \mu \) is a fuzzy left deductive system of \( X \) if it satisfies \((FD1)\) and \((FD3)\), and \( \mu \) is a fuzzy right deductive system of \( X \) if it satisfies \((FD2)\) and \((FD3)\).

**Example 3.1.** Let \( X = \{1, x, y, z\} \) be the set with the following Cayley tables:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>x</td>
<td>x</td>
<td>1</td>
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<tr>
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<td>y</td>
<td>1</td>
<td>y</td>
<td>z</td>
</tr>
<tr>
<td>z</td>
<td>z</td>
<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>

Then \((X; \odot, \ast, 1)\) is a BE-semigroup (see [1]). Let \( \mu \) be a fuzzy set in \( X \) defined by \( \mu(1) = t_0 \), \( \mu(x) = t_1 \), \( \mu(y) = \mu(z) = t_2 \), where \( t_0 > t_1 > t_2 \in [0, 1] \). Then \( \mu \) is a fuzzy deductive system of \( X \).

**Lemma 3.1.** If \( D \) is a fuzzy left (resp. right) deductive system of \( X \), then for all \( x \in X \)
\[
\mu_D(1) \geq \mu_D(x).
\]

**Proof.** Let \( x \in X \). Since \( D \) is a fuzzy left (resp. right) deductive system of \( X \), it follows that \( \mu_D(1) = \mu_D(1 \odot x) \geq \mu_D(x) \) (resp. \( \mu_D(1) = \mu_D(x \odot 1) \geq \mu_D(x) \)). \( \square \)

**Theorem 3.2.** Let \( D \) be a fuzzy left (resp. right) deductive system of \( X \). If there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} \mu_D(x_n) = 1 \), then \( \mu_D(1) = 1 \).

**Proof.** By Lemma 3.1, we have \( \mu_D(1) \geq \mu_D(x) \) for \( x \in X \). Consider \( 1 \geq \mu_D(1) \geq \lim_{n \to \infty} \mu_D(x_n) = 1 \). Therefore, \( \mu_D(1) = 1 \). \( \square \)

**Theorem 3.3.** Let \( \mu \) be a fuzzy left (resp. right) deductive system of \( X \). Then the set \( X_\mu = \{ x \in X \mid \mu(x) = \mu(1) \} \) is a left (resp. right) deductive system of \( X \).

**Proof.** Let \( \mu \) be a fuzzy left deductive system of \( X \). Let \( X_\mu = \{ x \in X \mid \mu(x) = \mu(1) \} \). If \( x \in X \) and \( y \in X_\mu \), then \( \mu(y) = \mu(1) \). Since \( \mu(x \odot y) \geq \mu(y) = \mu(1) \), it follows that \( x \odot y \in X_\mu \) so that \( X \odot X_\mu \subseteq X_\mu \). Now let \( x, y \in X \) be such that \( y \in X_\mu \) and \( y \ast x \in X_\mu \). Then \( \mu(x) \geq \min \{ \mu(y), \mu(y \ast x) \} = \min \{ \mu(1), \mu(1) \} = \mu(1) \), and thus \( x \in X_\mu \). Therefore, \( X_\mu \) is a left deductive system of \( X \). Similarly we have the desired result for the right case. \( \square \)

**Corollary 3.4.** If \( \mu \) is a fuzzy deductive system of \( X \), then the set \( X_\mu = \{ x \in X \mid \mu(x) = \mu(1) \} \) is a deductive system of \( X \).
Theorem 3.5. Let \( \mu \) be a fuzzy set in \( X \). Then \( \mu \) is a fuzzy deductive system of \( X \) if and only if the nonempty level subset \( U(\mu, t) \) of \( \mu \) is a deductive system of \( X \).

Proof. Let \( \mu \) be a fuzzy deductive system of \( X \) and the level subset \( U(\mu, t) = \{ x \in A : \mu(x) \geq t \} \) of \( \mu \). Let \( x \in X \) and \( y \in U(\mu, t) \). Then \( \mu(y) \geq t \). Since \( \mu(x \circ y) \geq \mu(y) \geq t \), it follows that \( x \circ y \in U(\mu, t) \) so that \( X \circ U(\mu, t) \subseteq U(\mu, t) \). Let \( y \in X \) and \( x \in U(\mu, t) \).

Then \( \mu(x) \geq t \). Since \( \mu(x \circ y) \geq \mu(x) \geq t \), it follows that \( x \circ y \in U(\mu, t) \) so that \( U(\mu, t) \circ X \subseteq U(\mu, t) \). Now, let \( x, y \in X \) be such that \( y \in U(\mu, t) \) and \( y \ast x \in U(\mu, t) \).

Then \( \mu(x \ast y) \geq \mu(y) \geq t \), and thus \( x \in U(\mu, t) \). Therefore, \( U(\mu, t) \) is a deductive system of \( X \).

Conversely, assume that the nonempty level set \( U(\mu, t) \) of \( \mu \) is a deductive system of \( X \) for every \( t \in [0, 1] \). If \( \mu(x_0 \circ y_0) < \mu(y_0) \) for some \( x_0, y_0 \in X \), then by taking \( t_0 = \frac{1}{2}(\mu(x_0 \circ y_0) + \mu(y_0)) \) we have \( \mu(x_0 \circ y_0) < t_0 < \mu(y_0) \). Thus \( y_0 \in U(\mu, t_0) \) and \( x_0 \circ y_0 \notin U(\mu, t_0) \), a contradiction and so \( \mu(x_0 \circ y_0) \geq \mu(y_0) \) for all \( x_0, y_0 \in X \).

If \( \mu(x_1 \circ y_1) < \mu(x_1) \) for some \( x_1, y_1 \in X \), then by taking \( t_1 = \frac{1}{2}(\mu(x_1 \circ y_1) + \mu(x_1)) \) we have \( \mu(x_1 \circ y_1) < t_1 < \mu(x_1) \). Thus \( x_1 \in U(\mu, t_1) \) and \( x_1 \circ y_1 \notin U(\mu, t_1) \), which is also a contradiction and so \( \mu(x_1 \circ y_1) \geq \mu(x_1) \) for all \( x_1, y_1 \in X \). Next, if \( \mu(x_2) < \min \{ \mu(y_2), \mu(y_2 \ast x_2) \} \) for some \( x_2, y_2 \in X \), then by taking \( t_2 = \frac{1}{2}(\mu(x_2) + \min \{ \mu(y_2), \mu(y_2 \ast x_2) \}) \) we have \( \mu(x_2) < t_2 < \min \{ \mu(y_2), \mu(y_2 \ast x_2) \} \). Thus \( y_2, y_2 \ast x_2 \in U(\mu, t_2) \) and \( x_2 \notin U(\mu, t_2) \), which is again a contradiction and so \( \mu(x_2) \geq \min \{ \mu(y_2), \mu(y_2 \ast x_2) \} \) for all \( x_2, y_2 \in X \). This completes the proof. \( \square \)

Theorem 3.6. Let \( \mu \) be a fuzzy deductive system of \( X \). Then

\[(\forall a, b \in X) \quad (a \leq b \Rightarrow \mu(a) \leq \mu(b))\]

Proof. Let \( a, b \in X \) be such that \( a \leq b \). Then \( a \ast b = 1 \). It follows from \( (FD3) \) and Lemma 3.1 that \( \mu(b) \geq \min \{ \mu(a), \mu(a \ast b) \} = \min \{ \mu(a), \mu(1) \} = \mu(a) \). Hence \( \mu(a) \leq \mu(b) \). \( \square \)

Definition 3.2. For a family of fuzzy sets \( \{ \mu_i \mid i \in I \} \) in a BE-semigroup \( X \), define the joint \( \bigvee_{i \in I} \mu_i \) and meet \( \bigwedge_{i \in I} \mu_i \) of \( \{ \mu_i \mid i \in I \} \) as follows:

\[\left( \bigvee_{i \in I} \mu_i \right)(x) = \sup \{ \mu_i(x) \mid i \in I \}, \quad \left( \bigwedge_{i \in I} \mu_i \right)(x) = \inf \{ \mu_i(x) \mid i \in I \},\]

for all \( x \in X \), where \( I \) is any index set.

Consider two fuzzy sets \( A \subset B \) in \( X \). Zadeh [16] gave a definition of fuzzy set inclusion with: \( A \subset B \iff \mu_A(x) \leq \mu_B(x), \forall x \in X \).

Theorem 3.7. The family of fuzzy deductive systems of \( X \) is a completely distributive lattice under the ordering of fuzzy set inclusion \( \subseteq \).

Proof. Let \( \{ \mu_i \mid i \in I \} \) be a family of fuzzy deductive systems of \( X \). Since \( [0, 1] \) is a completely distributive lattice with respect to the usual ordering in \([0, 1]\), it is sufficient to show that \( \bigwedge_{i \in I} \mu_i \) is a fuzzy deductive systems of \( X \). For any \( x, y \in X \), we have

\[\left( \bigwedge_{i \in I} \mu_i \right)(x \circ y) = \inf \{ \mu_i(x \circ y) \mid i \in I \} \geq \inf \{ \mu_i(x) \mid i \in I \} = \left( \bigwedge_{i \in I} \mu_i \right)(x),\]

\[\left( \bigwedge_{i \in I} \mu_i \right)(x) = \inf \{ \mu_i(x) \mid i \in I \} \geq \inf \{ \mu_i(y) \mid i \in I \} = \left( \bigwedge_{i \in I} \mu_i \right)(y),\]
\[
\left( \bigwedge_{i \in I} \mu_i \right)(x) = \inf \{ \mu_i(x) \mid i \in I \} \\
\geq \inf \{ \min \{ \mu_i(y), \mu_i(y \circ x) \} \mid i \in I \} \\
= \min \{ \inf \{ \mu_i(y) \mid i \in I \}, \inf \{ \mu_i(y \circ x) \mid i \in I \} \}
\]

Hence \( \bigwedge_{i \in I} \mu_i \) is a fuzzy deductive system of \( X \), completing the proof. \( \square \)

**Theorem 3.8.** Let \( D \) be a subset of \( X \). Suppose that \( \mu \) is a fuzzy set in \( X \) defined by

\[
\mu(x) = \begin{cases} 
\alpha & \text{if } x \in D, \\
\beta & \text{otherwise}, 
\end{cases}
\]

where \( \alpha > \beta \) in \([0,1]\). Then \( \mu \) is a fuzzy deductive system if and only if \( D \) is a deductive system of \( X \). Moreover, \( X_\mu = D \).

**Proof.** Let \( \mu \) be a fuzzy deductive system. Let \( x \in D \) and \( y \in X \). Then \( \mu(x \circ y) \geq \mu(x) = \alpha \) and so \( x \circ y \in D \), that is, \( D \circ X \subseteq D \). Let \( y \in D \) and \( x \in X \). Then \( \mu(x \circ y) \geq \mu(y) = \alpha \) and so \( x \circ y \in D \), that is, \( X \circ D \subseteq D \). Now let \( a, x \in X \) be such that \( a \in D \) and \( a \circ x \in D \). Then \( \mu(x) \geq \min \{ \mu(a), \mu(a \circ x) \} = \min \{ \alpha, \alpha \} = \alpha \) and so \( x \in D \). Thus \( D \) is a deductive system of \( X \).

Conversely, suppose that \( D \) is a deductive system of \( X \). Let \( x, y \in X \). If at least one of \( x, y \in D \), then \( \mu(x \circ y) = \alpha \geq \mu(y) \) and \( \mu(x \circ y) = \alpha \geq \mu(x) \). If \( x \not\in D \) and \( y \not\in D \), then \( \mu(x \circ y) \geq \beta = \mu(x) = \mu(y) \). In order to prove \( \mu(x) \geq \min \{ \mu(y), \mu(y \circ x) \} \), we consider two cases:

1. If \( x \in D \), then the inequality is obvious.
2. If \( x \not\in D \) implies that \( y \not\in D \) or \( y \circ x \not\in D \), so that \( \mu(y) = \beta \) or \( \mu(y \circ x) = \beta \) which implies \( \mu(x) \geq \beta = \min \{ \mu(y), \mu(y \circ x) \} \) and hence \( \mu \) is a fuzzy deductive system. Moreover, we have

\[
X_\mu = \{ x \in X \mid \mu(x) = \mu(1) \} = \{ x \in X \mid \mu(x) = \alpha \} = D.
\]

\( \square \)

**Corollary 3.9.** Let \( X \) be a BE-semigroup and \( \chi_D \) be the characteristic function of a subset \( D \subset X \). Then \( \chi_D \) is a fuzzy deductive system if and only if \( D \) is a deductive system.

**Definition 3.3.** Let \( \xi : X \to Y \) be a mapping of BE-semigroups. If \( \mu \) is a fuzzy set of \( Y \), then the fuzzy subset \( \nu = \mu \circ \xi \) in \( X \) (i.e., the fuzzy subset defined by \( \mu^\xi(x) = \nu(x) = \mu(\xi(x)) \) for all \( x \in X \)) is called the preimage of \( \mu \) under \( \xi \).

**Theorem 3.10.** Let \( \xi : X \to Y \) be a BE-homomorphism of BE-semigroups. If \( \mu \) is a fuzzy deductive system of \( Y \), then \( \mu^\xi \) is a fuzzy deductive system of \( X \).

**Proof.** Let \( x, y \in X \). Then we have

\[
\mu^\xi(x \circ y) = \mu(\xi(x \circ y)) = \mu(\xi(x) \circ \xi(y)) \geq \mu(\xi(x)) = \mu^\xi(x),
\]

\[
\mu^\xi(x \circ y) = \mu(\xi(x \circ y)) = \mu(\xi(x) \circ \xi(y)) \geq \mu(\xi(y)) = \mu^\xi(y),
\]

\( \square \)
Let $\mu^\xi$ be a fuzzy deductive system of $X$. \hfill \Box

**Theorem 3.11.** Let $\mu$ be a fuzzy set of $Y$ and let $\xi : X \to Y$ be a BE-epimorphism of BE-semigroups. If $\mu^\xi$ is a fuzzy deductive system of $X$, then $\mu$ is a fuzzy deductive system of $Y$.

**Proof.** For any $x, y \in Y$, there exist $a, b \in X$ such that $x = \xi(a)$ and $y = \xi(b)$. It follows that

$$
\mu(x \circ y) = \mu(\xi(a) \circ \xi(b)) = \mu(\xi(a \circ b)) = \mu^\xi(a \circ b) \geq \mu^\xi(a) = \mu(x),
$$

and

$$
\mu(x \circ y) = \mu(\xi(a) \circ \xi(b)) = \mu(\xi(a \circ b)) = \mu^\xi(a \circ b) \geq \mu^\xi(b) = \mu(\xi(b)) = \mu(y),
$$

hence $\mu$ is a fuzzy deductive system of $Y$. \hfill \Box

**Definition 3.4.** A BE-semigroup $X$ is said to satisfy the ascending (resp. descending) chain condition if for every ascending (resp. descending) sequence $A_1 \subseteq A_2 \subseteq A_3 ...$ (resp. $A_1 \supseteq A_2 \supseteq A_3 ...$) of deductive systems of $X$, there exists a natural number $n$ such that $A_n = A_k$ for all $n \geq k$. If $X$ satisfies the ascending chain condition, we say $X$ is a Noetherian BE-semigroup.

**Theorem 3.12.** Let $X$ be a BE-semigroup. If every fuzzy deductive system of $X$ has finite number of values, then $X$ is Noetherian.

**Proof.** Suppose that $X$ is not Noetherian. Then, there exists a strictly descending chain $X = A_1 \supseteq A_2 \supseteq A_3 ...$ of deductive systems of $X$. Define a fuzzy set $\mu$ in $X$ by

$$
\mu(x) = \begin{cases} 
\frac{n + 1}{n} & \text{if } x \in A_n - A_{n+1}, \\
1 & \text{if } x \in \bigcap_{n=1}^{\infty} A_n,
\end{cases}
$$

for all $x \in X$. We prove that $\mu$ is a fuzzy deductive system. Let $x, y \in X$.

If $x \circ y \in \bigcap_{n=1}^{\infty} A_n$, then obviously $\mu(x \circ y) = 1 \geq \min \{\mu(x), \mu(y)\}$.

If $x \circ y \notin \bigcap_{n=1}^{\infty} A_n$, then $x \circ y \in A_t - A_{t+1}$ for some $t \in \mathbb{N}^*$. If $x \in \bigcap_{n=1}^{\infty} A_n$, or $y \in \bigcap_{n=1}^{\infty} A_n$, then $x \circ y \notin \bigcap_{n=1}^{\infty} A_n$, a contradiction. Hence $x \notin \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$.

So $x \in A_m - A_{m+1}$ and $y \in A_j - A_{j+1}$ for some $m, j \in \mathbb{N}^*$. Without loss of generality, we assume that $m \leq j$. Then clearly, $y \in A_m$. It follows that $x \circ y \in A_m$. If $t < m$,
Let $A_m \subseteq A_{m+1} \subseteq A_t$ and so $x \circ y \in A_{t+1}$, a contradiction. Hence $m \leq t$. Thus

$$\mu(x \circ y) = \frac{m}{m+1} \geq \min \{\mu(x), \mu(y)\} \leq \frac{m}{m+1}.$$ 

Let $x, y \in X$. Suppose that $y \circ x \in A_k - A_{k+1}$ and $y \in A_r - A_{r+1}$ for some $k, r \in \mathbb{N}^*$. Without loss of generality, we assume that $k \leq r$. Then clearly $y \in A_k$. Hence $\mu(x) \geq \frac{k}{k+1} = \min \{\mu(y \circ x), \mu(y)\}$.

If $y \circ x, y \in \bigcap_{n=1}^{\infty} A_n$, then $x \in \bigcap_{n=1}^{\infty} A_n$. Thus $\mu(x) = 1 \geq \min \{\mu(y \circ x), \mu(y)\}$.

If $y \circ x \notin \bigcap_{n=1}^{\infty} A_n$ and $y \in \bigcap_{n=1}^{\infty} A_n$, then there exists $k \in \mathbb{N}^*$ such that $y \circ x \in A_k - A_{k+1}$.

It follows that $x \in A_k$ and so we have $\mu(x) \geq \frac{k}{k+1} = \min \{\mu(y \circ x), \mu(y)\}$.

If $y \circ x \in \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$, then there exists $i \in \mathbb{N}^*$ such that $y \in A_i - A_{i+1}$. It follows that $x \in A_i$. Hence $\mu(x) \geq \frac{i}{i+1} = \min \{\mu(y \circ x), \mu(y)\}$. Therefore, $\mu$ is a fuzzy deductive system and has infinite number of different values, which is a contradiction.

**Definition 3.5.** Let $\mu_1, \mu_2, \ldots, \mu_n$ be $n$ fuzzy subsets of BE-semigroups $X_1, X_2, \ldots, X_n$, respectively. Then the direct product of finite fuzzy subsets of BE-semigroup is denoted by $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ and is defined as $\mu_1 \times \mu_2 \times \ldots \times \mu_n : X_1 \times X_2 \times \ldots \times X_n \rightarrow [0, 1]$ by $(\mu_1 \times \mu_2 \times \ldots \times \mu_n)(s_1, s_2, \ldots, s_n) = \min\{\mu_1(s_1), \mu_2(s_2), \ldots, \mu_n(s_n)\}$.

**Theorem 3.13.** Let $\mu_1, \mu_2, \ldots, \mu_n$ be $n$ fuzzy left (resp. right) deductive systems of BE-semigroups $X_1, X_2, \ldots, X_n$, respectively. Then $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ is a fuzzy left (resp. right) deductive system of BE-semigroup $X_1 \times X_2 \times \ldots \times X_n$.

**Proof.** Let $\mu_1, \mu_2, \ldots, \mu_n$ be $n$ fuzzy left deductive systems of BE-semigroups $X_1, X_2, \ldots, X_n$, respectively and let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X_1 \times X_2 \times \ldots \times X_n$. Then

$$(\mu_1 \times \mu_2 \times \ldots \times \mu_n)((x_1, x_2, \ldots, x_n) \circ (y_1, y_2, \ldots, y_n))$$

$$= (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x_1 \circ y_1, x_2 \circ y_2, \ldots, x_n \circ y_n)$$

$$= \min \{\mu_1(x_1 \circ y_1), \mu_2(x_2 \circ y_2), \ldots, \mu_n(x_n \circ y_n)\}$$

$$\geq \min \{\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_n)\}$$

$$= (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(y_1, y_2, \ldots, y_n),$$

and

$$(\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x_1, x_2, \ldots, x_n)$$

$$= \min \{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\}$$

$$\geq \min \{\min \{\mu_1(y_1), \mu_1(y_1 \times x_1)\}, \ldots, \min \{\mu_n(y_n), \mu_n(y_n \times x_n)\}\}$$

$$\geq \min \{\min \{\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_n)\}, \min \{\mu_1(y_1 \times x_1), \ldots, \mu_n(y_n \times x_n)\}\}$$

$$= \min \{\min \{\mu_1(y_1, y_2, \ldots, y_n), \mu_1(x_1, x_2, \ldots, x_n)\}, \min \{\mu_1(y_1 \times x_1, y_2, \ldots, y_n), \mu_1(x_1, x_2, \ldots, x_n)\}\}. $$

Consequently, $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ is a fuzzy left deductive system of BE-semigroup $X_1 \times X_2 \times \ldots \times X_n$. Similarly we have the desired result for the right case.

**Definition 3.6.** A fuzzy deductive system $\mu$ of $X$ is said to be normal if there exists $x \in X$ such that $\mu(x) = 1$.

Let $\mathfrak{D}(X)$ denote the set of all normal fuzzy deductive system of $X$.

**Theorem 3.14.** Let $\mu$ be a fuzzy deductive system of $X$ and let $\mu^+$ be a fuzzy set in $X$ defined by $\mu^+(x) = \mu(x) + 1 - \mu(1)$ for all $x \in X$. Then $\mu^+ \in \mathfrak{D}(X)$ and $\mu \subseteq \mu^+$. 
Proof. Clearly, \( \mu^+(1) = 1 \). Let \( x, y \in X \). Then \( \mu^+(x \odot y) = \mu(x \odot y) + 1 - \mu(1) \geq \mu(y) + 1 - \mu(1) = \mu^+(y) \). Similarly, we have that \( \mu^+(x \odot y) \geq \mu^+(x) \). Let \( z, w \in X \). Then

\[
\mu^+(z) = \mu(z) + 1 - \mu(1) \\
\geq \min \{ \mu(w), \mu(w * z) \} + 1 - \mu(1) \\
= \min \{ \mu(w) + 1 - \mu(1), \mu(w * z) + 1 - \mu(1) \} \\
= \min \{ \mu^+(w), \mu^+(w * z) \}.
\]

Therefore, \( \mu^+ \in \mathcal{D}(X) \), and obviously \( \mu \subseteq \mu^+ \).

\[\Box\]

**Corollary 3.15.** If \( \mu \) is a fuzzy deductive system of \( X \) satisfying \( \mu^+(s) = 0 \) for some \( s \in X \), then \( \mu(s) = 0 \).

**Theorem 3.16.** Let \( \mu \in \mathcal{D}(X) \) be non-constant such that is a maximal element of the poset \( \left( \mathcal{D}(X), \subseteq \right) \). Then \( \mu \) takes only the values 0 and 1.

**Proof.** Since \( \mu \) is normal, we have \( \mu(1) = 1 \). Let \( x \in X \) be such that \( \mu(x) \neq 1 \). We have to prove that \( \mu(x) = 0 \). If not, then there exists \( a \in X \) such that \( 0 < \mu(a) < 1 \). Define a fuzzy set \( \nu \) in \( X \) by \( \nu(x) = \frac{\mu(x) + \mu(a)}{2} \), for all \( x \in X \). Clearly, \( \nu \) is well-defined. Let \( x, y \in X \). Then

\[
\nu(x \odot y) = \frac{\mu(x \odot y) + \mu(a)}{2} \\
\geq \frac{\mu(y) + \mu(a)}{2} = \nu(y).
\]

In a similar way we get \( \nu(x \odot y) \geq \nu(x) \). Let \( x \in X \). Then

\[
\nu(x) = \frac{\mu(x) + \mu(a)}{2} \\
\geq \frac{\min \{ \mu(y), \mu(y * x) \} + \mu(a)}{2} \\
= \min \left\{ \frac{\mu(y) + \mu(a)}{2}, \frac{\mu(y * x) + \mu(a)}{2} \right\} \\
= \min \{ \nu(y), \nu(y * x) \}.
\]

Hence \( \nu \) is a fuzzy deductive system of \( X \). By Theorem 3.14 \( \nu^+ \in \mathcal{D}(X) \), where \( \nu^+ \) is defined by \( \nu^+(x) = \nu(x) + 1 - \nu(1) \), for all \( x \in X \). Note that

\[
\nu^+(a) = \nu(a) + 1 - \nu(1) \\
= \frac{\mu(a) + \mu(a)}{2} + 1 - \frac{\mu(1) + \mu(a)}{2} \\
= \frac{\mu(a) + \mu(a)}{2} + 1 - \frac{1 + \mu(a)}{2} \\
= \frac{\mu(a) + 1}{2} > \frac{\mu(a)}{2},
\]

and \( \nu^+(a) < 1 = \nu^+(1) \). It follows that \( \nu^+ \) is non-constant, and \( \mu \) is not a maximal element of \( \left( \mathcal{D}(X), \subseteq \right) \). This is a contradiction. \[\Box\]
4. Quotient self-distributive BE-semigroups induced by fuzzy deductive system

Let $D$ be a deductive system of a self-distributive BE-semigroup $X$. We define a relation $\sim_D$ on $X$ as follows:

$$x \sim_D y \text{ if and only if } x \ast y \in D \text{ and } y \ast x \in D.$$ 

Then $\sim_D$ is an equivalence relation on $X$ (see [4]). We denote the equivalence class containing $x$ by $D_x$ and the set of all equivalence classes in $X$ by $X/D$, that is,

$$D_x = \{ y \in X \mid y \sim_D x \} \quad \text{and} \quad X/D = \{ D_x \mid x \in X \}.$$ 

Define binary operations $\circ$ and $\ast$ on $X/D$ by $D_x \circ D_y = D_{x \circ y}$ and $D_x \ast D_y = D_{x \ast y}$ for all $D_x, D_y \in X/D$.

Then $(X/D, \circ, \ast, D_1)$ is a self-distributive BE-semigroup (see [4]). Let $\mu$ be a non-constant fuzzy deductive system of a self-distributive BE-semigroup $X$ and define a binary relation, denoted by $x \sim_\mu y$, on $X$ as follows:

$$x \sim_\mu y \text{ if and only if } \mu(x \ast y) = \mu(1) \text{ and } \mu(y \ast x) = \mu(1),$$

for every $x, y \in X$.

**Lemma 4.1.** $\sim_\mu$ is an equivalence relation of a self-distributive BE-semigroup $X$.

**Proof.** For any $x \in X$, we have $\mu(x \ast x) = \mu(1)$. Hence $x \sim_\mu x$. The symmetry of $\sim_\mu$ follows directly from the definition. For any $x, y, z \in X$, if $x \sim_\mu y$ and $y \sim_\mu z$, then $\mu(x \ast y) = \mu(y \ast z) = \mu(z \ast z) = \mu(1)$ and so $x \ast y, y \ast x, z \ast y, y \ast z \in X_\mu$. Since $\mu((y \ast z) \ast (x \ast y)) = \mu(1)$, by Corollary 3.4, we have $(x \ast y) \ast (x \ast z) \in X_\mu$ and $x \ast z \in X_\mu$, that is, $\mu(x \ast z) = \mu(1)$. Similarly, we have $\mu(z \ast x) = \mu(1)$. Therefore, $\sim_\mu$ is an equivalence relation on $X$. 

**Theorem 4.2.** $\sim_\mu$ is a congruence relation on a self-distributive BE-semigroup $X$.

We denote $\mu_x = \{ y \in X \mid y \sim_\mu x \}$ the equivalence class containing $x$ and $X/\mu = \{ \mu_x \mid x \in X \}$ the set of all equivalence classes of $X$. Define binary operations $\circ$ and $\ast$ on $X/\mu$ by $\mu_x \circ \mu_y = \mu_{x \circ y}$ and $\mu_x \ast \mu_y = \mu_{x \ast y}$. Note that $\mu_x = \mu_y$ if and only if $x \sim_\mu y$.

**Theorem 4.3.** If $\mu$ is a fuzzy deductive system of a self-distributive BE-semigroup $X$, then $(X/\mu, \circ, \ast, \mu_1)$ is a self-distributive BE-semigroup.

**Proof.** Clearly $(X/\mu, \circ, \ast, \mu_1)$ is a BE-algebra. Let $\mu_x = \mu_y$ and $\mu_y = \mu_z$. Then $x \ast y, y \ast x, u \ast v, v \ast u \in X_\mu$. Since $X_\mu$ is a deductive system, we have $(x \circ u) \ast (x \circ v) = x \circ (u \ast v) \in X_\mu$ and $(x \circ v) \ast (x \circ v) = x \circ (v \ast v) \in X_\mu$. Thus $(x \circ u) \sim_\mu (x \circ v) \sim_\mu (x \circ v)$. On the other hand, $(x \circ v) \ast (y \circ v) = (x \circ y) \circ v \in X_\mu$ and $(y \circ v) \ast (y \circ v) = (y \circ x) \circ v \in X_\mu$. Hence $(x \circ v) \sim_\mu (y \circ v)$, and so $\mu_{x \circ v} = \mu_{y \circ v}$. This shows that $\circ$ is well-defined. Therefore, it is easy to prove that $(X/\mu, \circ)$ is a semigroup. Moreover, for any $\mu_x, \mu_y, \mu_z \in X/\mu$, we obtain $\mu_x \circ (\mu_y \circ \mu_z) = \mu_x \circ (\mu_z \circ \mu_y) = \mu_x \circ \mu_{x \circ (y \circ z)} = \mu_{x \circ (y \circ z)} \circ \mu_x = (\mu_x \circ \mu_y) \circ \mu_z$. Similarly, $\mu_x \circ (\mu_y \circ \mu_z) = (\mu_x \circ \mu_y) \circ \mu_z$. Thus, $X/\mu$ is a BE-semigroup. Let $\mu_x, \mu_y, \mu_z \in X/\mu$. Then $\mu_x \circ (\mu_y \circ \mu_z) = \mu_x \circ \mu_{x \circ (y \circ z)} = \mu_{x \circ (y \circ z)} \circ \mu_x = (\mu_x \circ \mu_y) \circ \mu_z = (\mu_x \circ \mu_y) \circ \mu_z$. Therefore, $(X/\mu, \circ, \ast, \mu_1)$ is a self-distributive BE-semigroup.

**Theorem 4.4.** (BE-Homomorphism Theorem) Let $X$ and $Y$ be self-distributive BE-semigroups, $\xi : X \to Y$ a BE-epimorphism and $\mu$ a fuzzy deductive system. Then $X/(\mu \circ \xi) \equiv Y/\mu$. 

Proof. By Theorem 3.10 we have that $\mu \circ \xi$ is a fuzzy deductive system. Then, by Theorem 4.3, $(X/(\mu \circ \xi), \oplus, \ominus, (\mu \circ \xi)_1)$ and $(Y/\mu, \oplus', \ominus', \mu_1)$ are self-distributive BE-semigroups. Define $\psi : X/(\mu \circ \xi) \to Y/\mu$ by

$$
\psi((\mu \circ \xi)_x) = \mu_{\xi(x)}.
$$

For any $(\mu \circ \xi)_x, (\mu \circ \xi)_y \in X/(\mu \circ \xi)$, we have

$$(\mu \circ \xi)_x = (\mu \circ \xi)_y \iff (\mu \circ \xi)(x * y) = (\mu \circ \xi)(y * x) = (\mu \circ \xi)(1)$$
$$\iff \mu(\xi(x * y)) = \mu(\xi(y * x)) = \mu(\xi(1))$$
$$\iff \mu(\xi(x)) \circ \xi(y) = \mu(\xi(y)) \circ \xi(x) = \mu(1)$$
$$\iff \mu_{\xi(x)} = \mu_{\xi(y)}$$

Hence $\psi$ is well-defined and injective. For all $(\mu \circ \xi)_x, (\mu \circ \xi)_y \in X/(\mu \circ \xi)$, we get

$$
\psi((\mu \circ \xi)_x \oplus (\mu \circ \xi)_y) = \psi((\mu \circ \xi)_{x+y}) = \mu_{\xi(x+y)} = \mu_{\xi(x) \circ \xi(y)} = \psi((\mu \circ \xi)_x) \ominus' \psi((\mu \circ \xi)_y),
$$

and

$$
\psi((\mu \circ \xi)_x \otimes (\mu \circ \xi)_y) = \psi((\mu \circ \xi)_{x\circ y}) = \mu_{\xi(x \circ y)} = \mu_{\xi(x) \circ \xi(y)} = \psi((\mu \circ \xi)_x) \ominus' \psi((\mu \circ \xi)_y).
$$

So $\psi$ is a BE-homomorphism of self-distributive BE-semigroups. Let $\mu_z \in Y/\mu$. Since $\xi$ is a BE-epimorphism, there exists $x \in X$ such that $\xi(x) = z$. So $\psi((\mu \circ \xi)_x) = \mu_{\xi(x)} = \mu_z$. Hence $\psi$ is a BE-epimorphism. Therefore, $X/(\mu \circ \xi) \cong Y/\mu$. □

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References


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