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Fuzzy deductive systems in BE-semigroups

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Abstract. In this paper, we introduce the notion of fuzzy deductive systems and investigate some of their properties. Also we give the construction of quotient self-distributive BE-semigroup X/μ induced by a fuzzy deductive system μ and discuss their interesting properties.

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1. Introduction

Imai and Iséki introduced two classes of abstract algebras, namely, BCK-algebras and BCI-algebras [5], [6]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [11], Neggers and Kim introduced the notion of d-algebras which is a generalization of BCK-algebras. Moreover, Jun et al. [7] introduced a new notion, called a BH-algebra, which is a generalization of BCK/BCI-algebras. Recently, as another generalization of BCK-algebras, the notion of a BE-algebra was introduced by Kim and Kim [9]. They provided an equivalent condition of the filters in BE-algebras using the notion of upper sets. In [2], [3], Ahn and So introduced the notion of ideals in BE-algebras and proved several characterizations of such ideals. In [1], Ahn and Kim combined BE-algebras and semigroups and introduced the notion of BE-semigroups. Also, congruences and BE-Relations in BE-Algebras was studied by Yon et al. [15]. Recently, Handam introduced the notion of BE-homomorphisms between BE-semigroups [4].

The theory of fuzzy sets was first developed by Zadeh [16] and has been applied to many branches in mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [13] and he introduced the notion of fuzzy subgroups. In 1975, Zadeh [17] introduced the concept of interval valued fuzzy subset, where the values of the membership functions are intervals of numbers instead of the numbers. Later on, Song et al. [14] introduced the concept of a fuzzy ideals in BE-algebras. Recently, Rezaei and Saeid [12] introduced the concepts of fuzzy BE-subalgebras and fuzzy topological BE-algebras. In this paper, we introduce the concept of fuzzy deductive systems and investigate some of their properties. We give the construction of quotient self-distributive BE-semigroup X/μ via a fuzzy deductive system μ. In addition, we establish a generalization of fundamental BE-homomorphism theorem in self-distributive BE-semigroups by using fuzzy deductive systems.

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2. Preliminaries

In this section we cite some elementary aspects that will be used in the sequel of this paper.

Definition 2.1. [9]. An algebra $(X, *, 1)$ of type $(2, 0)$ is called a *BE-algebra* if
$(BE1)$ $x * x = 1$ for all $x \in X$,
$(BE2)$ $x * 1 = 1$ for all $x \in X$,
$(BE3)$ $1 * x = x$ for all $x \in X$,
$(BE4)$ $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

Example 2.1. [9]. Let $X = \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	0	a	b	c	d	0
0	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

We can define a relation "$\leq$"on X by $x \leq y$ if and only if $x * y = 1$.

In an BE-algebra, the following identities are true (see [9]):
$(a1)$ $x * (y * x) = 1$.
$(a2)$ $x * ((x * y) * y)) = 1$.

Definition 2.2. [9]. A BE-algebra $(X, *, 1)$ is said to be *self-distributive* if $x*(y*z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Example 2.2. [9]. Let $X = \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
0	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then it is easy to see that X is a self-distributive BE-algebra.

Definition 2.3. [1]. An algebraic system $(X; \odot, *, 1)$ is called a *BE-semigroup* if it satisfies the following:
(i) $(X; \odot)$ is a semigroup,
(ii) $(X; *, 1)$ is a BE-algebra,
(iii) the operation "$\odot$" is distributive (on both sides) over the operation "$*$", that is,

$$x \odot (y * z) = (x \odot y) * (x \odot z) \text{ and } (x * y) \odot z = (x \odot z) * (y \odot z) \text{ for all } x, y, z \in X.$$

Example 2.3. [1]. Define two operations "$\odot$" and "$*$" on a set $X = \{1, a, b, c\}$ as follows:

$\odot$	1	a	b	c
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
c	1	a	b	c

$*$	1	a	b	c
0	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	c

It is easy to see that $(X;\odot,*,1)$ is a BE-semigroup.

Definition 2.4. [4]. A BE-semigroup $(X;\odot,*,1)$ is said to be *self-distributive BE-semigroup* if X is self-distributive BE-algebra.

Proposition 2.1. [1]. *Let $(X;\odot,*,1)$ be a BE-semigroup. Then*
(i) $(\forall x \in X)$ $(1 \odot x = x \odot 1 = 1)$,
(ii) $(x, y, z \in X)$ $(x \leq y \Rightarrow x \odot z \leq y \odot z, z \odot x \leq z \odot y)$.

Definition 2.5. [1]. Let $(X;\odot,*,1)$ be a BE-semigroup. A nonempty subset D of X is called a *left (*resp.*, right) deductive system* if it satisfies:
$(ds1)$ $X \odot D \subseteq D$ *(*resp. $(D \odot X \subseteq D)$),
$(ds2)$ $(\forall a \in D)$ $((\forall x \in X)$ $(a * x \in D \Rightarrow x \in D)$.

Both left and right deductive system is a two sided deductive system or simply deductive system.

Example 2.4. [1]. Let $X = \{x, y, z, 1\}$ be a set with the following Cayley tables:

$\odot$	1	x	y	z
1	1	1	1	1
x	1	x	1	1
y	1	1	y	z
z	1	1	z	y

$*$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	1	1	z
z	1	1	1	1

It is easy to show that $(X;\odot,*,1)$ is a BE-semigroup and $D = \{1, x\}$ is an left deductive system of X.

Definition 2.6. [4]. Let X and Y be two BE-semigroups. A mapping $\psi : X \to Y$ is called a *BE-homomorphism* if for all $a, b \in X$, $\psi(a * b) = \psi(a) * \psi(b)$ and $\psi(a \odot b) = \psi(a) \odot \psi(b)$.

Proposition 2.2. [4]. *Suppose that $\psi : X \to Y$ is a BE-homomorphism of BE-semigroups. Then $\psi(1) = 1$.*

A BE-homomorphism ψ is called a *BE-monomorphism* (resp. *BE-epimorphism*) if it is injective (resp. surjective). A bijective BE-homomorphism is called a *BE-isomorphism.* For any BE-homomorphism $\psi : X \to Y$, the set $\{x \in X \mid \psi(x) = 1\}$ is called the *kernel* of ψ, denoted by $Ker(\psi)$ and the set $\{\psi(x) \mid x \in X\}$ is called the *image* of ψ, denoted by $Im(\psi)$. We denote by $Hom(X, Y)$ the set of all BE-homomorphisms of BE-semigroups from X to Y.

We now review some fuzzy logic concepts. The readers are referred to [10] for some basic definitions and results on fuzzy sets and fuzzy algebras, not given in this paper. Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0, 1]$. For any $t \in [0, 1]$, the set $U(\mu, t) = \{x \in A : \mu(x) \geq t\}$ is called level subset of μ. Let ξ be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B. The inverse image of B, denoted $\xi^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{\xi^{-1}(B)}$ defined by $\mu_{\xi^{-1}(B)}(x) = \mu_B(\xi(x))$ for all $x \in X$. Conversely, let A be a fuzzy set in X with membership function μ_A Then the image of A, denoted by ξ (A), is the fuzzy set in Y such that:

$$\mu_{\xi(A)}(y) = \begin{cases} \sup\limits_{z \in \xi^{-1}(y)} \mu_A(z), & \text{if } \xi^{-1}(y) = \{x : \xi(x) = y\} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy set A in a BE-semigroup X with the membership function μ_A is said to have the *sup property* if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that

$$\mu_A(x_0) = \sup_{t \in T} \mu_A(t).$$

3. Fuzzy deductive systems

In what follows, X denotes a BE-semigroup, A or μ_A denotes a fuzzy set A in X.

Definition 3.1. A fuzzy set μ in X is called a *fuzzy deductive system of* X if it satisfies the following conditions*:*
$(FD1)$ $\mu(x \odot y) \geq \mu(y)$ for all $x, y \in X$,
$(FD2)$ $\mu(x \odot y) \geq \mu(x)$ for all $x, y \in X$,
$(FD3)$ $\mu(x) \geq \min\{\mu(y), \mu(y * x)\}$ for all $x, y \in X$.

Note that μ is a *fuzzy left deductive system of* X if it satisfies $(FD1)$ and $(FD3)$, and μ is a *fuzzy right deductive system of* X if it satisfies $(FD2)$ and $(FD3)$.

Example 3.1. Let $X = \{1, x, y, z\}$ be the set with the following Cayley tables:

$\odot$	1	x	y	z
1	1	1	1	1
x	1	x	1	1
y	1	1	y	z
z	1	1	z	y

$*$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	1	1	z
z	1	1	1	1

Then $(X; \odot, *, 1)$ is a BE-semigroup (see [1]). Let μ be a fuzzy set in X defined by $\mu(1) = t_0$, $\mu(x) = t_1$, $\mu(y) = \mu(z) = t_2$, where $t_0 > t_1 > t_2$ in $[0, 1]$. Then μ is a fuzzy deductive system of X.

Lemma 3.1. *If D is a fuzzy left (resp. right) deductive system of X, then for all $x \in X$*

$$\mu_D(1) \geq \mu_D(x).$$

Proof. Let $x \in X$. Since D is a fuzzy left (resp. right) deductive system of X, it follows that $\mu_D(1) = \mu_D(1 \odot x) \geq \mu_D(x)$ (resp. $\mu_D(1) = \mu_D(x \odot 1) \geq \mu_D(x)$). □

Theorem 3.2. *Let D be a fuzzy left (resp. right) deductive system of X. If there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} \mu_D(x_n) = 1$, then $\mu_D(1) = 1$.*

Proof. By Lemma 3.1, we have $\mu_D(1) \geq \mu_D(x)$ for $x \in X$. Consider $1 \geq \mu_D(1) \geq \lim_{n \to \infty} \mu_D(x_n) = 1$. Therefore, $\mu_D(1) = 1$. □

Theorem 3.3. *Let μ be a fuzzy left (resp. right) deductive system of X. Then the set $X_\mu = \{x \in X \mid \mu(x) = \mu(1)\}$ is a left (resp. right) deductive system of X.*

Proof. Let μ be a fuzzy left deductive system of X. Let $X_\mu = \{x \in X \mid \mu(x) = \mu(1)\}$. If $x \in X$ and $y \in X_\mu$, then $\mu(y) = \mu(1)$. Since $\mu(x \odot y) \geq \mu(y) = \mu(1)$, it follows that $x \odot y \in X_\mu$ so that $X \odot X_\mu \subseteq X_\mu$. Now let $x, y \in X$ be such that $y \in X_\mu$ and $y * x \in X_\mu$. Then $\mu(x) \geq \min\{\mu(y), \mu(y * x)\} = \min\{\mu(1), \mu(1)\} = \mu(1)$, and thus $x \in X_\mu$. Therefore, X_μ is a left deductive system of X. Similarly we have the desired result for the right case. □

Corollary 3.4. *If μ is a fuzzy deductive system of X, then the set $X_\mu = \{x \in X \mid \mu(x) = \mu(1)\}$ is a deductive system of X.*

Theorem 3.5. *Let μ be a fuzzy set in X. Then μ is a fuzzy deductive system of X if and only if the nonempty level subset $U(\mu,t)$, $t \in Im(\mu)$ is a deductive system of X.*

Proof. Let μ be a fuzzy deductive system of X and the level subset $U(\mu,t) = \{x \in A : \mu(x) \geq t\}$ of μ. Let $x \in X$ and $y \in U(\mu,t)$. Then $\mu(y) \geq t$. Since $\mu(x \odot y) \geq \mu(y) \geq t$, it follows that $x \odot y \in U(\mu,t)$ so that $X \odot U(\mu,t) \subseteq U(\mu,t)$. Let $y \in X$ and $x \in U(\mu,t)$. Then $\mu(x) \geq t$. Since $\mu(x \odot y) \geq \mu(x) \geq t$, it follows that $x \odot y \in U(\mu,t)$ so that $U(\mu,t) \odot X \subseteq U(\mu,t)$. Now, let $x,y \in X$ be such that $y \in U(\mu,t)$ and $y * x \in U(\mu,t)$. Then $\mu(x) \geq \min\{\mu(y), \mu(y*x)\} \geq t$, and thus $x \in U(\mu,t)$. Therefore, $U(\mu,t)$ a deductive system of X.

Conversely, assume that the nonempty level set $U(\mu,t)$ of μ is a deductive system of X for every $t \in [0,1]$. If $\mu(x_0 \odot y_0) < \mu(y_0)$ for some $x_0, y_0 \in X$, then by taking $t_0 = \frac{1}{2}(\mu(x_0 \odot y_0) + \mu(y_0))$ we have $\mu(x_0 \odot y_0) < t_0 < \mu(y_0)$. Thus $y_0 \in U(\mu,t_0)$ and $x_0 \odot y_0 \notin U(\mu,t_0)$, a contradiction and so $\mu(x_0 \odot y_0) \geq \mu(y_0)$ for all $x_0, y_0 \in X$. If $\mu(x_1 \odot y_1) < \mu(x_1)$ for some $x_1, y_1 \in X$, then by taking $t_1 = \frac{1}{2}(\mu(x_1 \odot y_1) + \mu(x_1))$ we have $\mu(x_1 \odot y_1) < t_1 < \mu(x_1)$. Thus $x_1 \in U(\mu,t_1)$ and $x_1 \odot y_1 \notin U(\mu,t_1)$, which is also a contradiction and so $\mu(x_1 \odot y_1) \geq \mu(x_1)$ for all $x_1, y_1 \in X$. Next, if $\mu(x_2) < \min\{\mu(y_2), \mu(y_2 * x_2)\}$ for some $x_2, y_2 \in X$, then by taking $t_2 = \frac{1}{2}(\mu(x_2) + \min\{\mu(y_2), \mu(y_2 * x_2)\})$ we have $\mu(x_2) < t_2 < \min\{\mu(y_2), \mu(y_2 * x_2)\}$. Thus $y_2, y_2 * x_2 \in U(\mu,t_2)$ and $x_2 \notin U(\mu,t_2)$, which is again a contradiction and so $\mu(x_2) \geq \min\{\mu(y_2), \mu(y_2 * x_2)\}$ for all $x_2, y_2 \in X$. This completes the proof. □

Theorem 3.6. *Let μ be a fuzzy deductive system of X. Then*

$$(\forall a, b \in X) \quad (a \leq b \Rightarrow \mu(a) \leq \mu(b))$$

Proof. Let $a, b \in X$ be such that $a \leq b$. Then $a * b = 1$. It follows from $(FD3)$ and Lemma 3.1 that $\mu(b) \geq \min\{\mu(a), \mu(a*b)\} = \min\{\mu(a), \mu(1)\} = \mu(a)$. Hence $\mu(a) \leq \mu(b)$. □

Definition 3.2. For a family of fuzzy sets $\{\mu_i \mid i \in I\}$ in a BE-semigroup X, define the *joint* $\bigvee_{i\in I} \mu_i$ and *meet* $\bigwedge_{i\in I} \mu_i$ of $\{\mu_i \mid i \in I\}$ as follows:

$$\left(\bigvee_{i\in I} \mu_i\right)(x) = \sup\{\mu_i(x) \mid i \in I\}, \quad \left(\bigwedge_{i\in I} \mu_i\right)(x) = \inf\{\mu_i(x) \mid i \in I\},$$

for all $x \in X$, where I is any index set.

Consider two fuzzy sets A and B in X. Zadeh [16] gave a definition of fuzzy set inclusion with: $A \subset B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$, $\forall x \in X$.

Theorem 3.7. *The family of fuzzy deductive systems of X is a completely distributive lattice under the ordering of fuzzy set inclusion $\subset$.*

Proof. Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy deductive systems of X. Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\bigwedge_{i\in I} \mu_i$ is a fuzzy deductive systems of X. For any $x, y \in X$, we have

$$\left(\bigwedge_{i\in I} \mu_i\right)(x \odot y) = \inf\{\mu_i(x \odot y) \mid i \in I\} \geq \inf\{\mu_i(x) \mid i \in I\} = \left(\bigwedge_{i\in I} \mu_i\right)(x),$$

$$\left(\bigwedge_{i\in I} \mu_i\right)(x \odot y) = \inf\{\mu_i(x \odot y) \mid i \in I\} \geq \inf\{\mu_i(y) \mid i \in I\} = \left(\bigwedge_{i\in I} \mu_i\right)(y),$$

$$\begin{aligned}
\left(\bigwedge_{i\in I}\mu_i\right)(x) &= \inf\{\mu_i(x) \mid i\in I\} \\
&\geq \inf\{\min\{\mu_i(y),\mu_i(y*x)\} \mid i\in I\} \\
&= \min\{\inf\{\mu_i(y) \mid i\in I\},\inf\{\mu_i(y*x) \mid i\in I\}\} \\
&= \min\left\{\left(\bigwedge_{i\in I}\mu_i\right)(y),\left(\bigwedge_{i\in I}\mu_i\right)(y*x)\right\}.
\end{aligned}$$

Hence $\bigwedge_{i\in I}\mu_i$ is a fuzzy deductive system of X, completing the proof. □

Theorem 3.8. *Let D be a subset of X. Suppose that μ is a fuzzy set in X defined by*

$$\mu(x)=\begin{cases}\alpha & \text{if } x\in D,\\ \beta & \text{otherwise,}\end{cases}$$

where $\alpha>\beta$ in $[0,1]$. Then μ is a fuzzy deductive system if and only if D is a deductive system of X. Moreover, $X_\mu = D$.

Proof. Let μ be a fuzzy deductive system. Let $x\in D$ and $y\in X$. Then $\mu(x\odot y)\geq\mu(x)=\alpha$ and so $x\odot y\in D$, that is, $D\odot X\subseteq D$. Let $y\in D$ and $x\in X$. Then $\mu(x\odot y)\geq\mu(y)=\alpha$ and so $x\odot y\in D$, that is, $X\odot D\subseteq D$. Now let $a,x\in X$ be such that $a\in D$ and $a*x\in D$. Then $\mu(x)\geq\min\{\mu(a),\mu(a*x)\}=\min\{\alpha,\alpha\}=\alpha$ and so $x\in D$. Thus D is a deductive system of X.

Conversely, suppose that D is a deductive system of X. Let $x,y\in X$. If at least one of $x,y\in D$, then $\mu(x\odot y)=\alpha\geq\mu(y)$ and $\mu(x\odot y)=\alpha\geq\mu(x)$. If $x\notin D$ and $y\notin D$, then $\mu(x\odot y)\geq\beta=\mu(x)=\mu(y)$. In order to prove $\mu(x)\geq\min\{\mu(y),\mu(y*x)\}$, we consider two cases:
(1) If $x\in D$, then the inequality is obvious.
(2) If $x\notin D$ implies that $y\notin D$ or $y*x\notin D$, so that $\mu(y)=\beta$ or $\mu(y*x)=\beta$ which implies $\mu(x)\geq\beta=\min\{\mu(y),\mu(y*x)\}$ and hence μ is a fuzzy deductive system. Moreover, we have

$$X_\mu=\{x\in X\mid\mu(x)=\mu(1)\}=\{x\in X\mid\mu(x)=\alpha\}=D.$$

□

Corollary 3.9. *Let X be a BE-semigroup and χ_D be the characteristic function of a subset $D\subset X$. Then χ_D is a fuzzy deductive system if and only if D is a deductive system.*

Definition 3.3. Let $\xi : X\to Y$ be a mapping of BE-semigroups. If μ is a fuzzy set of Y, then the fuzzy subset $\nu=\mu\circ\xi$ in X (i.e. the fuzzy subset defined by $\mu^\xi(x)=\nu(x)=\mu(\xi(x))$ for all $x\in X$) is called the *preimage of μ under ξ*.

Theorem 3.10. *Let $\xi : X\to Y$ be a BE-homomorphism of BE-semigroups. If μ is a fuzzy deductive system of Y, then μ^ξ is a fuzzy deductive system of X.*

Proof. Let $x,y\in X$. Then we have

$$\mu^\xi(x\odot y)=\mu(\xi(x\odot y))=\mu(\xi(x)\odot\xi(y))\geq\mu(\xi(x))=\mu^\xi(x),$$

$$\mu^\xi(x\odot y)=\mu(\xi(x\odot y))=\mu(\xi(x)\odot\xi(y))\geq\mu(\xi(y))=\mu^\xi(y),$$

and

$$\begin{aligned}\mu^{\xi}(x) &= \mu(\xi(x)) \\ &\geq \min\{\mu(\xi(y)), \mu(\xi(y) * \xi(x))\} \\ &= \min\{\mu(\xi(y)), \mu(\xi(y * x))\} \\ &= \min\left\{\mu^{\xi}(y), \mu^{\xi}(y * x)\right\}.\end{aligned}$$

Therefore, μ^{ξ} is a fuzzy deductive system of X. □

Theorem 3.11. *Let μ be a fuzzy set of Y and let $\xi : X \to Y$ be a BE-epimorphism of BE-semigroups. If μ^{ξ} is a fuzzy deductive system of X, then μ is a fuzzy deductive system of Y.*

Proof. For any $x, y \in Y$, there exist $a, b \in X$ such that $x = \xi(a)$ and $y = \xi(b)$. It follows that

$$\mu(x \odot y) = \mu(\xi(a) \odot \xi(b)) = \mu(\xi(a \odot b)) = \mu^{\xi}(a \odot b) \geq \mu^{\xi}(a) = \mu(\xi(a)) = \mu(x),$$

$$\mu(x \odot y) = \mu(\xi(a) \odot \xi(b)) = \mu(\xi(a \odot b)) = \mu^{\xi}(a \odot b) \geq \mu^{\xi}(b) = \mu(\xi(b)) = \mu(y),$$

and

$$\begin{aligned}\mu(x) &= \mu(\xi(a)) = \mu^{\xi}(a) \\ &\geq \min\left\{\mu^{\xi}(b), \mu^{\xi}(b * a)\right\} \\ &= \min\{\mu(\xi(b)), \mu(\xi(b * a))\} \\ &= \min\{\mu(\xi(b)), \mu(\xi(b) * \xi(a))\} \\ &= \min\{\mu(y), \mu(y * x)\}.\end{aligned}$$

Hence μ is a fuzzy deductive system of Y. □

Definition 3.4. A BE-semigroup X is said to *satisfy the ascending (*resp. *descending) chain condition* if for every ascending (resp. descending) sequence $A_1 \subseteq A_2 \subseteq A_3 ...$ (resp. $A_1 \supseteq A_2 \supseteq A_3 ...$) of deductive systems of X, there exists a natural number n such that $A_n = A_k$ for all $n \geq k$. If X satisfies the ascending chain condition, we say X is a *Noetherian BE-semigroup.*

Theorem 3.12. *Let X be a BE-semigroup. If every fuzzy deductive system of X has finite number of values, then X is Noetherian.*

Proof. Suppose that X is not Noetherian. Then, there exists a strictly descending chain $X = A_1 \supset A_2 \supset A_3 ...$ of deductive systems of X. Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in A_n - A_{n+1}, \\ 1, & \text{if } x \in \bigcap_{n=1}^{\infty} A_n, \end{cases}$$

for all $x \in X$. We prove that μ is a fuzzy deductive system. Let $x, y \in X$.
If $x \odot y \in \bigcap_{n=1}^{\infty} A_n$, then obviously $\mu(x \odot y) = 1 \geq \min\{\mu(x), \mu(y)\}$.
If $x \odot y \notin \bigcap_{n=1}^{\infty} A_n$, then $x \odot y \in A_t - A_{t+1}$ for some $t \in \mathbb{N}^*$. If $x \in \bigcap_{n=1}^{\infty} A_n$ or $y \in \bigcap_{n=1}^{\infty} A_n$, then $x \odot y \in \bigcap_{n=1}^{\infty} A_n$, a contradiction. Hence $x \notin \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$. So $x \in A_m - A_{m+1}$ and $y \in A_j - A_{j+1}$ for some $m, j \in N^*$. Without loss of generality, we assume that $m \leq j$. Then clearly, $y \in A_m$. It follows that $x \odot y \in A_m$. If $t < m$,

then $A_m \subseteq A_{t+1} \subset A_t$ and so $x \odot y \in A_{t+1}$, a contradiction. Hence $m \le t$. Thus $\mu(x \odot y) = \frac{t}{t+1} \ge \min\{\mu(x), \mu(y)\} = \frac{m}{m+1}$.
Let $x, y \in X$. Suppose that $y * x \in A_k - A_{k+1}$ and $y \in A_r - A_{r+1}$ for some $k, r \in N^*$. Without loss of generality, we assume that $k \le r$. Then clearly $y \in A_k$. Hence $\mu(x) \ge \frac{k}{k+1} = \min\{\mu(y * x), \mu(y)\}$.
If $y * x, y \in \bigcap_{n=1}^{\infty} A_n$, then $x \in \bigcap_{n=1}^{\infty} A_n$. Thus $\mu(x) = 1 \ge \min\{\mu(y * x), \mu(y)\}$.
If $y*x \notin \bigcap_{n=1}^{\infty} A_n$ and $y \in \bigcap_{n=1}^{\infty} A_n$, then there exists $k \in \mathbb{N}^*$ such that $y*x \in A_k - A_{k+1}$. It follows that $x \in A_k$ and so we have $\mu(x) \ge \frac{k}{k+1} = \min\{\mu(y * x), \mu(y)\}$.
If $y * x \in \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$, then there exists $i \in \mathbb{N}^*$ such that $y \in A_i - A_{i+1}$. It follows that $x \in A_i$. Hence $\mu(x) \ge \frac{i}{i+1} = \min\{\mu(y * x), \mu(y)\}$. Therefore, μ is a fuzzy deductive system and has infinite number of different values, which is a contradiction. □

Definition 3.5. Let $\mu_1, \mu_2, ..., \mu_n$ be n fuzzy subsets of BE-semigroups $X_1, X_2, ..., X_n$, respectively. Then the *direct product* of finite fuzzy subsets of BE-semigroup is denoted by $\mu_1 \times \mu_2 \times ... \times \mu_n$ and is defined as $\mu_1 \times \mu_2 \times ... \times \mu_n : X_1 \times X_2 \times ... \times X_n \to [0, 1]$ by $(\mu_1 \times \mu_2 \times ... \times \mu_n)(s_1, s_2, ..., s_n) = \min\{\mu_1(s_1), \mu_2(s_2), ..., \mu_n(s_n)\}$.

Theorem 3.13. *Let $\mu_1, \mu_2, ..., \mu_n$ be n fuzzy left (resp, right) deductive systems of BE-semigroups X_1, X_2,..., X_n, respectively. Then $\mu_1 \times \mu_2 \times ... \times \mu_n$ is a fuzzy left (resp, right) deductive system of BE-semigroup $X_1 \times X_2 \times ... \times X_n$.*

Proof. Let $\mu_1, \mu_2, ..., \mu_n$ be n fuzzy left deductive systems of BE-semigroups X_1, X_2,..., X_n, respectively and let $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in X_1 \times X_2 \times ... \times X_n$. Then

$$\begin{aligned}
&(\mu_1 \times \mu_2 \times ... \times \mu_n)((x_1, x_2, ..., x_n) \odot (y_1, y_2, ..., y_n)) \\
&= (\mu_1 \times \mu_2 \times ... \times \mu_n)(x_1 \odot y_1, x_2 \odot y_2, ..., x_n \odot y_n) \\
&= \min\{\mu_1(x_1 \odot y_1), \mu_2(x_2 \odot y_2), ..., \mu_n(x_n \odot y_n)\} \\
&\ge \min\{\mu_1(y_1), \mu_2(y_2), ..., \mu_n(y_n)\} \\
&= (\mu_1 \times \mu_2 \times ... \times \mu_n)(y_1, y_2, ..., y_n),
\end{aligned}$$

and

$$\begin{aligned}
&(\mu_1 \times \mu_2 ... \times \mu_n)(x_1, x_2, ..., x_n) \\
&= \min\{\mu_1(x_1), \mu_2(x_2), ..., \mu_n(x_n)\} \\
&\ge \min\{\min\{\mu_1(y_1), \mu_1(y_1 * x_1)\}, ..., \min\{\mu_n(y_n), \mu_n(y_n * x_n)\}\} \\
&= \min\{\min\{\mu_1(y_1), ..., \mu_n(y_n)\}, \min\{\mu_1(y_1 * x_1), ..., \mu_n(y_n * x_n)\}\} \\
&= \min\{(\mu_1 \times ... \times \mu_n)(y_1, y_2, ..., y_n), (\mu_1 \times ... \times \mu_n)(y_1 * x_1, y_2 * x_2, ..., y_n * x_n)\} \\
&= \min\{(\mu_1 \times ... \times \mu_n)(y_1, y_2, ..., y_n), (\mu_1 \times ... \times \mu_n)((y_1, y_2, ..., y_n) * (x_1, x_2, ..., x_n))\}.
\end{aligned}$$

Consequently, $\mu_1 \times \mu_2 \times ... \times \mu_n$ is a fuzzy left deductive system of BE-semigroup $X_1 \times X_2 \times ... \times X_n$. Similarly we have the desired result for the right case. □

Definition 3.6. A fuzzy deductive system μ of X is said to be *normal* if there exists $x \in X$ such that $\mu(x) = 1$.

Let $\mathfrak{D}(X)$ denote the set of all normal fuzzy deductive system of X.

Theorem 3.14. *Let μ be a fuzzy deductive system of X and let μ^+ be a fuzzy set in X defined by $\mu^+(x) = \mu(x) + 1 - \mu(1)$ for all $x \in X$. Then $\mu^+ \in \mathfrak{D}(X)$ and $\mu \subseteq \mu^+$.*

Proof. Clearly, $\mu^+(1) = 1$. Let $x, y \in X$. Then $\mu^+(x \odot y) = \mu(x \odot y) + 1 - \mu(1) \geq \mu(y) + 1 - \mu(1) = \mu^+(y)$. Similarly, we have that $\mu^+(x \odot y) \geq \mu^+(x)$. Let $z, w \in X$. Then

$$\begin{aligned} \mu^+(z) &= \mu(z) + 1 - \mu(1) \\ &\geq \min\{\mu(w), \mu(w * z)\} + 1 - \mu(1) \\ &= \min\{\mu(w) + 1 - \mu(1), \mu(w * z) + 1 - \mu(1)\} \\ &= \min\left\{\mu^+(w), \mu^+(w * z)\right\}. \end{aligned}$$

Therefore, $\mu^+ \in \mathfrak{D}(X)$, and obviously $\mu \subseteq \mu^+$. □

Corollary 3.15. *If μ is a fuzzy deductive system of X satisfying $\mu^+(s) = 0$ for some $s \in X$, then $\mu(s) = 0$.*

Theorem 3.16. *Let $\mu \in \mathfrak{D}(X)$ be non-constant such that is a maximal element of the poset $(\mathfrak{D}(X), \subseteq)$. Then μ takes only the values 0 and 1.*

Proof. Since μ is normal, we have $\mu(1) = 1$. Let $x \in X$ be such that $\mu(x) \neq 1$. We have to prove that $\mu(x) = 0$. If not, then there exists $a \in X$ such that $0 < \mu(a) < 1$. Define a fuzzy set ν in X by $\nu(x) = \frac{\mu(x)+\mu(a)}{2}$, for all $x \in X$. Clearly, ν is well-defined. Let $x, y \in X$. Then

$$\begin{aligned} \nu(x \odot y) &= \frac{\mu(x \odot y) + \mu(a)}{2} \\ &\geq \frac{\mu(y) + \mu(a)}{2} \\ &= \nu(y). \end{aligned}$$

In a similar way we get $\nu(x \odot y) \geq \nu(x)$. Let $x \in X$. Then

$$\begin{aligned} \nu(x) &= \frac{\mu(x) + \mu(a)}{2} \\ &\geq \frac{\min\{\mu(y), \mu(y * x)\} + \mu(a)}{2} \\ &= \min\left\{\frac{\mu(y) + \mu(a)}{2}, \frac{\mu(y * x) + \mu(a)}{2}\right\} \\ &= \min\{\nu(y), \nu(y * x)\}. \end{aligned}$$

Hence ν is a fuzzy deductive system of X. By Theorem 3.14 $\nu^+ \in \mathfrak{D}(X)$, where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(1)$, for all $x \in X$. Note that

$$\begin{aligned} \nu^+(a) &= \nu(a) + 1 - \nu(1) \\ &= \frac{\mu(a) + \mu(a)}{2} + 1 - \frac{\mu(1) + \mu(a)}{2} \\ &= \frac{\mu(a) + \mu(a)}{2} + 1 - \frac{1 + \mu(a)}{2} \\ &= \frac{\mu(a) + 1}{2} \\ &> \mu(a), \end{aligned}$$

and $\nu^+(a) < 1 = \nu^+(1)$. It follows that ν^+ is non-constant, and μ is not a maximal element of $(\mathfrak{D}(X), \subseteq)$. This is a contradiction. □

4. Quotient self-distributive BE-semigroups induced by fuzzy deductive system

Let D be a deductive system of a self-distributive BE-semigroup X. We define a relation " $\sim_D$ " on X as follows:

$$x \sim_D y \text{ if and only if } x * y \in D \text{ and } y * x \in D.$$

Then $\sim_D$ is an equivalence relation on X (see [4]). We denote the equivalence class containing x by D_x and the set of all equivalence classes in X by X/D, that is, $D_x = \{y \in X \mid y \sim_D x\}$ and $X/D = \{D_x \mid x \in X\}$. Define binary operations $\odot'$ and $*^{\tilde{}}$ on X/D by $D_x \odot' D_y = D_{x\odot y}$ and $D_x *^{\tilde{}} D_y = D_{x*y}$ for all $D_x, D_y \in$X/D. Then $(X/D, \odot', *^{\tilde{}}, D_1)$ is a self-distributive BE-semigroup (see [4]). Let μ be a non-constant fuzzy deductive system of a self-distributive BE-semigroup X and define a binary relation, denoted by $x \sim_\mu y$, on X as follows:

$$x \sim_\mu y \text{ if and only if } \mu(x * y) = \mu(1) \text{ and } \mu(y * x) = \mu(1),$$

for every $x, y \in X$.

Lemma 4.1. *$\sim_\mu$ is an equivalence relation of a self-distributive BE-semigroup X.*

Proof. For any $x \in X$, we have $\mu(x * x) = \mu(1)$. Hence $x \sim_\mu x$. The symmetry of $\sim_\mu$ follows directly from the definition. For any $x, y, z \in X$, if $x \sim_\mu y$ and $y \sim_\mu z$, then $\mu(x*y) = \mu(y*x) = \mu(y*z) = \mu(z*y) = \mu(1)$ and so $x*y, y*x, y*z, z*y \in X_\mu$. Since $\mu((y*z)*((x*y)*(x*z))) = \mu(1)$, by Corollary 3.4, we have $(x*y)*(x*z) \in X_\mu$ and so $x * z \in X_\mu$, that is, $\mu(x * z) = \mu(1)$. Similarly, we have $\mu(z * x) = \mu(1)$. Therefore, $\sim_\mu$ is an equivalence relation on X. □

Theorem 4.2. *$\sim_\mu$ is a congruence relation on a self-distributive BE-semigroup X.*

We denote $\mu_x = \{y \in X \mid y \sim_\mu x\}$ the equivalence class containing x and $X/\mu = \{\mu_x \mid x \in X\}$ the set of all equivalence classes of X. Define binary operations $\circledcirc$ and $\circledast$ on X/μ by $\mu_x \circledcirc \mu_y = \mu_{x\odot y}$ and $\mu_x \circledast \mu_y = \mu_{x*y}$. Note that $\mu_x = \mu_y$ if and only if $x \sim_\mu y$.

Theorem 4.3. *If μ is a fuzzy deductive system of a self-distributive BE-semigroup X, then $(X/\mu, \circledcirc, \circledast, \mu_1)$ is a self-distributive BE-semigroup.*

Proof. Clearly $(X/\mu, \circledast, \mu_1)$ is a BE-algebra. Let $\mu_x = \mu_y$ and $\mu_u = \mu_v$. Then $x*y, y*x, u*v, v*u \in X_\mu$. Since X_μ is a deductive system, we have $(x\odot u)*(x\odot v) = x\odot(u*v) \in X_\mu$ and $(x \odot v) * (x\odot u) = x \odot (v * u) \in X_\mu$. Thus $(x \odot u) \sim_\mu (x \odot v)$. On the other hand, $(x\odot v)*(y\odot v) = (x*y)\odot v \in X_\mu$ and $(y\odot v)*(x\odot v) = (y*x)\odot v \in X_\mu$. Hence $(x\odot v) \sim_\mu (y\odot v)$, and so $\mu_{x\odot u} = \mu_{y\odot v}$. This shows that $\circledcirc$ is well-defined. Therefore, it is easy to prove that $(X/\mu, \circledcirc)$ is a semigroup. Moreover, for any $\mu_x, \mu_y, \mu_z \in X/\mu$, we obtain $\mu_x \circledcirc (\mu_y \circledast \mu_z) = \mu_x \circledcirc \mu_{y*z} = \mu_{x\odot(y*z)} = \mu_{(x\odot y)*(x\odot z)} = \mu_{(x\odot y)} \circledast \mu_{(x\odot z)} = (\mu_x \circledcirc \mu_y) \circledast (\mu_x \circledcirc \mu_z)$. Similarly, $(\mu_x \circledast \mu_y) \circledcirc \mu_z = (\mu_x \circledcirc \mu_z) \circledast (\mu_y \circledcirc \mu_z)$. Thus, X/μ is a BE-semigroup. Let $\mu_x, \mu_y, \mu_z \in X/\mu$. Then $\mu_x \circledast (\mu_y \circledast \mu_z) = \mu_x \circledast \mu_{y*z} = \mu_{x*(y*z)} = \mu_{(x*y)*(x*z)} = \mu_{x*y} \circledast \mu_{x*z} = (\mu_x \circledast \mu_y) \circledast (\mu_x \circledast \mu_z)$. Therefore, $(X/\mu, \circledcirc, \circledast, \mu_1)$ is a self-distributive BE-semigroup. □

Theorem 4.4. *(BE-Homomorphism Theorem) Let X and Y be self-distributive BE-semigroups, $\xi : X \to Y$ a BE-epimorphism and μ a fuzzy deductive system. Then $X/(\mu \circ \xi) \cong Y/\mu$.*

Proof. By Theorem 3.10 we have that $\mu \circ \xi$ is a fuzzy deductive system. Then, by Theorem 4.3, $(X/(\mu \circ \xi), \circledast, \circledcirc, (\mu \circ \xi)_1)$ and $(Y/\mu, \circledast^{'}, \circledcirc^{'}, \mu_1)$ are self-distributive BE-semigroups. Define $\psi : X/(\mu \circ \xi) \to Y/\mu$ by

$$\psi((\mu \circ \xi)_x) = \mu_{\xi(x)}.$$

For any $(\mu \circ \xi)_x, (\mu \circ \xi)_y \in X/(\mu \circ \xi)$, we have

$$\begin{aligned}
(\mu \circ \xi)_x = (\mu \circ \xi)_y \quad &\Leftrightarrow \quad (\mu \circ \xi)(x * y) = (\mu \circ \xi)(y * x) = (\mu \circ \xi)(1) \\
&\Leftrightarrow \quad \mu(\xi(x * y)) = \mu(\xi(y * x)) = \mu(\xi(1)) \\
&\Leftrightarrow \quad \mu(\xi(x) * \xi(y)) = \mu(\xi(y) * \xi(x)) = \mu(1) \\
&\Leftrightarrow \quad \mu_{\xi(x)} = \mu_{\xi(y)}
\end{aligned}$$

Hence ψ is well-defined and injective. For all $(\mu \circ \xi)_x, (\mu \circ \xi)_y \in X/(\mu \circ \xi)$, we get

$$\begin{aligned}
\psi((\mu \circ \xi)_x \circledast (\mu \circ \xi)_y) &= \psi((\mu \circ \xi)_{x*y}) \\
&= \mu_{\xi(x*y)} \\
&= \mu_{\xi(x)*\xi(y)} \\
&= \mu_{\xi(x)} \circledast^{'} \mu_{\xi(y)} \\
&= \psi((\mu \circ \xi)_x) \circledast^{'} \psi((\mu \circ \xi)_y),
\end{aligned}$$

and

$$\begin{aligned}
\psi((\mu \circ \xi)_x \circledcirc (\mu \circ \xi)_y) &= \psi((\mu \circ \xi)_{x \odot y}) \\
&= \mu_{\xi(x \odot y)} \\
&= \mu_{\xi(x) \odot \xi(y)} \\
&= \mu_{\xi(x)} \circledcirc^{'} \mu_{\xi(y)} \\
&= \psi((\mu \circ \xi)_x) \circledcirc^{'} \psi((\mu \circ \xi)_y).
\end{aligned}$$

So ψ is a BE-homomorphism of self-distributive BE-semigroups. Let $\mu_z \in Y/\mu$. Since ξ is a BE-epimorphism, there exists $x \in X$ such that $\xi(x) = z$. So $\psi((\mu \circ \xi)_x) = \mu_{\xi(x)} = \mu_z$. Hence ψ is a BE-epimorphism. Therefore, $X/(\mu \circ \xi) \cong Y/\mu$. □

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