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The Dynamics of an Extensible Link with one and two Moving Ends

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ABSTRACT. *In this paper the dynamical behavior of an extensible link with one fixed and one moving end, and with two moving ends is presented. The equations of motion used for the analysis of the extensible link are written. A dynamic analysis is carried out in order to understand the changes under dynamic reconfiguration and an optimal design.*

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INTRODUCTION

The design and analyze of complex rotating mechanical systems with extensible links typically means the creation of physical prototypes for evaluation. To avoid time-consuming and high costs, simulations may be used to study the combined effect of the parameters on the dynamical behavior of the mechanical system.

The study of mechanical models with extensible members was considered in [5]. The Lagranges equations of motion have been presented and the model was analyzed with regard to the extensibility of the system. Dynamic analysis of planar mechanisms with rotating slider joint and clearance was studied in [13]. The design and dynamics of systems with lumped masses was studied in [11, 3]. The impact equations using Newtons law have been discussed in [1].

The application of algebraic geometry to an extensible robotic arm for motion planning, using polynomial equations with constraints was discussed in [7]. A FEM analysis of an extensible rod (jib) mounted on mobile platform, including jib profile and stable area of operation have been discussed in [2]. The free vibration of an extensible rotating beam was studied in[8]. The discrete vibration model of a extensible crane, and the related the influence of the crane hydraulic cylinder was discussed in [10].

The post buckling of an extensible rod under axial load, and its associated kinematics and equilibrium have been studied in [4]. A non-linear formulation of the equilibrium of elastic rods under compression and bending was discussed in [9]. Hockling problems in very long extensible rods, such as the self-contacting loop, was analyzed in [14]. The shape, force, and moments that occur during hockling have been discussed.

The aim of this study is a better understanding of system dynamics when an extensible link is considered. Other aim includes an optimal design of such mechanical systems. In this regard, the dynamical behavior of an extensible link with one fixed and one moving end, and with two moving ends is considered.

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SYSTEM MODEL

In this section, extensible links as shown in Fig. 1 with one fixed end and one moving end - crank type extensible links, and two moving ends - connecting extensible links) are considered. The next assumptions are made: (i) the links are rigid. (ii) the motions is planar.

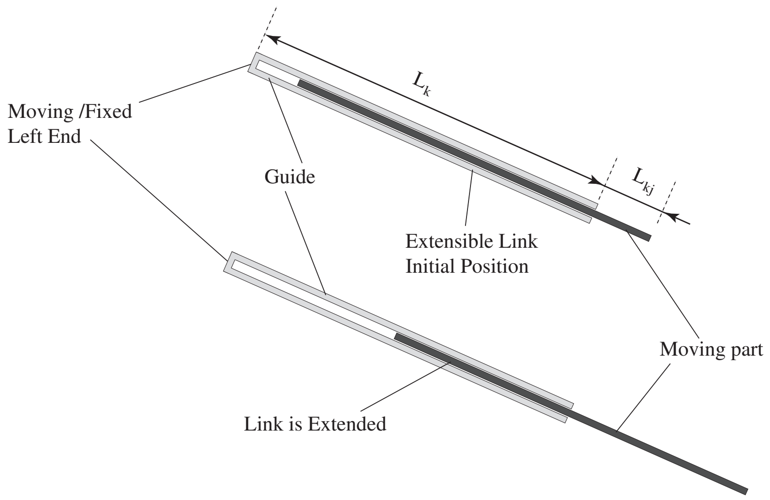


FIGURE 1. Extensible Link with Moving Ends .

Extensible Link with one fixed end - crank type extensible link. The extensible link shown in Fig. 1 is rotating about the fixed end. The position vector of the center of the mass C is given by $\mathbf{r}_C = x_C\imath + y_C\jmath$, the velocity vector is the derivative with respect to time of the position $\mathbf{v}_C = \dot{\mathbf{r}}_C = \dot{x}_C\imath + \dot{y}_C\jmath$, the acceleration vector is the double derivative with respect to time that is, $\mathbf{a}_C = \ddot{\mathbf{r}}_C = \ddot{x}_C\imath + \ddot{y}_C\jmath$. The extensible link is changing its length based on the next formula

$$L_{1e} = L_1 + L_{11} + L_{11}\cos(\theta_1), \tag{1}$$

where θ_1 vary from 0 to 2π, and $L_{11} < \dfrac{L_1}{2}$. The position and velocity of the left end of the extensible link are

$$\begin{aligned} x_l &= 0,\ y_l = 0, \\ v_{x_l} &= 0,\ v_{y_l} = 0. \end{aligned} \tag{2}$$

The position and velocity of the right end of the extensible link are

$$\begin{aligned} x_r &= L_{1e}\cos(\theta_1),\ \ y_r = L_{1e}\sin(\theta_1), \\ v_{x_r} &= -L_{11}\dot{\theta}_1\cos(\theta_1)\sin(\theta_1) - L_{2e}\dot{\theta}_1\sin(\theta_1), \\ v_{y_r} &= -L_{11}\dot{\theta}_1\sin(\theta_1)\sin(\theta_1) - L_{2e}\dot{\theta}_1\cos(\theta_1). \end{aligned} \tag{3}$$

The position and velocity of the of the mass center of the extensible link are calculated as

$$\begin{aligned} x_C &= \frac{L_{1e}}{2}\cos(\theta_1),\ \ y_C = \frac{L_{1e}}{2}\sin(\theta_1), \\ v_{x_C} &= -\frac{L_{11}}{2}\dot{\theta}_1\cos(\theta_1)\sin(\theta_1) - \frac{L_{1e}}{2}\dot{\theta}_1\sin(\theta_1), \\ v_{y_C} &= -\frac{L_{11}}{2}\dot{\theta}_1\sin(\theta_1)\sin(\theta_1) - \frac{L_{1e}}{2}\dot{\theta}_1\cos(\theta_1). \end{aligned} \tag{4}$$

Extensible Link with one fixed end and one moving end along a circular trajectory. In this case the extensible link is rotating about a fixed end, and one end of the moving part is along a circular trajectory. A geometrical method is used to describe the motion. The extensible link shown in Fig. 2 is the link denoted by CA, the length of the radius $\overline{OA}$ is R, the length of $\overline{CA}$ is l, and the length of $\overline{CA'}$ is d. Using the cosine rule in $\triangle\ OAC$ one can write

$$l^2 = R^2 + (R-d)^2 - 2R(R-d)\cos(\theta) \tag{5}$$

In $\triangle\ OAD$ the angle $\sphericalangle\ ODA$ is $90°$ so one can write

$$\cos\theta = \frac{\overline{OC}+\overline{CD}}{R} = \frac{(R-d)+x}{R} \tag{6}$$

In $\triangle\ ACD$ the angle $\sphericalangle\ ADC$ is $90°$ so one can write

$$\cos\alpha = \frac{\overline{CD}}{l} = \frac{x}{l} \tag{7}$$

From Eq. (6) and Eq. (7) one can write

$$\cos\theta = \frac{l\cos\alpha + (R-d)}{R} \tag{8}$$

From Eq. (5) and Eq. (8) one can write

$$l^2 = R^2 + (R-d)^2 - 2R(R-d)\frac{l\cos\alpha + (R-d)}{R} \tag{9}$$

The position and velocity of the left end of the extensible link are

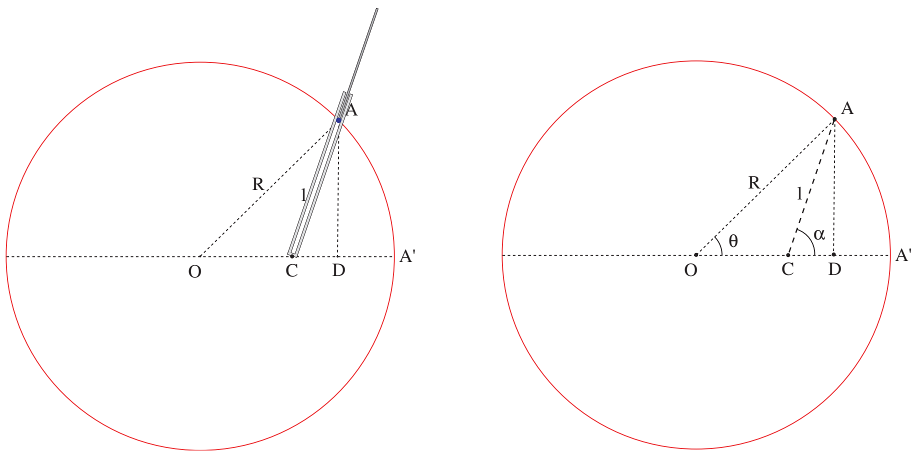


FIGURE 2. The extensible link with one fixed end and one moving end along a circular trajectory

$$\begin{aligned} x_l &= 0, \;\; y_l = R-d, \\ v_{x_l} &= 0, \;\; v_{y_l} = 0. \end{aligned} \tag{10}$$

The position and velocity of the right end of the extensible link are

$$\begin{aligned} x_r &= l\cos(\alpha)+d_1, \;\; y_r = l\sin(\alpha)+d_1, \\ v_{x_r} &= \dot{l}\cos(\alpha) - l\dot{\alpha}\sin(\alpha), \\ v_{y_r} &= \dot{l}\sin(\alpha) + l\dot{\alpha}\cos(\alpha) \end{aligned} \tag{11}$$

Extensible Link with one fixed end - Gray type extensible link. In this case the extensible link is rotating about a fixed end, with the moving end generated as in Gray [6]. The position and velocity of the left end of the extensible link are

$$\begin{aligned} x_l &= 0,\ \ y_l = 0, \\ v_{x_l} &= 0,\ \ v_{y_l} = 0. \end{aligned} \tag{12}$$

The position and velocity of the right end of the extensible link are

$$\begin{aligned} x_r &= L_1\cos(\theta_1) + (L_2 - L_{21})\cos(\theta_2),\ \ y_r = L_1\sin(\theta_1) - (L_2 - L_{21})\sin(\theta_2), \\ v_{x_r} &= -L_1\dot{\theta}_1\sin(\theta_1) - (L_2 - L_{21})\dot{\theta}_2\sin(\theta_2), \\ v_{y_r} &= L_1\dot{\theta}_1\cos(\theta_1) - (L_2 - L_{21})\dot{\theta}_2\cos(\theta_2). \end{aligned} \tag{13}$$

The position and velocity of the center of the mass of the extensible link are

$$\begin{aligned} x_C &= \frac{L_1}{2}\cos(\theta_1) + \frac{L_2 - L_{21}}{2}\cos(\theta_2),\ \ y_C = \frac{L_1}{2}\sin(\theta_1) - \frac{L_2 - L_{21}}{2}\sin(\theta_2), \\ v_{x_C} &= -\frac{L_1}{2}\dot{\theta}_1\sin(\theta_1) - \frac{L_2 - L_{21}}{2}\dot{\theta}_2\sin(\theta_2), \\ v_{y_C} &= \frac{L_1}{2}\dot{\theta}_1\cos(\theta_1) - \frac{L_2 - L_{21}}{2}\dot{\theta}_2\cos(\theta_2). \end{aligned} \tag{14}$$

The position of the right end of the extensible link can be expresses as [6]

$$\begin{aligned} x_r &= \cos\theta_1 + \left(1 - \frac{L_{21}}{L_2}\right)\sqrt{L_2^2 - \sin^2\theta_1}, \\ y_r &= \frac{L_{21}}{L_2}\sin\theta_1, \end{aligned} \tag{15}$$

where the length L_1 in Gray [6] formulation was considered to be equal with the unity, i.e., $L_1 = 1$, and $\cos(\theta_2) = \dfrac{L_1}{L_{21} - L_2}\cos(\theta_1)$. An relatively equivalent formulation can be obtained using [12].

Extensible Link with two moving ends - connecting type extensible link. For the extensible link shown in Fig. 1 the left end is rotating about a fixed point and the right end is sliding along a sliding direction. The extensible link is changing the length based on the next low

$$L_{2e} = L_2 + L_{21} + L_{21}\cos(\theta_1), \tag{16}$$

where θ_1 vary from 0 to 2π, and $L_{21} < \dfrac{L_2}{2}$. The position and velocity of the left end of the extensible link are

$$\begin{aligned} x_l &= L_1\cos\theta_1,\ \ y_l = L_1\sin\theta_1, \\ v_{x_l} &= -L_1\dot{\theta}_1\sin\theta_1,\ v_{y_l} = L_1\dot{\theta}_1\cos\theta_1. \end{aligned} \tag{17}$$

where $\sin\theta_2 = \dfrac{L1}{L2e}\sin\theta_1$. The position and velocity of the right end of the extensible link are

$$\begin{aligned} x_r &= L_1\cos(\theta_1) + L_{2e}\cos(\theta_2),\ \ y_r = L_1\sin(\theta_1) - L_{2e}\sin(\theta_2), \\ v_{x_r} &= -L_1\dot{\theta}_1\sin(\theta_1) - L_{21}\dot{\theta}_1\cos(\theta_2)\sin(\theta_1) - L_{2e}\dot{\theta}_2\sin(\theta_2), \\ v_{y_r} &= L_1 L_1\dot{\theta}_1\cos(\theta_1) - L_{21}\dot{\theta}_1\sin(\theta_1)\sin(\theta_2) - L_{2e}\dot{\theta}_2\cos(\theta_2), \end{aligned} \tag{18}$$

The position and velocity of the of the mass center of the extensible link (connecting link) are calculated with respect to the left end and right end of the link

$$
\begin{aligned}
x_C &= L_1\cos(\theta_1)+\frac{L_{2e}}{2}\cos(\theta_2), \;\; y_C = L_1\sin(\theta_1)-\frac{L_{2e}}{2}\sin(\theta_2),\\
v_{x_C} &= -L_1\dot\theta_1\sin(\theta_1)-\frac{L_{21}}{2}\dot\theta_1\cos(\theta_2)\sin(\theta_1)-\frac{L_{2e}}{2}\dot\theta_2\sin(\theta_2),\\
v_{y_C} &= L_1\dot\theta_1\cos(\theta_1)-\frac{L_{21}}{2}\dot\theta_1\sin(\theta_1)\sin(\theta_2)-\frac{L_{2e}}{2}\dot\theta_2\cos(\theta_2).
\end{aligned} \tag{19}
$$

EQUATIONS OF MOTION

For this model, the moving part of the extensible link translate parallel to its support called the line of translation. The Lagrange differential equation of motion with no impact can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right)-\frac{\partial T}{\partial q_i}=Q,$$

where T is the total kinetic energy of the system, Q is the generalized force and $q_i = \theta_1$ are the generalized coordinates. The total kinetic energy is $T=\sum_{i=1}^{n} T_i$ where $\boldsymbol{\omega}_i = \dot\theta_i\mathbf{k}$ and $T_i = \frac{1}{2}m_i\mathbf{v}_{C_i}^2+\frac{1}{2}I_{C_i}\boldsymbol{\omega}_i^2$ are the angular velocity and respectively the kinetic energy of each i-th component.

The generalized forces can be written as

$$Q_i = \sum_{j=1}^{n}\frac{\partial \mathbf{r}_{C_j}}{\partial\theta_i}\cdot\mathbf{G}_j+\frac{\partial\boldsymbol{\omega}_i}{\partial\dot\theta_i}\cdot\mathbf{M}_i,$$

where $\mathbf{G}_i = -m_i g\mathbf{J}$ is the force acting on the i-th component.

One can express the generalized coordinate and generalized force vectors in column vector format as

$$
\begin{aligned}
\{q\} &= [q_1 \quad q_2 \quad \ldots \quad q_n]^T\\
\{Q\} &= [Q_1 \quad Q_2 \quad \ldots \quad Q_n]^T,
\end{aligned} \tag{20}
$$

Using the vector format, one can write Lagrange's equations as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\{q\}}\right)-\frac{\partial L}{\partial\{q\}}=\{Q\}^T. \tag{21}$$

RESULTS

In this section, results from computer simulations are presented. To study the dynamics either the trajectory for the center of mass or the trajectory of the end of the extensible link is considered.

For the extensible link with one fixed end - one fixed end and one moving end along a circular trajectory - the trajectory of the motion is plotted for the moving end. The values used in the simulation are $R = 4.1$ and $d = 1.1$ m. The trajectory of the moving end is shown in Fig. (3). The Ox and Oy trajectories are shown in Fig. (4.a) and .

For the extensible link with two moving ends - the connecting type extensible link - the trajectory of the motion is plotted for the mass center of the extensible link. The next values are considered for the simulation: $L_{21} = 0.2$ m, $L_2 = 2.8$ m, and $L_1 = 1$. The trajectory is shown in Fig. (5).

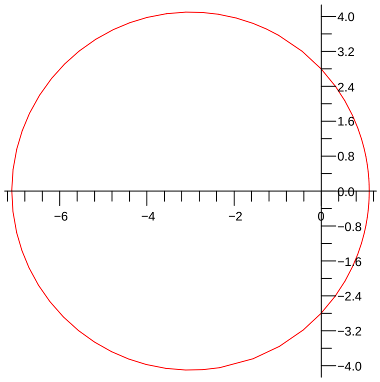
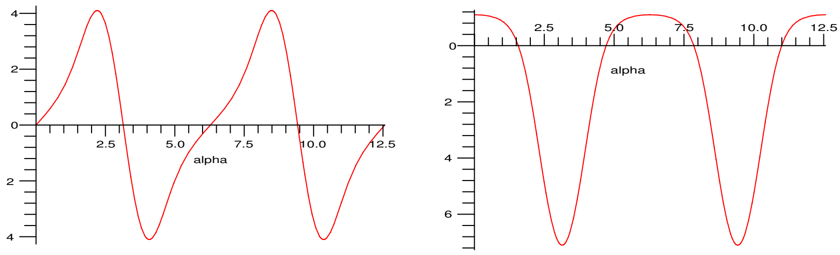


FIGURE 3. Trajectory of an extensible link with one fixed end and one moving end along a circular path



(a) (b)

FIGURE 4. The Ox and Oy trajectories of an extensible link with one fixed end and one moving end along a circular path

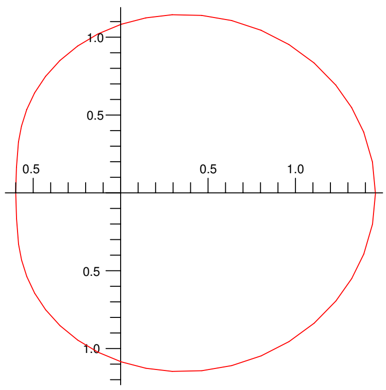


FIGURE 5. Trajectory of an extensible link with two moving ends

For the extensible link with one fixed end - the Gray type extensible link - the trajectory of the motion is plotted for the mass center of the extensible link. The next values are considered for the simulation: $L_{21} = 0.95$ and $L_2 = 1$. The trajectory is shown in Fig. (6).

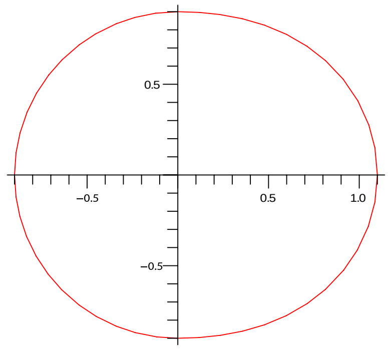


FIGURE 6. Trajectory of an Gray type extensible link

CONCLUSIONS

In this paper the dynamical behavior of an extensible link with one moving end and two moving ends was analyzed. For the values used in this study elliptic and oval trajectory of the motion have been observed. It was also observed that the trajectory can be apriori established in some cases.

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