On a Friedrichs-type inequality

NEVEN ELEZOVIĆ, JOSIP PEČARIĆ, AND MARJAN PRALJIK

Abstract. We extend an inequality proved by Rao & Śikić [5] to the class of naturally defined convex functions and derive some related inequalities. Using exponential convexity, we refine the Friedrichs-type inequality proved by Rao & Śikić [5].

2010 Mathematics Subject Classification. 26D15.
Key words and phrases. Friedrichs inequality, convex function.

1. Introduction

One of the results Rao & Śikić [5] obtained was the following inequality for a class of convex functions (inequality (65), pg 122)

**Theorem 1.** Let \( \Phi : (0, +\infty) \to (0, +\infty) \) be a convex function for which a positive Borel \( \sigma \)-finite measure \( \eta \) exists such that

\[
\Phi(\tau) = \int_0^\tau \varphi(t) dt, \quad \text{for every } \tau \in (0, +\infty),
\]

where

\[
\varphi(t) = \eta([0, t]), \quad \text{for every } t \in (0, +\infty).
\]

Furthermore, let \( \Omega \) be a bounded, open and connected set in \( \mathbb{R}^n \) and let \( f \in C^1(\Omega) \) be such that \( \text{supp}(f) \subset \Omega \). Then

\[
\Phi(|f(x)|) \leq \frac{1}{\omega_n} \int_\Omega \frac{\|\nabla f(y) \cdot (x - y)\|}{\|x - y\|^n} \varphi(|f(y)|) \, dy,
\]

where \( \nabla f = (\partial f / \partial x_1, ..., \partial f / \partial x_n) \) and \( \omega_n \) is the area of the surface of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \).

Furthermore, using Theorem 1, Rao & Śikić proved Friedrichs-type inequality

\[
\int_\Omega |f(x)|^p \, dx \leq C \int_\Omega \|\nabla f(x)\|^p \, dx,
\]

with constant \( C = p \cdot \text{diam}(\Omega) \).

The goal of this paper is to extend the inequality from Theorem 1 to the class of all convex functions on \( (0, +\infty) \) and, by using exponential convexity, to refine the Friedrichs-type inequality.

Received February 16, 2012.
2. Main results

We will use the following lemma proved by Rao & Šikić [5]

**Lemma 2.** Let $\Omega$ be a bounded, open and connected set in $\mathbb{R}^n$ and let $f \in C^4(\Omega)$ be such that $\text{supp}(f) \subset \Omega$. Then, for every $x \in \mathbb{R}^n$ and $u \geq 0$ the following inequality holds

$$|f(x)| \leq u + \frac{1}{\omega_n} \int_{\Omega} \frac{\| \nabla f(y) \cdot (x - y) \|}{\| x - y \|^n} \mathbf{1}_{\{f(y) \geq u\}} \, dy.$$

The following theorem states the main result

**Theorem 3.** Let $\Omega$ and $f$ be as in Lemma 2, let $R = \sup_{x \in \Omega} |f(x)|$ and let $\Phi(0, R) \to \mathbb{R}$ be a convex function with $\varphi$ denoting the right-continuous version of its derivative. Let $z > 0$ and $x \in B_z$, where

$$B_z = \{ y \in \Omega : |f(y)| \geq z \}.$$

Then the following inequality holds

$$\Phi(|f(x)|) - \Phi(z) \leq \frac{1}{\omega_n} \int_{B_z} \frac{\| \nabla f(y) \cdot (x - y) \|}{\| x - y \|^n} \varphi(|f(y)|) \, dy + \varphi(z)(|f(x)| - z) \varphi(z).$$

If $\Phi$ is a concave function, then the above inequality is reversed.

**Proof.** Integration by parts gives

$$\Phi(|f(x)|) - \Phi(z) = \int_z^{\|f(x)|} \varphi(u) du = u \varphi(u) \bigg|_z^{\|f(x)|} - \int_z^{\|f(x)|} ud\varphi(u)$$

$$= |f(x)| \varphi(|f(x)|) - z \varphi(z) - \int_z^{\|f(x)|} (u \pm |f(x)|) d\varphi(u)$$

$$= \int_z^{\|f(x)|} (|f(x)| - u) d\varphi(u) + \varphi(z)(|f(x)| - z).$$

Since $d\varphi$ is a positive measure, using Lemma 2 we get

$$\Phi(|f(x)|) - \Phi(z) \leq \frac{1}{\omega_n} \int_z^{\|f(x)|} \int_{\Omega} \frac{\| \nabla f(y) \cdot (x - y) \|}{\| x - y \|^n} \mathbf{1}_{\{f(y) \geq u\}} \, dy \, d\varphi(u) + \varphi(z)(|f(x)| - z). \quad (1)$$

Using Fubini’s theorem and nonnegativity of the integrand, we further get

$$\frac{1}{\omega_n} \int_z^{\|f(x)|} \int_{\Omega} \frac{\| \nabla f(y) \cdot (x - y) \|}{\| x - y \|^n} \mathbf{1}_{\{f(y) \geq u\}} \, dy \, d\varphi(u) =$$

$$\leq \frac{1}{\omega_n} \int_{\Omega} \left[ \frac{\| \nabla f(y) \cdot (x - y) \|}{\| x - y \|^n} \right] \int_z^{\|f(x)|} \mathbf{1}_{\{f(y) \geq u\}} d\varphi(u) \, dy$$

$$= \frac{1}{\omega_n} \int_{\Omega} \left[ \frac{\| \nabla f(y) \cdot (x - y) \|}{\| x - y \|^n} \right] \int_z^{\infty} \mathbf{1}_{\{f(y) \geq u\}} d\varphi(u) \, dy$$

$$= \frac{1}{\omega_n} \int_{\Omega} \mathbf{1}_{\{ \|f(y)\| \leq \| \nabla f(y) \cdot (x - y) \| \}} \left[ \varphi(|f(y)|) - \varphi(z) \right] \mathbf{1}_{B_z}(y) \, dy$$

$$= \frac{1}{\omega_n} \int_{B_z} \frac{\| \nabla f(y) \cdot (x - y) \|}{\| x - y \|^n} \varphi(|f(y)|) \, dy - \frac{\varphi(z)}{\omega_n} \int_{B_z} \frac{\| \nabla f(y) \cdot (x - y) \|}{\| x - y \|^n} \, dy.$$
Plugging the last inequality in (1) and rearranging finishes the proof.

The following corollary gives the integral version of the inequality

**Corollary 4.** Let $B = \bigcup_{\Omega \ni y \in C} B_z = \{y \in \Omega : f(y) \neq 0\}$, $C \subset B$ and $z : C \to (0, +\infty)$. If $x \in B_z$ for every $x \in C$, then for a finite measure $\mu$ on $C$ the following inequality holds

$$\int_C \left( \Phi(|f(x)|) - \Phi(z(x)) \right) d\mu(dx) \leq \frac{1}{\omega_n} \int_C \int_{B_z(x)} \frac{|\nabla f(y) \cdot (x - y)|}{|x - y|^n} \varphi(|f(y)|) dy d\mu(dx)$$

$$- \int_C \varphi(z(x)) \left( |f(x)| - \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x - y)|}{|x - y|^n} dy \right) d\mu(dx) - \int_C z(x) \varphi(z(x)) d\mu(dx).$$

In particular, for $C = B_z$ and $z(x) \equiv z$ the following inequality holds

$$\int_{B_z} \Phi(|f(x)|) d\mu(dx) - \Phi(z) d\mu(B_z) \leq \frac{1}{\omega_n} \int_{B_z} \varphi(|f(x)|) \left( \int_{B_z} \frac{|\nabla f(x) \cdot (y - x)|}{|y - x|^n} d\mu(dy) \right) dx$$

$$+ \varphi(z) \left( |f(x)| - \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x - y)|}{|x - y|^n} dy \right) d\mu(dx) - z\varphi(z) d\mu(B_z).$$

**Proof.** The first inequality of the corollary follows by integrating the inequality from Theorem 3 with respect to the measure $\mu$.

The second inequality follows by taking $C = B_z$ and $z(x) \equiv z$ in the first inequality and applying Fubini’s theorem on the first integral of the right-hand side.

**Corollary 5.** Under the assumptions of Corollary 4, for $p \in \mathbb{R}\setminus\{0, 1\}$ the following inequality holds

$$\frac{1}{p(p-1)} \int_{B_z} |f(x)|^p d\mu(dx) \leq \frac{1}{(p-1)\omega_n} \int_{B_z} |f(x)|^{p-1} \left( \int_{B_z} \frac{|\nabla f(x) \cdot (y - x)|}{|y - x|^n} d\mu(dy) \right) dx$$

$$+ \frac{z^{p-1}}{p-1} \int_{B_z} \left( |f(x)| - \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x - y)|}{|x - y|^n} dy \right) d\mu(dx) - z^p d\mu(B_z).$$

**Proof.** The inequality follows by applying Corollary 4 to the function $\Phi(\tau) = \frac{\tau^p}{p(p-1)}$.

The following corollary takes into account properties of the second term on the right-hand side of the inequality from Theorem 3

**Corollary 6.** Under the assumptions of Theorem 3, if $\varphi(z)$ is nonnegative, then the following inequality holds

$$\Phi(|f(x)|) - \Phi(z) \leq \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x - y)|}{|x - y|^n} \varphi(|f(y)|) dy - z\varphi(z).$$

**Proof.** For functions $f$ that satisfy the assumptions of the corollary, the well-known formula

$$f(x) = \frac{1}{\omega_n} \int_{\Omega} \frac{\nabla f(y) \cdot (x - y)}{|x - y|^n} dy$$
Under the assumptions of Theorem 3, if
\[ |f(x)| \leq \frac{1}{\omega_n} \int_{\Omega} \frac{\|\nabla f(y) \cdot (x - y)\|}{\|x - y\|^n} \, dy. \]
Since \( \varphi(z) \geq 0 \), the second term on the right-hand side of the inequality from Theorem 3 is nonpositive, so the claim of the corollary follows.

If Theorem 3 holds for some \( z > 0 \), then it holds for every \( z' \), \( 0 < z' \leq z \). Letting \( z' \rightarrow 0 \), we can get further inequalities.

In the proof of the following corollary we will use the fact that for a bounded and connected open set \( \Omega \) the following inequality holds
\[ \frac{1}{\omega_n} \int_{\Omega} \frac{dx}{\|x - y\|^{n-1}} \leq \frac{\text{diam}(\Omega)}{2} \]
(2)

**Theorem 7.** Under the assumptions of Theorem 3, if \( \varphi(0^+) \) is finite, then the following inequality holds
\[ \Phi(|f(x)|) - \Phi(0^+) \leq \frac{1}{\omega_n} \int_{\Omega} \frac{\|\nabla f(y) \cdot (x - y)\|}{\|x - y\|^n} \varphi(|f(y)|) \, dy \\
+ \varphi(0^+) \left( |f(x)| - \frac{1}{\omega_n} \int_{\Omega} \frac{\|\nabla f(y) \cdot (x - y)\|}{\|x - y\|^n} \, dy \right). \]

Furthermore, for a finite measure \( \mu \) on \( \Omega \) the following inequality holds
\[ \int_{\Omega} \Phi(|f(x)|) \mu(dx) - \Phi(0^+) \mu(\Omega) \leq \\
\frac{1}{\omega_n} \int_{\Omega} \varphi(|f(x)|) \left( \int_{\Omega} \frac{\|\nabla f(x) \cdot (y - x)\|}{\|y - x\|^n} \mu(dy) \right) dx \\
+ \varphi(0^+) \int_{\Omega} \left( |f(x)| - \frac{1}{\omega_n} \int_{\Omega} \frac{\|\nabla f(y) \cdot (x - y)\|}{\|x - y\|^n} \, dy \right) \mu(dx). \]

**Proof.** Since \( \varphi(0^+) \) is finite, we have \( \lim_{z \rightarrow 0} z \varphi(z) = 0 \), so the last term on the right-hand side of the inequality from Theorem 3 vanishes as \( z \rightarrow 0 \).

Since \( f \in C^1(\Omega) \) has a compact support \( \varphi(0^+) \) is finite, both functions \( \nabla f \) and \( \varphi(|f|) \) are bounded. Therefore
\[ \left| \frac{1}{\omega_n} \frac{\|\nabla f(y) \cdot (x - y)\|}{\|x - y\|^n} \varphi(|f(y)|) \right| \leq \frac{1}{\omega_n} \frac{\|\nabla f(y)\|}{\|x - y\|^{n-1}} \left| \varphi(|f(y)|) \right| \\
\leq \|\nabla f\|_{L^\infty} \|\varphi(|f|)\|_{L^\infty} \frac{1}{\omega_n} \frac{1}{\|x - y\|^{n-1}}. \]

Taking into account (2), we see that the integrand in the first integral of the inequality from Theorem 3 is dominated by an integrable function. Similarly, the integrand in the second integral is dominated as well, so by the dominated convergence theorem the right-hand side of the inequality from Theorem 3 converges to
\[ \frac{1}{\omega_n} \int_{B} \frac{\|\nabla f(y) \cdot (x - y)\|}{\|x - y\|^n} \varphi(|f(y)|) \, dy \\
+ \varphi(0^+) \left( |f(x)| - \frac{1}{\omega_n} \int_{B} \frac{\|\nabla f(y) \cdot (x - y)\|}{\|x - y\|^n} \, dy \right). \]
as \( z \rightarrow 0 \), where \( B = \bigcup_{z > 0} B_z = \{ y \in \Omega : f(y) \neq 0 \} \). Since \( \nabla f = 0 \) on the set \( B^c = \{ f = 0 \} \), the integrals over \( B \) can be replaced with integrals over \( \Omega \), which proves the first inequality.
The second inequality follows from the first by integrating with respect to the measure \( \mu \) and applying Fubini’s theorem on the first integral on the right-hand side.

**Corollary 8.** Under the assumptions of Theorem 7, for \( p > 1 \) the following inequality holds

\[
\int \Omega |f(x)|^p \mu(dx) \leq \frac{\rho}{\omega_n} \int \Omega |f(x)|^{p-1} \left( \int \Omega \frac{|\nabla f(x) \cdot (y - x)|}{\|y - x\|^n} \mu(dy) \right) dx.
\]

**Proof.** The inequality follows by applying Theorem 7 to the function \( \Phi(\tau) = \tau^p \). \( \square \)

Taking use of inequality (2), we can state the following corollary

**Corollary 9.** Under the assumptions of Theorem 7, if \( \mu(dx) = dx \) is the Lebesgue measure and \( \varphi \) is nonnegative, then the following inequality holds

\[
\int \Omega \Phi(|f(x)|) dx - \Phi(0+) \mu(\Omega) \leq \frac{1}{\omega_n} \int \Omega \varphi(|f(x)|) \left( \int \Omega \frac{|\nabla f(y) \cdot (x - y)|}{\|x - y\|^n} dy \right) dx.
\]

**Proof.** Since \( \varphi \) is nonnegative, we have

\[
\frac{1}{\omega_n} \int \Omega \varphi(|f(x)|) \left( \int \Omega \frac{|\nabla f(x) \cdot (y - x)|}{\|y - x\|^n} dy \right) dx \\
\leq \frac{1}{\omega_n} \int \Omega \varphi(|f(x)|) \left( \int \Omega \frac{dy}{\|y - x\|^n} \right) dx \\
\leq \frac{\text{diam}(\Omega)}{2} \int \Omega \varphi(|f(x)|) \|\nabla f(x)\| dx,
\]

and the claim of the corollary follows from the second inequality of Theorem 7. \( \square \)

**Corollary 10.** Under the assumptions of Corollary 9, for \( p > 1 \) the following two inequalities hold:

\[
\int \Omega |f(x)|^p dx \leq \frac{p \cdot \text{diam}(\Omega)}{2} \int \Omega |f(x)|^{p-1} \|\nabla f(x)\| dx
\]

and

\[
\left[ \int \Omega |f(x)|^p dx \right]^\frac{1}{p} \leq \frac{p \cdot \text{diam}(\Omega)}{2} \left[ \int \Omega \|\nabla f(x)\|^p dx \right]^\frac{1}{p}.
\]

**Proof.** The first inequalities follows from Corollary 9 applied to the function \( \Phi(\tau) = \tau^p \).

The second inequality follows by applying Hölder’s inequality on the right-hand side integral of the first inequality.

The second inequality from the last corollary can be restated as

\[
\|f\|_{L^p(\Omega)} \leq \frac{p \cdot \text{diam}(\Omega)}{2} \|\nabla f\|_{L^p(\Omega)}
\]

and represents a Friedrichs-type inequality in which the \( L^p \) norm of a function is bounded by the \( L^p \) norm of its gradient. Inequality (3) is a special case of inequality proven by Friedrichs [2], which in turn is a special case of Sobolev inequality (see [3]).
3. Exponential convexity

In this section we will use well known results from exponential convexity to derive new inequalities and refine some inequalities from the previous section (see [1]). We will also prove mean value theorems and generate Cauchy-type means and prove their monotonicity.

Let $\Omega$, $f$, $x$, $z$, $\mu$ and $C$ be as in Theorem 3 or Corollary 4 and let us define the following class of functions $A_k = A_k(\Omega, f, x, z, \mu, C)$ with

$$A_1(\Phi) = \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x - y)|}{\|x - y\|^n} \varphi(|f(y)|) \, dy - \Phi(|f(x)|) + \Phi(0^+)$$

$$\quad + \varphi(z)(|f(x)| - \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x - y)|}{\|x - y\|^n} \, dy) - z\varphi(z),$$

$$A_2(\Phi) = \frac{1}{\omega_n} \int_{C} \int_{B_z(x)} \frac{|\nabla f(y) \cdot (x - y)|}{\|x - y\|^n} \varphi(|f(y)|) \, dy \mu(dx)$$

$$\quad - \int_{C} \varphi(z(x))(|f(x)| - \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x - y)|}{\|x - y\|^n} \, dy) \mu(dx)$$

$$\quad - \int_{C} z(x) \varphi(z(x)) \mu(dx) - \int_{C} \left( \Phi(|f(x)|) - \Phi(z(x)) \right) \mu(dx),$$

$$A_3(\Phi) = \frac{1}{\omega_n} \int_{\Omega} \frac{|\nabla f(y) \cdot (x - y)|}{\|x - y\|^n} \varphi(|f(y)|) \, dy - \Phi(|f(x)|) + \Phi(0^+)$$

$$\quad + \varphi(0^+)(|f(x)| - \frac{1}{\omega_n} \int_{\Omega} \frac{|\nabla f(y) \cdot (x - y)|}{\|x - y\|^n} \, dy)$$

$$A_4(\Phi) = \frac{1}{\omega_n} \int_{\Omega} \varphi(|f(x)|) \left( \int_{\Omega} \frac{|\nabla f(x) \cdot (y - x)|}{\|y - x\|^n} \mu(dy) \right) \, dx$$

$$\quad + \varphi(0^+)(|f(x)| - \frac{1}{\omega_n} \int_{\Omega} \frac{|\nabla f(y) \cdot (x - y)|}{\|x - y\|^n} \, dy) \mu(dx)$$

$$\quad - \int_{\Omega} \Phi(|f(x)|) \mu(dx) + \Phi(0^+) \mu(\Omega).$$

Linear functional $A_k$, $k = 1, \ldots, 4$, depend on the choices of $\Omega$, $f$, $x$, $z$, $\mu$ and $C$, but if they are clear from the context, we will omit them from the notation.

Let us denote by $\Phi_p$ the following class of functions

$$\Phi_p(\tau) = \begin{cases} \frac{\tau^p}{p(p-1)}, & p \neq 0, 1 \\ -\log \tau, & p = 0 \\ \tau \log \tau, & p = 1 \end{cases}$$

and let us define functions $\psi_k : I_k \rightarrow \mathbb{R}_+$ by

$$\psi_k(p) = A_k(\Phi_p)$$

with $I_1 = I_2 = \mathbb{R}$ and $I_3 = I_4 = (1, +\infty)$. Notice that $\Phi''_p(\tau) = \tau^{p-2}$, so the functions $\Phi_p$ are convex. By Theorems 3 and 7 and Corollaries 4 and 8, the functions $\psi_k$ are,
indeed, well-defined and nonnegative. It is straightforward to check that all of the functions $\psi_k$ are continuous.

**Lemma 11.** For each $k \in \{1, 2, \ldots, 4\}$, the function $\psi_k$ is exponentially convex.

**Proof.** Let $n \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $p_i \in I_k$, $1 \leq i \leq n$, be arbitrary. Define the function $\Phi$ by

$$\Phi(\tau) = \sum_{i,j=1}^{n} \xi_i \xi_j \Phi_{p_i+p_j}(\tau).$$

Since

$$\Phi''(\tau) = \sum_{i,j=1}^{n} \xi_i \xi_j \frac{p_i+p_j}{2} - 2 \left( \sum_{i=1}^{n} \xi_i \frac{p_i}{2} - 1 \right)^2 \geq 0,$$

the function $\Phi$ is convex.

Furthermore, if $k = 3$ or $4$, we have

$$\varphi(0+) = \left| \sum_{i,j=1}^{n} \xi_i \xi_j \varphi_{p_i+p_j}(0+) \right| < +\infty,$$

so $\Phi$ satisfies the assumptions of Theorem 7. Hence, by Theorems 3 and 7 and Corollaries 4 and 8, for each $k$ we have

$$0 \leq A_k(\Phi) = \sum_{i,j=1}^{n} \xi_i \xi_j A_k \left( \frac{p_i+p_j}{2} \right) = \sum_{i,j=1}^{n} \xi_i \xi_j \psi_k \left( \frac{p_i+p_j}{2} \right).$$

Since $\psi_k$ are continuous in addition to satisfying the above condition, it follows that $\psi_k$ are exponentially convex functions. \qed

Due to the properties of exponentially convex functions, the following corollary is a direct consequence of the previous lemma

**Corollary 12.** For $\psi_k$, $k = 1, \ldots, 4$, defined by (5) the following statements hold

(i) For all $n \in \mathbb{N}$ and $p_i \in I_k$, $1 \leq i \leq n$ the matrix $[\psi_k(\frac{p_i+p_j}{2})]_{i,j=1}^{n}$ is positive semidefinite, so

$$\det \left[ \psi_k \left( \frac{p_i+p_j}{2} \right) \right]_{i,j=1}^{n} \geq 0.$$

(ii) For $p, s, t \in I_k$ we have

$$\psi_k(p) \geq \left[ \psi_k(s) \right]_{i,j=1}^{n} \left[ \psi_k(t) \right]_{i,j=1}^{n} \quad \text{if } p < s < t \quad \text{or} \quad s < t < p$$

$$\psi_k(p) \leq \left[ \psi_k(s) \right]_{i,j=1}^{n} \left[ \psi_k(t) \right]_{i,j=1}^{n} \quad \text{if } s < p < t.$$

Notice that the first set of inequalities in Corollary 12(ii) are refinements of the inequalities in Corollaries 5 and 8. Indeed, the latter inequalities, in the notation introduced in this section, are

$$0 \leq \psi_k(p), \quad k = 2, 4, p \in I_k \setminus \{0, 1\},$$

while the right-hand sides of inequalities in Corollary 12(ii) are nonnegative.

Furthermore, inequalities from Corollary 12(ii) are refinements of the Friedrichs-type inequality from Corollary 10. Indeed, we have the following result
Corollary 13. Let \( \psi_4 \) be defined by (5) and let \( \Omega \) and \( f \) be as in Theorem 3. Then, for \( 1 < p < s < t \) or \( 1 < s < t < p \) the following inequality holds

\[
p(p - 1)[\psi_4(s)]^{\frac{1}{p - s}}[\psi_4(t)]^{\frac{1}{t - s}}\left[\int_\Omega |f(x)|^p dx\right]^{\frac{s}{p - s}} \leq \frac{p \cdot \text{diam}(\Omega)}{2} \left[\int_\Omega \|\nabla f(x)\|^p dx\right]^{\frac{1}{p}} - \left[\int_\Omega |f(x)|^p dx\right]^{\frac{1}{p}}.
\]

Proof. As in the proof of Corollary 9, one can show that

\[
\frac{1}{\omega_n} \int_\Omega |f(x)|^{p-1} \left( \int_\Omega |\nabla f(x) : (y - x)|^p dy \right) dx \leq \frac{\text{diam}(\Omega)}{2} \left[\int_\Omega |f(x)|^{p-1}\|\nabla f(x)\| dx - \int_\Omega |f(x)|^p dx\right]
\]

Therefore

\[
p(p - 1)\psi_4(p) \leq \frac{p \cdot \text{diam}(\Omega)}{2} \int_\Omega |f(x)|^{p-1}\|\nabla f(x)\| dx - \int_\Omega |f(x)|^p dx.
\]

Applying Hölder’s inequality on the first integral of the right-hand side and multiplying the inequality by \( \left[\int_\Omega |f(x)|^p dx\right]^{(1 - p)/p} \), while taking into account the first inequality from Corollary 12(ii), we get the claim of the corollary. \( \square \)

Next, we will state and prove Lagrange- and Cauchy-type mean value results.

Lemma 14. Let \( k \in \{1, ..., 4\} \), let \( \Omega, f \) and \( R \) be as in Theorem 3 and let \( \Psi \in \mathcal{C}^2((0, R]) \). If \( A_k(\Psi) \) is finite, \( A_k(\Phi_2) \neq 0 \) and the function \( \Psi \), when \( k = 3 \) or \( 4 \), satisfies the same limiting assumptions at zero as the function \( \Psi \) in Theorem 7, then there exists \( \xi_k \in [0, R] \) (provided \( \Psi''(0) = \lim_{z \to 0} \Psi''(z) \) exists when \( \xi_k = 0 \)) such that

\[
A_k(\Psi) = \Psi''(\xi_k)A_k(\Phi_2).
\]

Proof. Since \( \Phi_2 \) is a convex function, when \( A_k(\Phi_2) \neq 0 \) by Theorems 3 and 7 and Corollary 4, we have \( A_k(\Phi_2) > 0, k = 1, ..., 4 \). Let

\[
m = \inf_{\tau \in (0, +\infty)} \Psi''(\tau) \quad \text{and} \quad M = \sup_{\tau \in (0, +\infty)} \Psi''(\tau).
\]

If \( M < +\infty \), then the function \( M\Phi_2 - \Psi \) is convex since

\[
\frac{d^2}{d\tau^2}\left(\frac{M\tau^2}{2} - \Psi(\tau)\right) = M - \Psi''(\tau) \geq 0.
\]

By the assumptions of the lemma, the assumptions of Theorems 3 and 7 and Corollary 4 are satisfied and, hence,

\[
0 \leq A_k \left(M\Phi_2 - \Psi\right), \quad k = 1, ..., 4,
\]

i. e.

\[
A_k(\Psi) \leq MA_k(\Phi_2), \quad k = 1, ..., 4. \tag{6}
\]

If \( M = +\infty \), then inequality (6) holds trivially. Similarly, for a finite \( m \) the inequality

\[
mA_k(\Phi_2) \leq A_k(\Psi), \quad k = 1, ..., 4 \tag{7}
\]

holds since \( \Psi - m\Phi_2 \) is convex, while for \( m = -\infty \) inequality (7) holds trivially.

Finally, the existence of \( \xi_k, \ k = 1, ..., 4 \), follows from (6), (7) and continuity of \( \Psi'' \). \( \square \)
Lemma 15. Let $k \in \{1, \ldots, 4\}$. If $\Psi$ and $\tilde{\Psi}$ satisfy the assumptions of Lemma 14 and if $A_k(\Phi_2) \neq 0$, then there exists $\xi_k \in [0, R]$ such that
\[
\frac{\psi''(\xi_k)}{\psi''(\xi_k)} = \frac{A_k(\Psi)}{A_k(\tilde{\Psi})},
\]
provided that the denominators are nonzero.

Proof. Let us define a function $\phi$ by
\[
\phi(\tau) = \Psi(\tau) A_k(\tilde{\Psi}) - \tilde{\Psi}(\tau) A_k(\Psi).
\]
The function $\phi$ also satisfies Lemma 14 and, hence, there exists $\xi_k \in [0, R]$ such that $A_k(\phi) = \psi''(\xi_k) A_k(\tilde{\Psi}) - \tilde{\Psi}(\tau) A_k(\Psi)$, equality (8) follows. □

Equality (8) allows us to define various means. Indeed, if $\Psi''/\tilde{\Psi}''$ is an invertible function for functions $\Psi$ and $\tilde{\Psi}$ that satisfy the assumptions of Lemma 15,
\[
\xi_k = \left(\frac{\psi''}{\psi''}ight)^{-1} \left(\frac{A_k(\Psi)}{A_k(\tilde{\Psi})}\right)
\]
is a well-defined mean provided $\xi_k > 0$. In particular, for $\Psi = \Phi_p$ and $\tilde{\Psi} = \Phi_q$, recalling the definitions (4) and (5) of functions $\Phi_p$ and $\psi_k$, we can define means $E_{p,q}^k$ by
\[
E_{p,q}^k = \left(\frac{A_k(\Phi_p)}{A_k(\psi_k)}\right)^{\frac{1}{p-q}} = \left(\frac{\psi_k(p)}{\psi_k(q)}\right)^{\frac{1}{p-q}}
\]
for $p, q \in I_k$, $p \neq q$. Moreover, we can continuously extend these means to cover the case $p = q$ as well by calculating the limits $\lim_{p \to q} E_{p,q}^k$. For $k = 1$ or 2 we get
\[
E_{p,q}^k = \begin{cases}
\left(\frac{A_k(\Phi_p)}{A_k(\Phi_p)}\right)^{\frac{1}{p-q}}, & p \neq q \\
\exp\left\{\frac{1}{2}\right\} - \left(\frac{A_k(\Phi_p)}{A_k(\Phi_p)}\right)^{\frac{1}{p-q}}, & p = q \neq 0, 1 \\
\exp\left\{\frac{1}{2}\right\} - \left(\frac{A_k(\Phi_p)}{A_k(\Phi_p)}\right)^{\frac{1}{p-q}}, & p = q = 1 \\
\exp\left\{\frac{1}{2}\right\} - \left(\frac{A_k(\Phi_p)}{A_k(\Phi_p)}\right)^{\frac{1}{p-q}}, & p = q = 0
\end{cases}
\]

The means $E_{p,q}^k$ for $k = 3$ and $k = 4$ have the same form, but are defined only for $p > 1$ and $q > 1$.

Corollary 16. Let $k \in \{1, 2, 3, 4\}$ and $p, q, r, s \in I_k$ be such that $p \leq r$ and $q \leq s$. Then
\[
E_{p,q}^k \leq E_{r,s}^k.
\]

Proof. Since the functions $\psi_k$ are exponentially convex by Lemma 11, they are also log-convex.
Now, the inequality of the corollary follows directly from log-convexity of the functions $\psi_k$ and continuity of the means $E^k$. □
References


(Neven Elezović) Faculty of Electrical Engineering and Computing, University of Zagreb, Unsa 3, 10000 Zagreb, Croatia
E-mail address: neven.elez@fer.hr

(Josip Pečarić) Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
E-mail address: pecaric@element.hr

(Marjan Praljak) Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia
E-mail address: mpraljak@pbf.hr