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On a Friedrichs-type inequality

Neven Elezović, Josip Pečarić, and Marjan Praljak

Abstract. We extend an inequality proved by Rao & Šikić [5] to the class of naturally defined convex functions and derive some related inequalities. Using exponential convexity, we refine the Friedrichs-type inequality proved by Rao & Šikić [5].

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1. Introduction

One of the results Rao & Šikić [5] obtained was the following inequality for a class of convex functions (inequality (65), pg 122)

Theorem 1. *Let $\Phi : (0,+\infty) \to (0,+\infty)$ be a convex function for which a positive Borel σ-finite measure η exists such that*

$$\Phi(\tau) = \int_0^\tau \varphi(t)dt, \qquad \text{for every } \tau \in (0,+\infty),$$

where

$$\varphi(t) = \eta([0,t]), \quad \text{for every } t \in (0,+\infty).$$

Furthermore, let Ω be a bounded, open and connected set in $\mathbb{R}^n$ and let $f \in C^1(\Omega)$ be such that $\operatorname{supp}(f) \subset \Omega$*. Then*

$$\Phi\big(|f(x)|\big) \le \frac{1}{\omega_n} \int_\Omega \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \varphi\big(|f(y)|\big)\, dy,$$

where $\nabla f = (\partial f/\partial x_1, ..., \partial f/\partial x_n)$ and ω_n is the area of the surface of the unit sphere S^{n-1} in $\mathbb{R}^n$.

Furthermore, using Theorem 1, Rao & Šikić proved Friedrichs-type inequality

$$\int_\Omega |f(x)|^p dx \le C \int_\Omega \|\nabla f(x)\|^p dx,$$

with constant $C = p \cdot \operatorname{diam}(\Omega)$.

The goal of this paper is to extend the inequality from Theorem 1 to the class of all convex functions on $(0,+\infty)$ and, by using exponential convexity, to refine the Friedrichs-type inequality.

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2. Main results

We will use the following lemma proved by Rao & Šikić [5]

Lemma 2. *Let Ω be a bounded, open and connected set in $\mathbb{R}^n$ and let $f \in C^1(\Omega)$ be such that* $\operatorname{supp}(f) \subset \Omega$. *Then, for every $x \in \mathbb{R}^n$ and $u \geq 0$ the following inequality holds*

$$|f(x)| \leq u + \frac{1}{\omega_n} \int_\Omega \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \mathbf{1}_{\{|f(y)| \geq u\}}\, dy.$$

The following theorem states the main result

Theorem 3. *Let Ω and f be as in Lemma 2, let $R = \sup_{x \in \Omega} |f(x)|$ and let $\Phi(0,R] \to \mathbb{R}$ be a convex function with φ denoting the right-continuous version of its derivative. Let $z > 0$ and $x \in B_z$, where*

$$B_z = \{y \in \Omega : |f(y)| \geq z\}.$$

Then the following inequality holds

$$\Phi\big(|f(x)|\big) - \Phi(z) \leq \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \varphi\big(|f(y)|\big)\, dy + \varphi(z)\Big(|f(x)| - \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n}\, dy\Big) - z\varphi(z).$$

If Φ is a concave function, then the above inequality is reversed.

Proof. Integration by parts gives

$$\begin{aligned}
\Phi(|f(x)|) - \Phi(z) &= \int_z^{|f(x)|} \varphi(u) du = u\varphi(u)\Big|_z^{|f(x)|} - \int_z^{|f(x)|} u d\varphi(u) \\
&= |f(x)|\varphi(|f(x)|) - z\varphi(z) - \int_z^{|f(x)|} (u \pm |f(x)|) d\varphi(u) \\
&= \int_z^{|f(x)|} (|f(x)| - u) d\varphi(u) + \varphi(z)(|f(x)| - z)
\end{aligned}$$

Since $d\varphi$ is a positive measure, using Lemma 2 we get

$$\Phi\big(|f(x)|\big) - \Phi(z) \leq \frac{1}{\omega_n} \int_z^{|f(x)|} \int_\Omega \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \mathbf{1}_{\{|f(y)| \geq u\}}\, dy\, d\varphi(u) + \varphi(z)(|f(x)| - z). \tag{1}$$

Using Fubini's theorem and nonnegativity of the integrand, we further get

$$\begin{aligned}
&\frac{1}{\omega_n} \int_z^{|f(x)|} \int_\Omega \frac{|\nabla f(y) \cdot (x-y)|}{|x-y\|^n} \mathbf{1}_{\{|f(y)| \geq u\}}\, dy\, d\varphi(u) = \\
&\quad = \int_\Omega \left[\frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \int_z^{|f(x)|} \mathbf{1}_{\{|f(y)| \geq u\}}\, d\varphi(u) \right] dy \\
&\quad \leq \int_\Omega \left[\frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \int_z^{+\infty} \mathbf{1}_{\{|f(y)| \geq u\}}\, d\varphi(u) \right] dy \\
&\quad = \int_\Omega \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \big[\varphi(|f(y)|) - \varphi(z)\big] \mathbf{1}_{B_z}(y)\, dy \\
&\quad = \frac{1}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \varphi\big(|f(y)|\big)\, dy - \frac{\varphi(z)}{\omega_n} \int_{B_z} \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n}\, dy.
\end{aligned}$$

Plugging the last inequality in (1) and rearranging finishes the proof. □

The following corollary gives the integral version of the inequality

Corollary 4. *Let* $B = \bigcup_{z\searrow 0} B_z = \{y \in \Omega : f(y) \neq 0\}$, $C \subset B$ *and* $z : C \to (0, +\infty)$. *If* $x \in B_{z(x)}$ *for every* $x \in C$, *then for a finite measure* μ *on* C *the following inequality holds*

$$\int_C \Big(\Phi\big(|f(x)|\big) - \Phi(z(x))\Big)\mu(dx) \leq \frac{1}{\omega_n}\int_C \int_{B_{z(x)}} \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\varphi\big(|f(y)|\big)\,dy\,\mu(dx) - \int_C \varphi(z(x))\Big(|f(x)| - \frac{1}{\omega_n}\int_{B_z} \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big)\mu(dx) - \int_C z(x)\varphi(z(x))\mu(dx).$$

In particular, for $C = B_z$ *and* $z(x) \equiv z$ *the following inequality holds*

$$\int_{B_z} \Phi\big(|f(x)|\big)\mu(dx) - \Phi(z)\mu(B_z) \leq \frac{1}{\omega_n}\int_{B_z} \varphi\big(|f(x)|\big)\Big(\int_{B_z} \frac{|\nabla f(x)\cdot(y-x)|}{\|y-x\|^n}\,\mu(dy)\Big)\,dx + \varphi(z)\int_{B_z}\Big(|f(x)| - \frac{1}{\omega_n}\int_{B_z} \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big)\mu(dx) - z\varphi(z)\mu(B_z).$$

Proof. The first inequality of the corollary follows by integrating the inequality from Theorem 3 with respect to the measure μ.

The second inequality follows by taking $C = B_z$ and $z(x) \equiv z$ in the first inequality and applying Fubini's theorem on the first integral of the right-hand side. □

Corollary 5. *Under the assumptions of Corollary 4, for* $p \in \mathbb{R}\backslash\{0,1\}$ *the following inequality holds*

$$\frac{1}{p(p-1)}\int_{B_z} |f(x)|^p \mu(dx) \leq \frac{1}{(p-1)\omega_n}\int_{B_z} |f(x)|^{p-1}\Big(\int_{B_z} \frac{|\nabla f(x)\cdot(y-x)|}{\|y-x\|^n}\,\mu(dy)\Big)\,dx + \frac{z^{p-1}}{p-1}\int_{B_z}\Big(|f(x)| - \frac{1}{\omega_n}\int_{B_z} \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big)\mu(dx) - \frac{z^p\mu(B_z)}{p}.$$

Proof. The inequality follows by applying Corollary 4 to the function $\Phi(\tau) = \frac{\tau^p}{p(p-1)}$. □

The following corollary takes into account properties of the second term on the right-hand side of the inequality from Theorem 3

Corollary 6. *Under the assumptions of Theorem 3, if* $\varphi(z)$ *is nonnegative, then the following inequality holds*

$$\Phi\big(|f(x)|\big) - \Phi(z) \leq \frac{1}{\omega_n}\int_{B_z} \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\varphi\big(|f(y)|\big)\,dy - z\varphi(z).$$

Proof. For functions f that satisfy the assumptions of the corollary, the well-known formula

$$f(x) = \frac{1}{\omega_n}\int_\Omega \frac{\nabla f(y)\cdot(x-y)}{\|x-y\|^n}dy$$

holds, so

$$|f(x)| \le \frac{1}{\omega_n}\int_\Omega \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}dy.$$

Since $\varphi(z) \ge 0$, the second term on the right-hand side of the inequality from Theorem 3 is nonpositive, so the claim of the corollary follows. □

If Theorem 3 holds for some $z > 0$, then it holds for every z', $0 < z' \le z$. Letting $z' \to 0$, we can get further inequalities.

In the proof of the following corollary we will use the fact that for a bounded and connected open set Ω the following inequality holds

$$\frac{1}{\omega_n}\int_\Omega \frac{dx}{\|x-y\|^{n-1}} \le \frac{\operatorname{diam}(\Omega)}{2} \tag{2}$$

Theorem 7. *Under the assumptions of Theorem 3, if $\varphi(0+)$ is finite, then the following inequality holds*

$$\Phi\big(|f(x)|\big) - \Phi(0+) \le \frac{1}{\omega_n}\int_\Omega \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\varphi\big(|f(y)|\big)\,dy + \varphi(0+)\Big(|f(x)| - \frac{1}{\omega_n}\int_\Omega \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big).$$

Furthermore, for a finite measure μ on Ω the following inequality holds

$$\int_\Omega \Phi\big(|f(x)|\big)\mu(dx) - \Phi(0+)\mu(\Omega) \le \frac{1}{\omega_n}\int_\Omega \varphi\big(|f(x)|\big)\Big(\int_\Omega \frac{|\nabla f(x)\cdot(y-x)|}{\|y-x\|^n}\,\mu(dy)\Big)\,dx + \varphi(0+)\int_\Omega \Big(|f(x)| - \frac{1}{\omega_n}\int_\Omega \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big)\mu(dx).$$

Proof. Since $\varphi(0+)$ is finite, we have $\lim_{z\to 0} z\varphi(z) = 0$, so the last term on the right-hand side of the inequality from Theorem 3 vanishes as $z \to 0$.

Since $f \in C^1(\Omega)$ has a compact support $\varphi(0+)$ is finite, both functions ∇f and $\varphi(|f|)$ are bounded. Therefore

$$\Big|\frac{1}{\omega_n}\frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\varphi\big(|f(y)|\big)\Big| \le \frac{1}{\omega_n}\frac{\|\nabla f(y)\|}{\|x-y\|^{n-1}}\big|\varphi(|f(y)|)\big| \le \|\nabla f\|_{L^\infty}\|\varphi(|f|)\|_{L^\infty}\frac{1}{\omega_n\|x-y\|^{n-1}}.$$

Taking into account (2), we see that the integrand in the first integral of the inequality from Theorem 3 is dominated by an integrable function. Similarly, the integrand in the second integral is dominated as well, so by the dominated convergence theorem the right-hand side of the inequality from Theorem 3 converges to

$$\frac{1}{\omega_n}\int_B \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\varphi\big(|f(y)|\big)\,dy + \varphi(0+)\Big(|f(x)| - \frac{1}{\omega_n}\int_B \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big).$$

as $z \to 0$, where $B = \bigcup_{z\searrow 0} B_z = \{y \in \Omega : f(y) \ne 0\}$. Since $\nabla f = 0$ on the set $B^c = \{f = 0\}$, the integrals over B can be replaced with integrals over Ω, which proves the first inequality.

The second inequality follows from the first by integrating with respect to the measure μ and applying Fubini's theorem on the first integral on the right-hand side. □

Corollary 8. *Under the assumptions of Theorem 7, for $p > 1$ the following inequality holds*

$$\int_\Omega |f(x)|^p \mu(dx) \le \frac{p}{\omega_n} \int_\Omega |f(x)|^{p-1} \Big(\int_\Omega \frac{|\nabla f(x) \cdot (y-x)|}{\|y-x\|^n} \mu(dy) \Big) \, dx.$$

Proof. The inequality follows by applying Theorem 7 to the function $\Phi(\tau) = \tau^p$. □

Taking use of inequality (2), we can state the following corollary

Corollary 9. *Under the assumptions of Theorem 7, if $\mu(dx) = dx$ is the Lebesgue measure and φ is nonnegative, then the following inequality holds*

$$\int_\Omega \Phi\big(|f(x)|\big) \, dx - \Phi(0+)\mu(\Omega) \le \frac{\operatorname{diam}(\Omega)}{2} \int_\Omega \varphi\big(|f(x)|\big) \|\nabla f(x)\| \, dx \\ + \varphi(0+) \int_\Omega \Big(|f(x)| - \frac{1}{\omega_n} \int_\Omega \frac{|\nabla f(y) \cdot (x-y)|}{\|x-y\|^n} \, dy \Big) dx.$$

Proof. Since φ is nonnegative, we have

$$\begin{aligned} \frac{1}{\omega_n} \int_\Omega \varphi\big(|f(x)|\big) \Big(\int_\Omega \frac{|\nabla f(x) \cdot (y-x)|}{\|y-x\|^n} \, dy \Big) \, dx \\ \le \frac{1}{\omega_n} \int_\Omega \varphi\big(|f(x)|\big) \|\nabla f(x)\| \Big(\int_\Omega \frac{dy}{\|y-x\|^{n-1}} \Big) \, dx \\ \le \frac{\operatorname{diam}(\Omega)}{2} \int_\Omega \varphi\big(|f(x)|\big) \|\nabla f(x)\| \, dx, \end{aligned}$$

and the claim of the corollary follows from the second inequality of Theorem 7. □

Corollary 10. *Under the assumptions of Corollary 9, for $p > 1$ the following two inequalities hold:*

$$\int_\Omega |f(x)|^p \, dx \le \frac{p \cdot \operatorname{diam}(\Omega)}{2} \int_\Omega |f(x)|^{p-1} \|\nabla f(x)\| \, dx$$

and

$$\Big[\int_\Omega |f(x)|^p \, dx \Big]^{\frac{1}{p}} \le \frac{p \cdot \operatorname{diam}(\Omega)}{2} \Big[\int_\Omega \|\nabla f(x)\|^p \, dx \Big]^{\frac{1}{p}}.$$

Proof. The first inequalities follows from Corollary 9 applied to the function $\Phi(\tau) = \tau^p$.

The second inequality follows by applying Hölder's inequality on the right-hand side integral of the first inequality. □

The second inequality from the last corollary can be restated as

$$\|f\|_{L^p(\Omega)} \le \frac{p \cdot \operatorname{diam}(\Omega)}{2} \|\nabla f\|_{L^p(\Omega)} \tag{3}$$

and represents a Friedrichs-type inequality in which the L^p norm of a function is bounded by the L^p norm of its gradient. Inequality (3) is a special case of inequality proven by Friedrichs [2], which in turn is a special case of Sobolev inequality (see [3]).

3. Exponential convexity

In this section we will use well known results from exponential convexity to derive new inequalities and refine some inequalities from the previous section (see [1]). We will also prove mean value theorems and generate Cauchy-type means and prove their monotonicity.

Let Ω, f, x, z, μ and C be as in Theorem 3 or Corollary 4 and let us define the following four linear functionals: $A_k = A_{k;\Omega,f,x,z,\mu,C}$ with

$$\begin{aligned} A_1(\Phi) = {} & \frac{1}{\omega_n}\int_{B_z} \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\varphi\big(|f(y)|\big)\,dy - \Phi\big(|f(x)|\big) + \Phi(z) \\ & + \varphi(z)\Big(|f(x)| - \frac{1}{\omega_n}\int_{B_z}\frac{|\nabla f(y)\cdot x-y)|}{\|x-y\|^n}\,dy\Big) - z\varphi(z), \end{aligned}$$

$$\begin{aligned} A_2(\Phi) = {} & \frac{1}{\omega_n}\int_C\int_{B_{z(x)}} \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\varphi\big(|f(y)|\big)\,dy\,\mu(dx) \\ & - \int_C \varphi(z(x))\Big(|f(x)| - \frac{1}{\omega_n}\int_{B_z}\frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big)\mu(dx) \\ & - \int_C z(x)\varphi(z(x))\mu(dx) - \int_C\Big(\Phi\big(|f(x)|\big) - \Phi(z(x))\Big)\mu(dx), \end{aligned}$$

$$\begin{aligned} A_3(\Phi) = {} & \frac{1}{\omega_n}\int_{\Omega} \frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\varphi\big(|f(y)|\big)\,dy - \Phi\big(|f(x)|\big) + \Phi(0+) \\ & + \varphi(0+)\Big(|f(x)| - \frac{1}{\omega_n}\int_{\Omega}\frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big) \end{aligned}$$

$$\begin{aligned} A_4(\Phi) = {} & \frac{1}{\omega_n}\int_{\Omega}\varphi\big(|f(x)|\big)\Big(\int_{\Omega}\frac{|\nabla f(x)\cdot(y-x)|}{\|y-x\|^n}\,\mu(dy)\Big)\,dx \\ & + \varphi(0+)\int_{\Omega}\Big(|f(x)| - \frac{1}{\omega_n}\int_{\Omega}\frac{|\nabla f(y)\cdot(x-y)|}{\|x-y\|^n}\,dy\Big)\mu(dx) \\ & - \int_{\Omega}\Phi\big(|f(x)|\big)\mu(dx) + \Phi(0+)\mu(\Omega). \end{aligned}$$

Linear functional A_k, $k = 1, ..., 4$, depend on the choices of Ω, f, x, z, μ and C, but if they are clear from the context, we will omit them from the notation.

Let us denote by Φ_p the following class of functions

$$\Phi_p(\tau) = \begin{cases} \frac{\tau^p}{p(p-1)}, & p \neq 0, 1 \\ -\log\tau, & p = 0 \\ \tau\log\tau, & p = 1 \end{cases} \tag{4}$$

and let us define functions $\psi_k : I_k \to \mathbb{R}_+$ by

$$\psi_k(p) = A_k(\Phi_p) \tag{5}$$

with $I_1 = I_2 = \mathbb{R}$ and $I_3 = I_4 = (1, +\infty)$. Notice that $\Phi_p''(\tau) = \tau^{p-2}$, so the functions Φ_p are convex. By Theorems 3 and 7 and Corollaries 4 and 8, the functions ψ_k are, indeed, well-defined and nonnegative. It is straightforward to check that all of the functions ψ_k are continuous.

Lemma 11. *For each $k \in \{1, 2, ..., 4\}$, the function ψ_k is exponentially convex.*

Proof. Let $n \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $p_i \in I_k$, $1 \le i \le n$, be arbitrary. Define the function Φ by

$$\Phi(\tau) = \sum_{i,j=1}^{n} \xi_i \xi_j \Phi_{\frac{p_i+p_j}{2}}(\tau).$$

Since

$$\Phi''(\tau) = \sum_{i,j=1}^{n} \xi_i \xi_j \tau^{\frac{p_i+p_j}{2}-2} = \Big(\sum_{i=1}^{n} \xi_i \tau^{\frac{p_i}{2}-1}\Big)^2 \ge 0,$$

the function Φ is convex.

Furthermore, if $k = 3$ or 4, we have

$$\varphi(0+) = \Big|\sum_{i,j=1}^{n} \xi_i \xi_j \varphi_{\frac{p_i+p_j}{2}}(0+)\Big| < +\infty,$$

so Φ satisfies the assumptions of Theorem 7. Hence, by Theorems 3 and 7 and Corollaries 4 and 8, for each k we have

$$0 \le A_k(\Phi) = \sum_{i,j=1}^{n} \xi_i \xi_j A_k\Big(\Phi_{\frac{p_i+p_j}{2}}\Big) = \sum_{i,j=1}^{n} \xi_i \xi_j \psi_k\Big(\frac{p_i+p_j}{2}\Big).$$

Since ψ_k are continuous in addition to satisfying the above condition, it follows that ψ_k are exponentially convex functions. □

Due to the properties of exponentially convex functions, the following corollary is a direct consequence of the previous lemma

Corollary 12. *For ψ_k, $k = 1, ..., 4$, defined by (5) the following statements hold*

(i) For all $n \in \mathbb{N}$ and $p_i \in I_k$, $1 \le i \le n$ the matrix $[\psi_k(\frac{p_i+p_j}{2})]_{i,j=1}^n$ is positive semidefinite, so

$$\det\Big[\psi_k\Big(\frac{p_i+p_j}{2}\Big)\Big]_{i,j=1}^{n} \ge 0.$$

(ii) For $p, s, t \in I_k$ we have

$$\psi_k(p) \ge \big[\psi_k(s)\big]^{\frac{t-p}{t-s}} \big[\psi_k(t)\big]^{\frac{p-s}{t-s}} \qquad \text{if } p < s < t \quad \text{or} \quad s < t < p$$
$$\psi_k(p) \le \big[\psi_k(s)\big]^{\frac{t-p}{t-s}} \big[\psi_k(t)\big]^{\frac{p-s}{t-s}} \qquad \text{if } s < p < t.$$

Notice that the first set of inequalities in Corollary 12(ii) are refinements of the inequalities in Corollaries 5 and 8. Indeed, the latter inequalities, in the notation introduced in this section, are

$$0 \le \psi_k(p), \qquad k = 2, 4, p \in I_k \backslash \{0, 1\},$$

while the right-hand sides of inequalities in Corollary 12(ii) are nonnegative.

Furthermore, inequalities from Corollary 12(ii) are refinements of the Friedrichs-type inequality from Corollary 10. Indeed, we have the following result

Corollary 13. *Let ψ_4 be defined by (5) and let Ω and f be as in Theorem 3. Then, for $1 < p < s < t$ or $1 < s < t < p$ the following inequality holds*

$$p(p-1)\big[\psi_4(s)\big]^{\frac{t-p}{t-s}}\big[\psi_4(t)\big]^{\frac{p-s}{t-s}}\left[\int_\Omega |f(x)|^p\,dx\right]^{\frac{1-p}{p}} \leq \frac{p\cdot\operatorname{diam}(\Omega)}{2}\left[\int_\Omega \|\nabla f(x)\|^p\,dx\right]^{\frac{1}{p}} - \left[\int_\Omega |f(x)|^p\,dx\right]^{\frac{1}{p}}.$$

Proof. As in the proof of Corollary 9, one can show that

$$\frac{1}{\omega_n}\int_\Omega |f(x)|^{p-1}\left(\int_\Omega \frac{|\nabla f(x)\cdot(y-x)|}{\|y-x\|^n}\,dy\right)dx \leq \frac{\operatorname{diam}(\Omega)}{2}\int_\Omega |f(x)|^{p-1}\|\nabla f(x)\|\,dx$$

Therefore

$$p(p-1)\psi_4(p) \leq \frac{p\cdot\operatorname{diam}(\Omega)}{2}\int_\Omega |f(x)|^{p-1}\|\nabla f(x)\|\,dx - \int_\Omega |f(x)|^p\,dx.$$

Applying Hölder's inequality on the first integral of the right-hand side and multiplying the inequality by $\big[\int_\Omega |f(x)|^p\,dx\big]^{(1-p)/p}$, while taking into account the first inequality from Corollary 12(ii), we get the claim of the corollary. □

Next, we will state and prove Lagrange- and Cauchy-type mean value results.

Lemma 14. *Let $k \in \{1,...,4\}$, let Ω, f and R be as in Theorem 3 and let $\Psi \in C^2((0,R])$. If $A_k(\Psi)$ is finite, $A_k(\Phi_2) \neq 0$ and the function Ψ, when $k = 3$ or 4, satisfies the same limiting assumptions at zero as the function Φ in Theorem 7, then there exists $\xi_k \in [0,R]$ (provided $\Psi''(0) = \lim_{z\to 0}\Psi''(z)$ exists when $\xi_k = 0$) such that*

$$A_k(\Psi) = \Psi''(\xi_k)A_k(\Phi_2).$$

Proof. Since Φ_2 is a convex function, when $A_k(\Phi_2) \neq 0$ by Theorems 3 and 7 and Corollary 4, we have $A_k(\Phi_2) > 0$, $k = 1,...,4$. Let

$$m = \inf_{\tau\in(0,+\infty)} \Psi''(\tau) \quad\text{and}\quad M = \sup_{\tau\in(0,+\infty)} \Psi''(\tau).$$

If $M < +\infty$, then the function $M\Phi_2 - \Psi$ is convex since

$$\frac{d^2}{d\tau^2}\Big(M\frac{\tau^2}{2} - \Psi(\tau)\Big) = M - \Psi''(\tau) \geq 0.$$

By the assumptions of the lemma, the assumptions of Theorems 3 and 7 and Corollary 4 are satisfied and, hence,

$$0 \leq A_k\Big(M\Phi_2 - \Psi\Big), \qquad k = 1,...,4,$$

i. e.

$$A_k(\Psi) \leq MA_k(\Phi_2), \qquad k = 1,...,4. \tag{6}$$

If $M = +\infty$, then inequality (6) holds trivially. Similarly, for a finite m the inequality

$$mA_k(\Phi_2) \leq A_k(\Psi), \qquad k = 1,...,4 \tag{7}$$

holds since $\Psi - m\Phi_2$ is convex, while for $m = -\infty$ inequality (7) holds trivially.

Finally, the existence of ξ_k, $k = 1,...,4$, follows from (6), (7) and continuity of Ψ''. □

Lemma 15. *Let $k \in \{1, ...4\}$. If Ψ and $\widetilde{\Psi}$ satisfy the assumptions of Lemma 14 and if $A_k(\Phi_2) \neq 0$, then there exists $\xi_k \in [0, R]$ such that*

$$\frac{\Psi''(\xi_k)}{\widetilde{\Psi}''(\xi_k)} = \frac{A_k(\Psi)}{A_k(\widetilde{\Psi})}, \tag{8}$$

provided that the denominators are nonzero.

Proof. Let us define a function ϕ by

$$\phi(\tau) = \Psi(\tau)A_k(\widetilde{\Psi}) - \widetilde{\Psi}(\tau)A_k(\Psi).$$

The function ϕ also satisfies Lemma 14 and, hence, there exists $\xi_k \in [0, R]$ such that $A_k(\phi) = \phi''(\xi_k)A_k(\Phi_2)$. Since $A_k(\phi) = 0$ and $\phi''(\xi_k) = \Psi''(\xi_k)A_k(\widetilde{\Psi}) - \widetilde{\Psi}''(\xi_k)A_k(\Psi)$, equality (8) follows. □

Equality (8) allows us to define various means. Indeed, if $\Psi''/\widetilde{\Psi}''$ is an invertible function for functions Ψ and $\widetilde{\Psi}$ that satisfy the assumptions of Lemma 15,

$$\xi_k = \left(\frac{\Psi''}{\widetilde{\Psi}''}\right)^{-1}\left(\frac{A_k(\Psi)}{A_k(\widetilde{\Psi})}\right)$$

is a well-defined mean provided $\xi_k > 0$. In particular, for $\Psi = \Phi_p$ and $\widetilde{\Psi} = \Phi_q$, recalling the definitions (4) and (5) of functions Φ_p and ψ_k, we can define means $E^k_{p,q}$ by

$$E^k_{p,q} = \left(\frac{A_k(\Phi_p)}{A_k(\Phi_q)}\right)^{\frac{1}{p-q}} = \left(\frac{\psi_k(p)}{\psi_k(q)}\right)^{\frac{1}{p-q}}$$

for $p, q \in I_k$, $p \neq q$. Moreover, we can continuously extend these means to cover the case $p = q$ as well by calculating the limits $\lim_{p\to q} E^k_{p,q}$. For $k = 1$ or 2 we get

$$E^k_{p,q} = \begin{cases} \left(\frac{A_k(\Phi_p)}{A_k(\Phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left\{\frac{1-2p}{p(p-1)} - \frac{A_k(\Phi_0\Phi_p)}{A_k(\Phi_p)}\right\}, & p = q \neq 0, 1 \\ \exp\left\{-1 - \frac{A_k(\Phi_0\Phi_1)}{2A_k(\Phi_1)}\right\}, & p = q = 1 \\ \exp\left\{1 - \frac{A_k(\Phi_0^2)}{2A_k(\Phi_0)}\right\}, & p = q = 0 \end{cases} \tag{9}$$

The means $E^k_{p,q}$ for $k = 3$ and $k = 4$ have the same form, but are defined only for $p > 1$ and $q > 1$.

Corollary 16. *Let $k \in \{1, 2, 3, 4\}$ and $p, q, r, s \in I_k$ be such that $p \leq r$ and $q \leq s$. Then*

$$E^k_{p,q} \leq E^k_{r,s}.$$

Proof. Since the functions ψ_k are exponentially convex by Lemma 11, they are also log-convex.

Now, the inequality of the corollary follows directly from log-convexity of the functions ψ_k and continuity of the means E^k. □

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(Neven Elezović) Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia
E-mail address: neven.elez@fer.hr

(Josip Pečarić) Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
E-mail address: pecaric@element.hr

(Marjan Praljak) Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia
E-mail address: mpraljak@pbf.hr