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Slant submanifolds of Lorentzian almost contact manifolds

Khushwant Singh, Siraj Uddin, S.S. Bhatia, and M.A. Khan

Abstract. In this paper we study slant submanifolds of Lorentzian almost contact manifolds. We consider the submanifold M as a space-like and define the slant angle on M and thus we obtain some characterization results.

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1. Introduction

Slant submanifolds were introduced by B.Y. Chen in [5, 6]. These submanifolds are the generalization of both holomorphic and totally real submanifolds of an almost Hermitian manifold. Since then many research articles have appeared on these submanifolds in different known spaces. A. Lotta [7] defined and studied slant submanifolds in contact geometry. Later on, J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez studied slant submanifolds of Sasakian manifolds [4]. Recently, Atceken [2] studied slant and semi-slant submanifolds of an almost paracontact metric manifold.

In this paper, we study slant submanifolds of Lorentzian almost contact manifolds. In section 2, we review some formulae for Lorentzian almost contact manifolds and their submanifolds. In section 3, we define a slant submanifold assuming that it is space-like except ξ. We obtain some characterization results for slant submanifolds of a Lorentzian almost contact manifold. The section 4, has been devoted to the study of slant submanifolds of Lorentzian Sasakian manifolds.

2. Preliminaries

Let $\bar{M}$ be a $(2n+1)-$dimensional manifold with an almost contact structure and compatible Lorentzian metric, $(\bar{M}, \phi, \xi, \eta, g)$ that is, ϕ is $(1,1)$ tensor field, ξ is a structure vector field, η is 1-form and g is Lorentzian metric on $\bar{M}$ satisfying [1]

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \ \eta(X) = -g(X, \xi) \tag{2.2}$$

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ denotes the Lie algebra of tangent vector fields on $\bar{M}$. An almost contact manifold with Lorentzian metric g is called a *Lorentzian almost contact manifold*. From (2.2), it follows that

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.3}$$

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A Lorentzian almost contact manifold is *Lorentzian Sasakian* if [1]

$$(\bar{\nabla}_X\phi)Y = -g(X,Y)\xi - \eta(Y)X. \tag{2.4}$$

It is easy to compute from (2.4) that

$$\bar{\nabla}_X\xi = -\phi X. \tag{2.5}$$

Now, let M be a submanifold of $\bar{M}$, we denote the induced Lorentzian metric on M by the same symbol g. Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections on the ambient manifold $\bar{M}$ and the submanifold M, respectively with respect to the Lorentzian metric g then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.6}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.7}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X,Y), V). \tag{2.8}$$

for any $x \in M$, $X \in T_xM$ and $V \in T^{\perp}_xM$, we write

$$\phi X = TX + NX \tag{2.9}$$

$$\phi V = tV + nV \tag{2.10}$$

where TX(resp. tV) denotes the tangential component of ϕX(resp. ϕV) and NX (resp. nV) denotes the normal component of ϕX (resp. ϕV).

3. Slant submanifolds

Throughout, this section we consider a submanifold M of a Lorentzian manifold $\bar{M}$ such that for all $X \in TM$, $g(X,X) > 0$ or $g(X,X) = 0$ i.e., all the tangent vectors on M are Space-like or null like, we shall call these type of submanifolds as *space-like* and also we assume that the structure vector field ξ is tangent to the submanifold M. For any $x \in M$ and $X \in T_xM$, if the vector field X and ξ are linearly independent then the angle $\theta(X) \in [0, \pi/2]$ between ϕX and T_xM is well defined, if $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_xM$, then M is *slant* in $\bar{M}$. The constant angle $\theta(X)$ is then called the *slant angle* of M in $\bar{M}$ and which in short we denote by Sla(M). The tangent bundle TM at every point $x \in M$ is decomposed as

$$TM = D \oplus \langle\xi\rangle$$

where $\langle\xi\rangle$ is the one dimensional distribution orthogonal to the slant distribution D on M and spanned by the structure vector field ξ.

For any $x \in M$ and $X \in T_XM$ we put $\phi X = TX + NX$ where $TX \in T_XM$ and $NX \in T^{\perp}_xM$. Thus, there is an endomorphism $T : T_xM \longrightarrow T_xM$, whose square T^2 will be denoted by Q. Then tensor fields on M of the type $(1,1)$ determined by their endomorphisms shall be denoted by same letters T and Q. It is easy to show that for every $x \in M$ and $X, Y \in T_xM$, $g(TX,Y) = -g(X,TY)$, which implies that Q is symmetric. Moreover, in the following steps we can prove that the eigenvalue of Q always belong to $[-1, 0]$. For any $X \in T_xM- < \xi >$, we get

$$g(QX,X) = -\|TX\|^2$$

but,

$$\|TX\| \leq \|QX\|$$

$$\|TX\| \leq \mu\|X\|, \text{ and } \mu \in [0,1].$$

Thus we obtain

$$g(QX, X) = -\mu^2(X)\|X\|^2.$$

That is,

$$g(QX, X) = \lambda(X)\|X\|^2$$

where $-1 \leq \lambda(X) \leq 0$ and λ depends on X. In other words, each eigenvalue of Q lies in $[-1, 0]$ and each eigenvalue has even multiplicity.

Now, we have the following theorem.

Theorem 3.1. *Let $x \in M$ and $X \in T_xM$ be an eigenvector of Q with eigenvalue $\lambda(X)$. Suppose X is linearly independent from ξ_x, then,*

$$\cos\theta(X) = \sqrt{-\lambda(X)}\frac{\|X\|}{\|\phi X\|}. \tag{3.1}$$

Proof. For any $X \in TM$ we have

$$\|TX\|^2 = g(TX, TX) = -\lambda(X)\|X\|^2. \tag{3.2}$$

On the other hand by definition of $\theta(X)$, we have

$$\begin{aligned}\cos\theta(X) &= \frac{g(\phi X, TX)}{\|TX\|\|\phi X\|} \\ &= \frac{g(TX, TX)}{\|TX\|\|\phi X\|} = -\lambda(X)\frac{\|X\|^2}{\|TX\|\|\phi X\|} = \frac{\|TX\|^2}{\|\phi X\|}.\end{aligned}$$

Again, using (3.2), we obtain

$$\cos(\theta)(X) = \sqrt{-\lambda(X)}\frac{\|X\|}{\|\phi X\|}.$$

This completes the proof. □

The following characterization theorem gives the existence of eigenvalues of the endomorphism Q.

Theorem 3.2. *Let M be a slant submanifold of a Lorentzian almost contact manifold $\bar{M}$ and $\theta = Sla(M) \neq \pi/2$, then Q admits the real number $-\cos^2\theta$ as the only non-vanishing eigenvalue, for any $x \in M$. Moreover the related eigenspace H satisfies $H \subset D$, where $D = Span(\xi_x)^{\perp} \subset T_xM$.*

Proof. Let $x \in M$, from equation (3.1) $Ker(Q) \neq T_xM$, otherwise $Sla(M) = \pi/2$ which contradicts the assumption. So let λ be an arbitrary non-vanishing eigenvalue of Q and let H be the corresponding eigenspace. Now, we have $dim(D) = 2n$ and $dim(H)$ is even, which shows that $dim(H \cap D) \geq 1$. Let $X \in H \cap D$ is a unit vector, then ϕX is also unit vector then from equation (3.1) we obtain

$$\cos\theta = \sqrt{-\lambda(X)},$$

which proves the first part. Moreover, for any $X \in H$, formula (3.1) yields $\|\phi X\| = \|X\|$ which imply that $g(X, \xi) = 0$, hence $H \subset D$. □

We have noted that, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant. In case of invariant submanifold $T = \phi$ and so

$$T^2 = \phi^2 = -I + \eta \otimes \xi.$$

While in case of anti-invariant submanifold, $T^2 = 0$. In fact, we have the following general result which characterize slant submanifolds.

Theorem 3.3. *Let M be a submanifold of a Lorentzian almost contact manifold $\bar{M}$ such that $\xi \in TM$. Then, M is slant submanifold if and only if there exist a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{3.3}$$

Furthermore, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Proof. Necessary condition is obvious, we have to prove the sufficient condition. Suppose that there exist a constant λ such that $T^2 = \lambda(-I + \eta \otimes \xi)$, then for any $X \in TM - \langle \xi \rangle$, we have

$$\begin{aligned} \cos\theta(X) &= \frac{g(\phi X, TX)}{\|TX\|\|\phi X\|} = \frac{g(TX, TX)}{\|TX\|\|\phi X\|} \\ &= -\frac{g(X, T^2X)}{\|TX\|\|\phi X\|} = \lambda\frac{\|\phi X\|}{\|TX\|}. \end{aligned} \tag{3.4}$$

On the other hand $\cos\theta(X) = \frac{\|TX\|}{\|\phi X\|}$ and therefore by using (3.4) we obtain that $\lambda = \cos^2\theta$. Hence, $\theta(X)$ is a constant angle of M i.e, M is slant. □

Now, we have the following corollary, which can be easily verified.

Corollary 3.1. *Let M be a slant submanifold of a Lorentzian almost contact manifold $\bar{M}$ with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(TX, TY) = \cos^2\theta(g(X,Y) + \eta(X)\eta(Y)) \tag{3.5}$$

$$g(NX, NY) = \sin^2\theta(g(X,Y) + \eta(X)\eta(Y)). \tag{3.6}$$

Proof. For any $X, Y \in TM$, we have

$$g(TX, TY) = -g(X, T^2Y).$$

Then by virtue of (3.3), we obtain (3.5). The proof of (3.6) follows from (2.2) and (2.9). □

4. Slant submanifolds of Lorentzian Sasakian manifolds

In this section we assume that M is a slant submanifold tangent to the structure vector filed ξ of a Lorentzian Sasakian manifold $\bar{M}$ and obtain some interesting results using curvature tensor. First, we have the following example of a Lorentzian Sasakian manifold.

Consider the following structure on $\mathbb{R}^{2n+1}$:

$$\phi_0\left(\sum_{i=1}^{n} X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i} + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{n} Y_i\frac{\partial}{\partial x^i} - \sum_{i=1}^{n} X_i\frac{\partial}{\partial y^i} + \sum_{i=1}^{n} Y_i y^i\frac{\partial}{\partial z}$$

$$\xi = 2\frac{\partial}{\partial z},\ \eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right)$$

and

$$g = -\eta\otimes\eta + \frac{1}{4}\left(\sum_{i=1}^{n} dx^i\otimes dx^i + \sum_{i=1}^{n} dy^i\otimes dy^i\right).$$

Now, consider the vector fields basis on $\mathbb{R}^5$ as follows

$$\left\{2\frac{\partial}{\partial y^1},\ 2\frac{\partial}{\partial y^2},\ 2\left(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}\right),\ 2\left(\frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial z}\right),\ \frac{\partial}{\partial z}\right\}.$$

Then $(\mathbb{R}^5, \phi_0, \xi, \eta, g)$ is a Lorentzian Sasakian manifold [3].

Now, we prove the following characterization theorem for a slant submanifold of Lorentzian Sasakian manifolds.

Theorem 4.1. *Let M be a slant submanifold of a Lorentzian Sasakian manifold $\bar{M}$. Then Q is parallel if and only if M is anti-invariant.*

Proof. Let θ be slant angle of M in $\bar{M}$, then for any $X, Y \in TM$ by (3.3), we have

$$T^2Y = QY = \cos^2\theta(-Y + \eta(Y)\xi) \tag{4.1}$$

and

$$Q\nabla_X Y = \cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi). \tag{4.2}$$

Taking the covariant derivative of (4.1) with respect to $X \in TM$, we get

$$\nabla_X QY = \cos^2\theta(-\nabla_X Y + X\eta(Y)\xi + \eta(Y)\nabla_X\xi). \tag{4.3}$$

Also, we have

$$X\eta(Y) = -Xg(Y,\xi) = -g(\nabla_X Y, \xi) - g(Y, \nabla_X\xi) = \eta(\nabla_X Y) - g(Y, \nabla_X\xi).$$

Using (2.5), (2.6), (2.9) and the above fact in (4.3), we derive

$$\nabla_X QY = \cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi + g(Y, TX)\xi - \eta(Y)TX). \tag{4.4}$$

Then from (4.2) and (4.4), we obtain

$$(\bar{\nabla}_X Q)Y = \nabla_X QY - Q\nabla_X Y = \cos^2\theta(g(TX, Y)\xi - \eta(Y)TX). \tag{4.5}$$

Thus, the assertion follows from (4.5). □

Now, we shall investigate the existence of a slant submanifold using curvature tensor.

Lemma 4.1. *Let M be a submanifold of Lorentzian Sasakian manifold $\bar{M}$ such that ξ is tangent to M. Then for any $X, Y \in TM$, we have*

$$R(X,Y)\xi = (\nabla_Y T)X - (\nabla_X T)Y \tag{4.6}$$

where R is the curvature tensor field associated to the metric induced by $\bar{M}$ on M. Moreover,

$$R(\xi, X)\xi = QX - (\nabla_\xi T)X \tag{4.7}$$

$$R(X, \xi, X, \xi) = g(QX, X). \tag{4.8}$$

Proof. For any $X \in TM$ then from (2.5) and (2.9) we have

$$TX = \nabla_X\xi.$$

Using this fact in the formula $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y$, we obtain

$$(\nabla_X T)Y = -\nabla_X\nabla_Y\xi + \nabla_{\nabla_X Y}\xi.$$

Similarly,

$$(\nabla_Y T)X = \nabla_Y TX - T\nabla_Y X = -\nabla_Y\nabla_X\xi + \nabla_{\nabla_Y X}\xi.$$

Substituting these equations in the definition of $R(X,Y)\xi$ it is easy to get (4.6). Rewriting (4.6) for $X = \xi$ and $Y = X$, we obtain

$$R(\xi, X)\xi = (\nabla_X T)\xi - (\nabla_\xi T)X = QX - (\nabla_\xi T)X$$

which proves (4.7). Now taking the inner product with X in (4.7), we get

$$R(\xi, X, \xi, X) = g(QX, X) - g((\nabla_\xi T)X, X). \tag{4.9}$$

The second term of right hand side in (4.9) will be identically zero as follows

$$\begin{aligned} g((\nabla_\xi T)X, X) &= g(\nabla_\xi TX, X) - g(T\nabla_\xi X, X) \\ &= -g(TX, \nabla_\xi X) + g(\nabla_\xi X, TX) = 0. \end{aligned}$$

Then (4.8) follows from (4.9) using the above fact. □

Theorem 4.2. *Let M be a submanifold of a Lorentzian Sasakian manifold $\bar{M}$ such that the characteristic vector field ξ is tangent to M. If $\theta \in (0, \pi/2)$ then the following statements are equivalent*
(i) *M is slant with slant angle θ.*
(ii) *For any $x \in M$ the sectional curvature of any 2-plane of T_xM containing ξ_x equals $\cos^2\theta$.*

Proof. Assume that the statement (i) holds, then for any $X \perp \xi$ by Theorem 3.3, we have

$$QX = \cos^2\theta X$$

which by virtue of (4.8) yields

$$R(X, \xi, X, \xi) = \cos^2\theta. \tag{4.10}$$

Thus (ii) is proved.

Conversely, suppose that (ii) holds then for any $X \in TM$, we may write

$$X = X_\xi + X_\xi{}^\perp \tag{4.11}$$

where $X_\xi = \eta(X)\xi$ and $X_\xi{}^\perp$ is the component of X perpendicular to the ξ, using (4.10) and (4.11)

$$\frac{R(X_\xi{}^\perp, \xi, X_\xi{}^\perp, \xi)}{|X_\xi|^2} = \cos^2\theta,$$

or,

$$R(X_\xi{}^\perp, \xi, X_\xi{}^\perp, \xi) = \cos^2\theta |X_\xi|^2. \tag{4.12}$$

Let X be a unit vector filed such that $QX = 0$. Then from (4.8) and (4.12), we obtain

$$\cos^2\theta |X_\xi{}^\perp|^2 = 0. \tag{4.13}$$

If $\cos\theta \neq 0$, then from the above equation $X = X_\xi$. This proves that at each point $x \in M$,

$$Ker(Q) = \langle \xi_x \rangle. \tag{4.14}$$

Moreover, let A be the matrix of the endomorphism Q at $x \in M$, then for a unit vector field X on M, $QX = AX$, and as $Q(X_\xi) = 0, X = X_\xi$. Then by (4.8) and (4.12)

$$A = \cos^2\theta I. \tag{4.15}$$

Choosing $\lambda = \cos^2\theta$, for any $x \in M$, the above fact together with (4.14) and Theorem 3.3, verifies that M is slant in $\bar{M}$ with slant angle θ. Finally, suppose $\cos\theta = 0$ and X is an arbitrary unit vector field such that $QX = \lambda X$ where $\lambda \in C^\infty(M)$. Then, from (4.8) and (4.12), we get $g(QX, X) = 0$ that is $\lambda = 0$ and therefore $Q = 0$ which means that M is anti-invariant. □

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(Khushwant Singh, Satvinder Singh Bhatia) School of Mathematics and Computer Applications, Thapar University, Patiala 147004, India
E-mail address: khushwantchahil@gmail.com, ssbhatia@thapar.edu

(Siraj Uddin) Institute of Mathematical Sciences,Faculty of Science,University of Malaya, 50603 Kuala Lumpur, Malaysia
E-mail address: siraj.ch@gmail.com

(Meraj Ali Khan) Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia
E-mail address: meraj79@gmail.com