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Exact parametric solutions of the nonlinear Riccati ODE as well as of some relative classes of linear second order ODEs of variable coefficients

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Abstract. We present a methodology for constructing the exact parametric solutions of the nonlinear first order Riccati ordinary differential equation (ODE) and of some other equivalent classes of second order linear ODEs with variable coefficients. For illustration purposes we extract the general solutions of the strongly nonlinear first order kinematic Eulers ODEs describing the dynamics of a rigid body free to rotate about a fixed point.

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1. Introduction

The vast majority of the problems in mathematical physics and nonlinear mechanics whose dynamics can be described by nonlinear ordinary differential equations (ODEs), do not admit general exact solutions in terms of known (tabulated) functions. Closed form analytical solutions can only be obtained under certain restrictions and assumptions [1-7] that may impose both quantitative and qualitative biases. A celebrated representative of this class of equations is the Riccati equation. This is a first order ODE, reading

$$y_x^{'} = f_2(x)y^2 + f_1(x)y + f_0(x),\ \ f_2(x) \neq 0.$$

Generally speaking, the term "Riccati equation" refers to matrix equations with an analogous quadratic term in both continuous and discrete-time systems. These equations play a key role in many areas of engineering and science especially in control, optimization and systems theory [8, 21-23].

It is well known, since 1700, that in order to construct solutions of a Riccati equation [8, 11], one has to solve an equivalent (through admissible functional transformations) second order linear homogeneous ODE of variable coefficients of the following form

$$z_{xx}^{''} + f_2(x)z_x^{'} + f_1(x)z = 0,\ \ f_1(x) \neq 0,\ \ f_2(x) \neq 0.$$

Here, we construct the exact parametric solution of a first order nonlinear Riccati ODE based on the work presented in [6,14]. Introducing a series of Theorems, Propositions, Corollaries and Lemmas, several of which are new, we construct the analytical solutions (without any restrictive assumptions), not only for the Riccati equation, but also for the equivalent classes of linear second order ODEs of variable coefficients. The

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main contribution of the current work is the introduction of several convenient functional transformations which allow us to define the solution of the Riccati equation through the construction of particular solutions of the Emden-Fowler linear type ODE [4]

$$y''_{xx} = -f(x)y,\ \ f(x) = \text{arbitrary} \neq 0.$$

To demonstrate our approach we extract the analytical solution of the well known three, first order, strongly nonlinear Euler kinematic equations [12-20] describing the dynamics of a rigid body which is free to rotate about a fixed point. Through a convenient decoupling procedure the above system of three equations results to a first order Riccati nonlinear ODE.

2. Some Basic Results

2.1. The first order nonlinear Riccati ODE [4, 9]. The general form of a first order Riccati ODE reads

$$y'_x = f_2(x)y^2 + f_1(x)y + f_0(x) \tag{2.1}$$

where, if $f_2 = 0$, one obtains a first order linear ODE, while if $f_0 = 0$, one derives a first order Bernoulli equation.

The symbols $d(\ )/\,dx = (\ )'_x, d^2(\ )/\,dx^2 = (\ )''_{xx}, \dots$ are used to denote total derivatives.

We state the following [3,4]:

Theorem 2.1. *(a) Given a particular solution $y_0 = y_0(x)$ of the Riccati equation (2.1), the general solution can be written as*

$$y(x) = y_0(x) + \Phi(x)\left[C - \int \Phi(x) f_2(x)\, dx\right]^{-1}, \tag{2.2}$$

where

$$\Phi(x) = \exp\left\{\int [2f_2(x)y_0(x) + f_1(x)]\ dx\right\},\ C = \textit{integration constant.}$$

To the particular solution $y_0(x)$ corresponds $C = \infty$.
(b) Using the admissible functional transformation

$$y(x) = E(x)u(x),\ E(x) = \exp\left(\int f_1(x)\, dx\right),$$

the Riccati equation (2.1) becomes of the normal type

$$u'_x = f_2(x)E(x)u^2(x) + \frac{f_0(x)}{E(x)}.$$

(c) The substitution

$$u(x) = \exp\left(-\int f_2(x) y\, dx\right),$$

reduces the general Riccati equation to a second linear homogenous ODE

$$f_2(x)u''_{xx} + [f_2(x) - f'_{2_x}(x) - f_1(x)f_2(x)]u'_x + f_0(x)f_2^2(x)u = 0.$$

2.2. The first-order nonlinear Abel ODE of the second kind [3, 4, 5]. Consider an Abel nonlinear ODE of the second kind with the general form

$$[y+g(x)]y_x^{'} = f_2(x)y^2 + f_1(x)y + f_0(x),\ f_i \neq 0,\ (i=0,1,2),\ g \neq 0. \tag{2.3}$$

Proposition 2.1. *If the variable coefficients of equation (2.3) satisfy the functional relation [3]*

$$f_1(x) = 2f_2(x)g(x) - g_x^{'}(x),$$

then a solution of this equation reads

$$y = -g(x) + E(x)\left\{2\int [f_0(x) + g(x)g_x^{'}(x) - f_2(x)g^2(x)]E^{-2}(x)\,dx\right\}^2,$$

where

$$E(x) = \exp\int f_2(x)\,dx.$$

Now, consider an Abel nonlinear ODE of the second kind with general form

$$[g_1(x)y + g_0(x)]y_x^{'} = f_2(x)y^2 + f_1(x)y + f_0(x). \tag{2.4}$$

Then the following proposition holds:

Proposition 2.2. *If the variable coefficients of equation (2.4) satisfy the functional relation [3,12]*

$$g_0(x)[2f_2(x) + g_{1_x}^{'}(x)] = g_1(x)[f_1(x) + g_{0_x}^{'}(x)],$$

then its general solution is given by the formula

$$\frac{g_1(x)y^2 + 2g_0(x)y}{g_1(x)I(x)} = 2\int \frac{f_0(x)}{g_1(x)I(x)}\,dx + C,$$

where C is an integration constant, while $I(x)$ is the integrating factor

$$I(x) = \exp\left(\int \frac{2f_2(x)}{g_1(x)}\,dx\right).$$

Referring now to the Abel equation (2.3), the following lemmas hold (see [3]).

Lemma 2.1. *The admissible transformation*

$$y + g(x) = u^{-1}(x),\ y + g(x) \neq 0,$$

reduces (2.3) to the equation

$$u_x^{'} = [f_2(x)g^2(x) - f_1(x)g(x) + f_0(x)]u^3 + [f_1(x) - 2f_2(x)g(x) + g_x^{'}(x)]u^2 + f_2(x)u = 0,$$

that is to an Abel ODE of the first kind of the form

$$y_x^{'} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x),$$

in which the free term $f_0(x)$ is missing.

Lemma 2.2. *The admissible functional transformation*

$$y = un(\xi),\ \xi = \int uf_2(x)\,dx,\ u = exp\left(\int f_1(x)\,dx\right),$$

reduces the Abel equation

$$f_x^{'} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y$$

to the degenerate incomplete Abel equation of the first order

$$n_\xi^{'} = g(\xi)n^3 + n^2,\ g(\xi) = u(x)\frac{f_3(x)}{f_2(x)}.$$

Furthermore, substituting [3]

$$\xi_t^{'} = -\frac{1}{tn(\xi)}, \ \xi_{tt}^{''} = \frac{1}{t^2} + \frac{n_\xi^{'}}{t^2 n^2},$$

the above mentioned incomplete Abel equation becomes a second order nonlinear ODE of the Emden-Fowler type [3], namely

$$\xi_{tt}^{''} = -t^2 g(\xi).$$

2.3. Linear ODEs of the second order [4]. A homogeneous linear ODE of the second order has the form

$$f_2(x)y_{xx}^{''} + f_1(x)y_x^{'} + f_0(x)y = 0. \tag{2.5}$$

Let $f_0 = y_0(x)$ be a nontrivial particular solution of this equation. Then, the general solution of (2.5) is given by the formula [4]:

$$y = y_0(x)\left[C_1 + C_2\int\frac{\exp(-F(x))}{y_0^2(x)}\,dx\right], \quad F(x) = \int\frac{f_1(x)}{f_2(x)}\,dx, \tag{2.6}$$

$$C_1, \quad C_2 = \text{integration constants.}$$

Proposition 2.3. *The substitution* $u(x) = \dfrac{y_x^{'}(x)}{y(x)}$ *brings equation (2.5) to the Riccati equation*

$$f_2(x)u_x^{'} + f_2(x)u^2 + f_1(x)u + f_0(x) = 0.$$

Proposition 2.4. *Assuming*

$$y(x) = u(x)\exp\left[-\frac{1}{2}\int\frac{f_1(x)}{f_2(x)}\,dx\right],$$

(2.5) results to the normal form [4, 9]

$$u_{xx}^{''} + f(x)u = 0, \ f(x) = \frac{f_0(x)}{f_2(x)} - \frac{1}{4}\left[\frac{f_1(x)}{f_2(x)}\right]^2 - \frac{1}{2}\left[\frac{f_1(x)}{f_2(x)}\right]_x^{'}.$$

A nonhomogeneous linear ODE of the second order has the general form

$$f_2(x)y_{xx}^{''} + f_1(x)y_x^{'} + f_0(x) = g(x). \tag{2.7}$$

Let $y_1(x)$ and $y_2(x)$ be two nontrivial linearly independent (i.e. $\dfrac{y_1(x)}{y_2(x)} \neq$ const.) solutions of the corresponding homogeneous ODE ($g = 0$). Then, the general solution of (2.7) (by using the Lagrange method) is given by

$$y = C_1y_1(x) + C_2y_2(x) + y_2(x)\int\frac{y_1(x)g(x)}{f_2(x)W(x)}\,dx - y_1(x)\int\frac{y_2(x)g(x)}{f_2(x)W(x)}\,dx, \tag{2.8}$$

where $W(x)$ is the Wronskian determinant, namely

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_{1_x}^{'}(x) & y_{2_x}^{'}(x) \end{vmatrix} = y_1(x)y_{2_x}^{'}(x) - y_2(x)y_{1_x}^{'}(x) \neq 0. \tag{2.9}$$

We state the following proposition.

Proposition 2.5. *Given a nontrivial solution* $t_1(x) \neq 0$ *of (2.5), formula (2.8) can be used for the construction of the general solution of the complete ODE (2.7)* ($g(x) \neq 0$) *with the second linearly independent solution* $y_2(x)$ *given by [4]:*

$$y_2 = y_1\int\frac{\exp[-F(x)]}{y_1^2(x)}\,dx, \ F(x) = \int\frac{f_1(x)}{f_2(x)}\,dx, \ W(x) = \exp[-F(x)].$$

2.4. The nonlinear second order Emden-Fowler and generalized Emden-Fowler ODEs [4, 6]. For the sake of presentation we use the symbolic notation $\{n, m\}$, $\{n, m, l\}$ to denote the Emden-Fowler and the generalized Emden-Fowler nonlinear ODEs respectively, that is

$$y^{''}_{xx} = Ax^n y^m, \ y^{''}_{xx} = Ax^n y^m (y^{'}_x)^l. \tag{2.10}$$

In the above equations the symbolic notations $\{n, m\}$, $\{n, m, l\}$ are rational numbers. We state the following:

Proposition 2.6. *With $m \neq 1, \ m \neq -2n-3$, the admissible functional transformation [4, 6]:*

$$\xi = \frac{2n+m+3}{m-1} x^{\frac{n+2}{m-1}} y, \ u = x^{\frac{n+2}{m-1}} \left(xy^{'}_x + \frac{n+2}{m-1} y \right),$$

reduces the Emden-Fowler equation $y^{''}_{xx} = Ax^n y^m$ to the Abel equation of the second kind of the normal form

$$uu^{'}_{\xi} - u = -\frac{(n+2)(n-m+1)}{(2n+m-1)^2}\xi + A\left(\frac{m-1}{2n+m+3}\right)^2 \xi^m.$$

We omit the proof of this proposition, because it is included in [4, p. 303], and we continue with the setup of the following results.

Lemma 2.3. *Assuming y as the independent and x as the dependent variable in (2.10), we obtain the generalized Emden-Fowler equation $x(y)$ [3,6]*

$$x^{''}_{yy} = -Ay^m x^n (x^{'}_y)^{3-l}.$$

Denote this transformation as $\mathcal{F}$ and represent it as

$$\{n, m, l\} \overset{\mathcal{F}}{\longleftrightarrow} \{m, n, 3-l\}$$

The twofold transformation $\mathcal{F}$ yields the original equation $y^{''}_{xx} = Ax^n y^m (y^{'}_x)^l$.

Consider now a second order nonlinear ODE of the generalized Emden-Fowler type

$$y^{''}_{xx} = f^n(x) y^m (y^{'}_x)^l, \ m \neq -l+1, \tag{2.11}$$

where $f(x)$ is a known smooth function of the independent variable x.

We state the following proposition.

Proposition 2.7. *For a given nonlinear ODE of the generalized Emden - Fowler type (2.11) there exist admissible functional transformations that lead to the construction of exact parametric solutions for this equation.*

Proof. We introduce the admissible functional transformation

$$\begin{gathered} U = f^{n-l+2} y^{m+l-1}, \ z = \frac{f}{y} y^{'}_x, \ y \neq 0, \\ -\infty < x < +\infty, \ -\infty < z < +\infty, \ m+l-1 \neq 0. \end{gathered} \tag{2.12}$$

The total differential of the first of these equations is

$$dU = \frac{U}{f}[(n-l+2)f^{'}_x + (m+l-1)z] \, dx, \tag{2.13}$$

while the total differential of the second one is given by

$$dz = \left(\frac{f}{y} y''_{xx} + \frac{f'_x - \frac{f}{y} y'_x}{y} y'_x \right) dx. \tag{2.14}$$

Rewriting the first of (2.12) as $U = f^n y^m f^{-l+2} y^{l-1}$, and using also the second of them, the initial equation (2.11) becomes

$$y''_{xx} = \frac{y}{f^2} z^l U \Leftrightarrow \frac{f}{y} y''_{xx} = \frac{1}{f} z^l U.$$

Taking into account the second equation appearing in (2.12), the total differential dz in (2.14) reads

$$dz = \left(\frac{U}{f} z^l + \frac{f'_x - z}{f} z \right) dx,$$

while dividing both expressions for dU, dz by parts one obtains

$$U'_z = \frac{U[(n-l+2)f'_x + (m+l-1)z]}{Uz^l + (f'_z - z)z}, \; Uz^l + (f'_x - z)z \neq 0. \tag{2.15}$$

At this point we introduce the ad hoc functional relation

$$g(z) = f'_x(x) = f'_z z'_x, \tag{2.16}$$

where $g(z)$ is a function of the independent variable z which must be determined.

Thus $U(z)$ is obtained by solving the Abel ODE of the second kind, that is

$$(Uz' + [g(z) - z]z)U'_z = U[(n-l+2)g(z) + (m+l-1)z]. \tag{2.17}$$

Making use of Proposition 2.2, one obtains the following solution for $g(z)$

$$g(z) = z^{-n+2l-3} \left(\frac{-m-2l+3}{n-2l+4} z^{n-2l+3} + \lambda \right), \quad \lambda = \text{first constant}, \tag{2.18}$$

and then, the general solution of (2.17) becomes

$$U(z) = -\frac{g(z) - z}{z^{l-1}} \pm \sqrt{\left(\frac{g(z) - z}{z^{l-1}} \right)^2 + \mu}, \quad \mu = \text{second constant}. \tag{2.19}$$

Finally, based on (2.16), equation (2.18) reads

$$g(z) = \frac{-m-2l+3}{n-2l+4} + \lambda z^{-n+2l-3} = f'_x(x), \tag{2.20}$$

that constitutes a functional relation between z and x. Since $f(x)$ is a known function, this relation provides the parametric expression $x = x(z, C_1)$.

Using equations (2.16), (2.20) we derive

$$g'_x = g'_z z'_x = f''_{xx}(x) \Leftrightarrow z'_x = \frac{f''_{xx}}{g'_z}, \tag{2.21}$$

or equivalently

$$z'_x = \frac{f''_{xx}}{\lambda(-n+2l-3)z^{-n+2l-4}}. \tag{2.22}$$

Thus, from (2.20) the independent variable x is expressed parametrically in terms of $z, (x = x(z, \lambda))$. The parametric expression of y can be extracted by (2.12) and (2.13) through the formulae

$$\begin{gathered} y_x^{'} = \frac{z}{f(x)} y, \\ U_x^{'} = U_z^{'} z_x^{'} = (n-l+2) f^{n-l+1} f_x^{'} y^{m+l-1} + (m+l-1) f^{n-l+2} y^{m+l-2} y_x^{'}. \end{gathered} \tag{2.23}$$

In fact, inserting the first of these equations into the second one, one gets

$$\frac{U_z^{'}}{[(n-l+2)f_x^{\prime} + (m+l-1)z] f^{n-l+1}} z_x^{'} = y^{n+l-1}, \tag{2.24}$$

where $z_x^{'}, x = x(z, \lambda)$ are estimated through (2.22), (2.20),(2.19), while, by means of (2.19), one finds

$$U_z^{'} = \frac{(g_z^{'} - 1) - (l-1)(g-z)}{z^{l-1}} \left[-1 \pm \frac{\dfrac{g-z}{z^{l-1}}}{\sqrt{\left(\dfrac{g-z}{z^{l-1}}\right)^2 + \mu}} \right], \tag{2.25}$$

where

$$\begin{gathered} g_z^{'} = (n-2l+3)\lambda z^{-n+2l-4}, \\ x = x(z, \lambda), \ \ \lambda, \mu = \text{constant parameters.} \end{gathered} \tag{2.26}$$

Equations (2.19), (2.20), (2.24) to (2.26) provide the exact parametric solutions of the ODE

$$y_{xx}^{''} = f^n(x) y^m (y_x^{'})^l.$$

The above construction completes the proof of Proposition 2.7. □

3. Exact parametric solutions of some unsolvable classes of first and second order nonlinear and linear ODEs

In this section we attempt to construct the exact parametric solutions for some unsolvable classes of the first and second order nonlinear and linear ODEs in mathematical physics and applied mathematics. We now state an analogous of Proposition 2.7.

Proposition 3.1. *The linear ODE*

$$y_{xx}^{''} = f(x) y,$$

with $f(x)$ a given smooth function,admits always a nontrivial particular exact parametric solution.

Proof. In the degenerate case when $n = m = 1$, $l = 0$ the generalized Emden-Fowler ODE (2.11) becomes a linear one of variable coefficients, while the coordinate transformation (2.12) is reduced to

$$U = f^3(x), \ \ z = \frac{f(x)}{y} y_z^{'}, \ \ (y \neq 0), \tag{3.1}$$

Thus, equation (2.24) is verified, while equation (2.17) is degenerated to the new Abel second kind type

$$(U + [g(z) - z]z) U_z^{'} = 3Uz, \ \ g(z) = f^{'}(x). \tag{3.2}$$

Therefore, the functional relation between the variable coefficients of the last equation (Proposition 2.2), performs the quintic equation

$$2z^5 - 5f_x^{'}(x)z^4 + 5\lambda = 0,\ \lambda = \text{constant}, \tag{3.3}$$

that furnishes

$$f_x^{'}(x) = \frac{2z^5 + 5\lambda}{5z^4},\ \lambda \neq 0, \tag{3.4}$$

or

$$z = F[f_x^{'}(x), \lambda],\ \lambda \neq 0, \tag{3.5}$$

where $F[f_x^{'}(x), \lambda]$ is a known function (solution of the above mentioned quintic equation for z). Using now the second equation of (3.1) together with (3.4) and (3.5), one gets the following nontrivial particular parametric solution of the given second order linear ODE:

$$\begin{gathered} x = f_x^{'-1}\left(\tfrac{2}{5}z + \tfrac{\lambda}{z^4}\right) = F\left(\tfrac{2}{5}z + \tfrac{\lambda}{z^4}\right), \\ y = \exp\left[\int \frac{\left(\frac{2}{5}z - \frac{4\lambda}{z^4}\right)F_n^{'}}{\mathcal{F}\left(\frac{2}{5}z + \frac{\lambda}{z^4}\right)}\,dz\right], \\ n = n(z) = \tfrac{2}{5}z + \tfrac{\lambda}{z^4},\ \lambda = \text{constant},\ -\infty < z = \text{parameter} < +\infty, \end{gathered} \tag{3.6}$$

in which

$$F \equiv f_x^{'-1},\ \mathcal{F} \equiv f_x^{'-1} \circ f,$$

are known smooth functions since f is a given smooth function, while

$$F_n^{'}\frac{dF}{dn} = \frac{d(f_x^{'-1})}{dn} = \Omega(n),$$

is also known function of its argument $n = \frac{2}{5}z + \frac{\lambda}{z^4}$. Moreover λ is a constant and $-\infty < z < +\infty$ is the basic parameter. □

One can establish the following:

Theorem 3.1. *For a second order linear ODE, $y_{xx}^{''} + f(x)y = 0$, where $f(x)$ is a given smooth function, the exact parametric solution including two arbitrary integration constants, can be constructed.*

Proof. Combining the results of Propositions 2.4 and 2.5, together with those of Proposition 3.1, one extracts the following exact parametric solution for the given linear ODE.

$$\begin{gathered}
F = (f_x^{'})^{-1} = \text{known smooth function} \\
\Omega(n) = \frac{dF}{dn},\ n = -\left(\tfrac{2}{5}z - \tfrac{\lambda}{z^4}\right); \\
\mathcal{F} \equiv f_x^{'-1} \circ f = \text{known smooth function}; \\
x = f_x^{'-1}\left[-\left(\tfrac{2}{5}z + \tfrac{\lambda}{z^4}\right)\right] = F\left[-\left(\tfrac{2}{5}z + \tfrac{\lambda}{z^4}\right)\right] \equiv F(n), \\
y = C_1 J_1 + C_2 J_2; \\
J_1 = \exp\left(-\int \frac{\left(\tfrac{2}{5}z - \tfrac{4\lambda}{z^4}\right)\Omega(n)}{\mathcal{F}\left[-\left(\tfrac{2}{5}z + \tfrac{\lambda}{z^4}\right)\right]}\, dz\right); \\
J_2 = J_1 \int (J_1)^{-2}\Omega(n)\left(\frac{2}{5}z - \frac{4\lambda}{z^5}\right) dz; \\
\lambda = \text{constant parameter}; \\
C_1, C_2 = \text{integration constants}; \\
-\infty < z = \text{parameter} < +\infty.
\end{gathered} \tag{3.7}$$

□

Theorem 3.2. *The exact parametric solution of the Riccati equation of the normal form $y_x^{'}(x) + y^2(x) + f(x) = 0$, where $f(x)$ is a given smooth function, results as follows*

$$\begin{gathered}
F \equiv (f_x^{'-1}) = \textit{known smooth function}; \\
\Omega(n) = F_n^{'} = \frac{df}{dn},\ n = -\left(\tfrac{2}{5}z + \tfrac{\lambda}{z^4}\right); \\
f_x^{'-1}\left[-\left(\tfrac{2}{5}z + \tfrac{\lambda}{z^4}\right)\right] = F\left[-\left(\tfrac{2}{5}z + \tfrac{\lambda}{z^4}\right)\right] \equiv F(n); \\
x = f_x^{'-1}(n) + F(n); \\
y = \frac{(J_1 + CJ_2)_z^{'}}{(J_1 + CJ_2)\left(-\tfrac{2}{5}z + \tfrac{4\lambda}{z^5}\right)\Omega(n)}; \\
J_1,\ J_2 = \textit{as in (3.7)}; \\
\lambda = \textit{constant parameter}; \\
C = \textit{integration constant}; \\
-\infty < z = \textit{parameter} < +\infty.
\end{gathered} \tag{3.8}$$

Proof. The prescribed parametric solution is extracted through the combination of Theorem 3.1 with Proposition 2.3, if we set $f_1 \equiv 0,\ f_0 \equiv f$. □

4. An Application of the proposed solutions

Consider the motion of a rigid body free to rotate about a fixed point . Convenient coordinates are the Euler angles $\psi(t),\ \theta(t),\ \phi(t)$ where t is the time. If $Ox_1y_1z_1$ is a coordinate system at rest with origin O and $Oxyz$ is the system of principal axes for O moving with the body, then the motion may be conceived as the result of three rotations, that is i) precession ϕ, ii) nodding motion θ and iii) spin ψ. The state of motion is a rotation with an instantaneous angular velocity ω the resultant of the vectors of $\psi_t^{'},\ \theta_t^{'},\ \phi_t^{'}$.

The projection of the ω-vector on the $Oxyz$ system furnishes the well known Euler kinematic equations [18, 23] determining the position of the body at any time t, that is

$$\begin{gathered} \phi_t^{'} \sin\theta \sin\psi + \theta_t^{'} \cos\psi = \omega_x, \\ \phi_t^{'} \sin\theta \cos\psi - \theta_t^{'} \sin\psi = \omega_y, \\ \phi_t^{'} \cos\theta + \psi_t^{'} = \omega_z. \end{gathered} \tag{4.1}$$

Here the $\omega_x,\ \omega_y,\ \omega_z$, resultants are considered to be known through the solution of the well known Euler dynamic equations [7, 12, 16-22].

The integration of the three nonlinear kinematic equations (4.1) in the most general case of response, loading and geometry is a very complex mathematical problem. Approximate analytic solutions, or analytic solutions for several special cases were extracted in [16] to [23]. Especially in [23] a successful attempt was made concerning the partial decoupling of the above system. It was proved [23] that equations (4.1) are equivalent to the following second-order nonlinear ODE for the nodding motion $\theta(t)$

$$\theta_{tt}^{''} + \sqrt{\omega^2 - \theta_t^{'2}} \left[\omega_z - \left(\omega^2 - \theta_t^{'2} \right)^{\frac{1}{2}} \cos\theta \right] = \frac{\omega_t^{'} \theta_t^{'} \pm \sqrt{(f^2 - \omega_t^{'2})(\omega^2 - \theta_t^{'2})}}{\omega}, \tag{4.2}$$

where

$$\theta_t^{'2} \le \omega^2 = \omega_x^2 + \omega_y^2,\ f^2 \ge {\omega_{x_t}^{'}}^2 + {\omega_{y_t}^{'}}^2$$

By means of several ad hoc assumptions the authors constructed approximate analytic solutions of equations (4.2) and thus of system (4.1). Quantitative and qualitative analysis coexist with the above constructions [12, 16, 23].

In what follows we will provide the general decoupling procedure concerning system (4.1). In addition, making use of convenient admissible functional transformations we will prove that system (4.1) results in the integration of a Riccati equation with respect to the spin $\psi(t)$.

For the main interval $\psi \in \left[0, \frac{\pi}{2}\right)$ multiplying the two first of equations (4.1) separately by $\sin\psi$ and $\cos\psi$ and adding the new resulting equations, one obtains

$$\phi_t^{'} \sin\theta = \omega_x \sin\psi + \omega_y \cos\psi. \tag{4.3}$$

Similarly, multiplying the two first of (4.1) by $\cos\psi$ and $\sin\psi$ respectively and substituting the results, we extract

$$\theta_t^{'} = \omega_x \cos\psi - \omega_y \sin\psi. \tag{4.4}$$

Thus, initial system (4.1) is equivalent to the following one

$$\phi_t^{'} \sin\theta = \omega_x \sin\psi + \omega_y \cos\psi, \tag{4.5}$$

$$\theta_t^{'} = \omega_x \cos\psi + \omega_y \sin\psi, \tag{4.6}$$

$$\phi_t^{'} \cos\theta + \psi_t^{'} = \omega_z. \tag{4.7}$$

On the other hand, combination of (4.5) and (4.7) results in

$$\tan\theta = \frac{\omega_x \sin\psi + \omega_y \cos\psi}{\omega_z - \psi_t^{'}}, \tag{4.8}$$

so that the system (4.5), (4.6), (4.7) becomes equivalent to the new one

$$\phi_t^{'} \sin\theta = \omega_x \sin\psi + \omega_y \cos\psi, \tag{4.9}$$

$$\theta_t^{'} = \omega_x \cos\psi - \omega_y \sin\psi, \tag{4.10}$$

$$\tan\theta = \frac{\omega_x \sin\psi + \omega_y \cos\psi}{\omega_z - \psi_t^{'}}. \tag{4.11}$$

Making use of the substitution

$$\tan\frac{\psi}{2} = u \Leftrightarrow \frac{1}{\cos^2\left(\frac{\psi}{2}\right)} \frac{\psi_t^{'}}{2} = u_t^{'}, \tag{4.12}$$

and setting

$$\begin{gathered} \frac{\omega_x \sin\psi + \omega_y \cos\psi}{\omega_z - \psi_t^{'}} = \tan\theta = \mu; \\ 0 \le \mu = \text{parameter} < +\infty, \end{gathered} \tag{4.13}$$

equation (4.11) becomes

$$\mu\omega_z - \mu\frac{2u_t^{'}}{1+u^2} = \omega_x \frac{2u}{1+u^2} + \omega_y \frac{1-u^2}{1+u^2},$$

that finally reads the following Riccati equation

$$u_t^{'} = \frac{1}{2}\left(\omega_z + \frac{\omega_y}{\mu}\right) u^2 - \frac{\omega_x}{\mu} u + \frac{1}{2}\left(\omega_z + \frac{\omega_y}{\mu}\right). \tag{4.14}$$

Thus, we have already proved that the problem under consideration, that is to say the decoupling and the solution of these three kinematic Euler equations (4.1), leads to the solution of the Riccati equation (4.14). We underline that the relative problem concerning the solution of the three dynamic Euler equations result in the solution of the three Abel equations of the second kind [7].

It is well known that for a given particular solution $u_0(t)$ of the Riccati equation (4.14) the general solution can be written as (Theorem 2.1a)

$$u(t) = u_0(t) + \Phi(t)\left[C - \int \Phi(t) \frac{\omega_z + \dfrac{\omega_y}{\mu}}{2}\, dt\right]^{-1}$$

where

$$\Phi(t) = \exp\left\{\int \left[\left(\omega_z + \frac{\omega_y}{\mu}\right) u_0(t) - \frac{\omega_x}{\mu}\right] dt\right\}.$$

In addition, the substitution (Theorem 2.1c)

$$p(y) = \exp\left(-\int \frac{\omega_z + \dfrac{\omega_y}{\mu}}{2} u\, dt\right), \tag{4.15}$$

reduces the Riccati equation (4.14) to the following second order linear homogeneous ODE

$$\left(\omega_z+\frac{\omega_y}{\mu}\right)p''_{tt}-\left\{\left(\omega_z+\frac{\omega_y}{\mu}\right)'_t-\left(\omega_z+\frac{\omega_y}{\mu}\right)\frac{\omega_x}{\mu}\right\}p'_t+ \\ +\frac{1}{4}\left(\omega_z+\frac{\omega_y}{\mu}\right)^2\left(\omega_z-\frac{\omega_y}{\mu}\right)p=0. \tag{4.16}$$

From now on, substituting

$$F_1(t)=\left(\omega_z+\frac{\omega_y}{\mu}\right)'_t-\left(\omega_z+\frac{\omega_y}{\mu}\right)\frac{\omega_x}{\mu},$$
$$F_2(t)=\left(\omega_z+\frac{\omega_y}{\mu}\right), \tag{4.17}$$
$$F_0(t)=\frac{1}{4}\left(\omega_z+\frac{\omega_y}{\mu}\right)^2\left(\omega_z-\frac{\omega_y}{\mu}\right),$$

the second order linear homogeneous ODE (4.16) becomes

$$F_2(t)p''_{tt}+F_1(t)p'_t+F_0(t)p=0, \tag{4.18}$$

while using Proposition 2.4, that is using the transformation

$$p(t)=r(t)\exp\left(-\frac{1}{2}\int\frac{F_1}{F_2}\,dt\right)=r(t)\left(\omega_z+\frac{\omega_y}{\mu}\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\int\frac{\omega_x}{\mu}\,dt\right), \tag{4.19}$$

one extracts the following degenerate linear ODE corresponding to the ODE (4.18)

$$r''_{tt}+F(t)r(t)=0, \tag{4.20}$$

where

$$F(t)=\frac{1}{4}\left[\omega_z^2-\left(\frac{\omega_y}{\mu}\right)^2\right]-\frac{1}{4}\left[\left(\ln\left|\omega_z+\frac{\omega_y}{\mu}\right|\right)'_t-\frac{\omega_x}{\mu}\right]^2-\frac{1}{2}\left[\left(\ln\left|\omega_z+\frac{\omega_y}{\mu}\right|\right)'_t-\frac{\omega_x}{\mu}\right]'_t. \tag{4.21}$$

Since through the solution of the Euler dynamic equations [7], $\omega_x(t);\omega_y(t);\omega_z(t)$ are known functions of time t, $F(t;\lambda,\mu)$ in (4.20) is also a known function of time t by way of substitutions (4.17) and (4.19). Therefore we are able now to construct an exact parametric solution of the second order linear ODE (4.20) by using Theorem 3.2. In conclusion we state the following:

Corollary 4.1. *The exact parametric solution of the second order linear ODE of variable coefficients (4.20) becomes as follows*

$$f^* \equiv \left(F_t^{'-1}\right) = \textit{known smooth function};$$

$$\Omega(n) = (f^*)_n^{'} = \frac{df^*}{dn}, \quad n = -\left(\frac{2}{5}z + \frac{\lambda}{z^4}\right);$$

$$t = F_t^{'-1}\left[-\left(\frac{2}{5}z + \frac{\lambda}{z^4}\right); \mu\right] = f^*\left[-\left(\frac{2}{5}z + \frac{\lambda}{z^4}\right); \mu\right] \equiv \Omega(n);$$

$$\mathcal{F} = F_t^{'-1} \circ F;$$
$$r = C_1 J_1 + C_2 J_2;$$

$$J_1 = \exp\left[-\int \frac{\left(\frac{2}{5}z - \frac{4\lambda}{z^4}\right)}{\mathcal{F}\left[-\left(\frac{2}{5}z + \frac{\lambda}{z^4}\right); \mu\right]}\Omega(n)\, dz\right]; \tag{4.22}$$

$$J_2 = J_1 \int (J_1)^{-2}\, \Omega(n) \left(\frac{2}{5}z - \frac{4\lambda}{z^5}\right) dz;$$

$$\lambda = \textit{constant parameter};$$

$$0 \leq \mu = \textit{first parameter} < +\infty;$$

$$C_1, C_2 = \textit{integration constants};$$

$$-\infty < z = \textit{parameter} < +\infty.$$

Proof. The proof is the results of Theorem 3.1 together with Theorem 3.2 if instead of the independent variable x, the variable t (t is the time) is introduced. □

Our aim is now to define the exact parametric solution of the Riccati equation (4.14) and thus the exact solution for the spin resultant $\psi(t)$. For this purpose using the expression for p among the transformations (4.15), (4.19), Corollary 4.1 and taking into account substitutions (4.17) and (4.12), we finally extract

$$p = r\exp\left(-\frac{1}{2}\int \frac{F_1}{F_2}\, dt\right), \; p = \exp\left(-\int F_2 u\, dt\right). \tag{4.23}$$

Functional relations (4.23), after differentiation and using also (4.12) perform a unique for the spin ψ equation, namely

$$\tan\frac{\psi}{2} = u = -\frac{1}{F_2}\frac{r_t^{'}}{r} + \frac{1}{2}\frac{F_1}{F_2^2}, \tag{4.24}$$

where F_1, F_2 as in (4.17). By now through (4.24) we finally extract the following expression for the spin result ψ

$$\tan\frac{\psi}{2} = \frac{1}{F_2}\left(\frac{1}{2}\frac{F_1}{F_2} - \frac{r_t^{'}}{r}\right), \tag{4.25}$$

where

$$F_1, F_2 = \text{as in equations (4.17).}$$

We state the following Corollary, giving in exact parametric form of the spin resultant ψ.

Corollary 4.2. *The exact parametric expression for the spin resultant ψ is given by:*

$$\tan\frac{\psi}{2} = \frac{1}{F_2}\left(\frac{1}{2}\frac{F_1}{F_2} - \frac{r_t^{'}}{r}\right), \tag{4.26}$$

where

$$F_1, F_2 = \textit{as in equations (4.17)};$$

$$\frac{r_t^{'}}{r} = \frac{(J_1 + CJ_2)_t^{'}}{J_1 + CJ_2} = \frac{(J_1 + CJ_2)_z^{'} z_t^{'}}{J_1 + CJ_2};$$

$$z_t^{'} = \frac{\Omega(n)}{\dfrac{2}{5} - \dfrac{4\lambda}{z^5}}; \tag{4.27}$$

$$\Omega(n) = \left((f^*)_n^{'}\right) = \frac{df^*}{dn},\ n = -\left(\frac{2}{5}z + \frac{\lambda}{z^4}\right);$$

$$f^* = (F_t^{'-1}) = \textit{known smooth function};$$

$$\mathcal{F} \equiv f^* \circ F = (F_t^{'-1}) \circ F = \textit{known smooth function};$$

$$t = F_t^{'-1}\left[-\left(\frac{2}{5}z + \frac{\lambda}{z^4}\right); \mu\right] = f^*\left[-\left(\frac{2}{5}z + \frac{\lambda}{z^4}\right); \mu\right] \equiv \Omega(n);$$

$$r = C_1 J_1 + C_2 J_2,$$

$$J_1 = \exp\left(-\int \frac{\left(\dfrac{2}{5}z - \dfrac{4\lambda}{z^4}\right)}{\mathcal{F}\left[-\left(\dfrac{2}{5}z + \dfrac{\lambda}{z^4}\right); \mu\right]}\Omega(n)\, dz\right),$$

$$J_2 = J_1 \int (J_1)^{-2}\,\Omega(n)\left(\frac{2}{5}z - \frac{4\lambda}{z^5}\right)\, dz; \tag{4.28}$$

$$\lambda = \textit{constant};$$

$$C = \textit{integration constant};$$

$$0 \le \mu = \textit{first parameter} < +\infty;$$

$$-\infty < z = \textit{parameter} < +\infty.$$

Proof. The proof results by the combination of Theorem 3.1 together with Theorem 3.2 as well as formulae (4.21) and (4.22) and (4.23).

The remaining other two component resultants, namely nodding motion θ and precession ϕ are obtained parametrically, after convenient integrations of equations (4.9) and (4.10) including suitable integration constants. ☐

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