Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 39(2), 2012, Pages 244–250
ISSN: 1223-6934

Hilbert's integral inequality in whole plane with general homogeneous kernel

Predrag Vuković

Abstract. The main objective of this paper is a study of some new generalizations of Hilbert's and Hardy-Hilbert's type inequalities. We build a new Hilbert's inequality with general homogeneous functions of degree $-2s$, $s \geq 0$ in whole plane. Also, we obtain the best possible constants when the parameters satisfy appropriate conditions.

2010 Mathematics Subject Classification. Primary 26D15; Secondary 47A07.
Key words and phrases. Hilbert's inequality, homogeneous kernel, the best possible constant, conjugate exponents.

1. Introduction

Although classical, Hilbert's inequality and its generalizations and modifications are still of a great interest. Zheng Zeng et al. in [4] considered the Hardy-Hilbert inequality with the integral in whole plane. They obtained the following result.

Let $1/p + 1/q = 1$, $p > 1$, $r \in (-1, 0)$, $0 < \alpha < \beta < \pi$. If $k_1 := \int_0^\infty u^{-1+r} \ln((1 + 2u\cos\alpha + u^2)/(1 + 2u\cos\beta + u^2))du$, $k_2 := \int_0^\infty u^{-1+r} \ln((1 - 2u\cos\alpha + u^2)/(1 - 2u\cos\beta + u^2))du$, then

$$\begin{aligned} k_1 &= \frac{4\pi \sin(r(\beta-\alpha)/2)\sin(r(\alpha+\beta)/2)}{r\sin r\pi}, \\ k_2 &= \frac{4\pi \sin(r(\beta-\alpha)/2)\sin(r\pi - r(\alpha+\beta)/2)}{r\sin r\pi}, \\ k &:= \int_{-\infty}^{\infty} |u|^{-1+r} \left| \ln \frac{1+2u\cos\alpha+u^2}{1+2u\cos\beta+u^2} \right| du \\ &= k_1 + k_2 = \frac{4\pi\sin(r(\beta-\alpha)/2)\cos((r/2)(\pi-\alpha-\beta))}{r\cos(r\pi/2)} \end{aligned} \tag{1}$$

If both functions, $f(x)$ and $g(x)$, are non-negative measurable functions and satisfy $0 < \int_{-\infty}^{\infty} |x|^{p(1+r)-1} f^p(x)dx < \infty$ and $0 < \int_{-\infty}^{\infty} |x|^{q(1-r)-1} g^q(x)dx < \infty$, then

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y) \left| \ln \frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2} \right| dxdy \\ &\quad < k \left(\int_{-\infty}^{\infty} |x|^{p(1+r)-1} f^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q(1-r)-1} g^q(x)dx \right)^{1/q}, \end{aligned} \tag{2}$$

Received March 13, 2012.
This research was supported by the Croatian Ministry of Science, Education, and Sports, under Research Grants 058–1170889–1050.

$$\int_{-\infty}^{\infty}|y|^{pr-1}\left(\int_{-\infty}^{\infty}f(x)\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dx\right)^p dy$$
$$< k^p\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx, \tag{3}$$

where the constant factor k is defined by (1). Inequalities (2) and (3) are equivalent, and where the constant factors k and k^p are the best possible.

Also, very recently Dongmei Xin et al. in [3] proved Hilbert-type inequalities with the homogeneous kernel of degree -2.

If $p > 1$, $1/p + 1/q = 1$, $|\lambda| < 1$, $0 < \alpha_1 < \alpha_2 < \pi$, $f, g \geq 0$, satisfying $0 < \int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx < \infty$ and $0 < \int_{-\infty}^{\infty}|y|^{q\lambda-1}g^q(y)dy < \infty$, then we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}f(x)g(y)dxdy$$
$$< k(\lambda)\left(\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|y|^{q\lambda-1}g^q(y)dy\right)^{1/q}, \tag{4}$$

$$\int_{-\infty}^{\infty}|y|^{p(1-\lambda)-1}\left(\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}f(x)dx\right)^p dy$$
$$< k^p(\lambda)\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx, \tag{5}$$

where the constant factors

$$k(\lambda) = \frac{\pi}{\sin\lambda\pi}\left[\frac{\sin\lambda\alpha_1}{\sin\alpha_1}+\frac{\sin\lambda(\pi-\alpha_2)}{\sin\alpha_2}\right] \quad (0 < |\lambda| < 1)$$

and $k^p(\lambda)$ are the best possible. Inequalities (4) and (5) are equivalent.

In this paper a generalization of above results for a general type of homogeneous kernels of degree of $-2s$, $s \geq 0$, is obtained. Recall that for a homogeneous function $K(x, y)$ of degree $-\lambda$, $\lambda \geq 0$, equality $K(tx, ty) = t^{-\lambda}K(x, y)$ is satisfied for every $t \in \mathbb{R}$.

Few years ago, M. Krnić and J. Pečarić [2], provided an unified treatment of the Hilbert and Hardy-Hilbert type inequalities in general form and extended them to cover the case when p and q are conjugate exponents. More precisely, they obtained the following two equivalent inequalities:

$$\int_{\Omega\times\Omega}K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \tag{6}$$
$$\leq\left[\int_\Omega\varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega\psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y) \leq \int_\Omega\varphi^p(x)F(x)f^p(x)d\mu_1(x), \tag{7}$$

where $p > 1$, μ_1, μ_2 are positive σ-finite measures, $K : \Omega\times\Omega \to \mathbb{R}$, $f, g, \varphi, \psi : \Omega \to \mathbb{R}$ are measurable, non-negative functions and

$$F(x) = \int_\Omega\frac{K(x,y)}{\psi^p(y)}d\mu_2(y) \quad\text{and}\quad G(y) = \int_\Omega\frac{K(x,y)}{\varphi^q(x)}d\mu_1(x). \tag{8}$$

The paper [2] also deals with the equality conditions in (6) and (7). More precisely, it is shown that the equality in (6) (and analogously in (7)) holds if and only if

$$f(x) = K_1\varphi^{-q}(x) \quad \text{and} \quad g(y) = K_2\psi^{-p}(y) \quad \text{a.e. on } \Omega, \tag{9}$$

for arbitrary constants K_1 and K_2.

Our results will be based on the mentioned results of Krnić and Pečarić. In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions.

2. Main results

In this section we develop an unified treatment of the Hilbert and Hardy-Hilbert type inequalities with general homogeneous kernels. At the beginning, we have to establish some basic notation and definitions.

We suppose that $K : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, $\lambda \geq 0$. Further, we define

$$k_1(\alpha) = \int_0^\infty K(1,t)t^{-\alpha}dt, \quad \text{and} \quad k_2(\alpha) = \int_0^\infty K(1,-t)t^{-\alpha}dt, \tag{10}$$

defined in terms of the function K.

To obtain main results we need the following lemma.

Lemma 2.1. *If $\lambda = 2s$, $s \geq 0$, and $K : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, then we have*

$$\int_{-\infty}^0 K(x,y)\frac{|x|^{\alpha-1+\lambda}}{|y|^\alpha}dy = k_1(\alpha), \quad \int_0^\infty K(x,y)\frac{|x|^{\alpha-1+\lambda}}{|y|^\alpha}dy = k_2(\alpha),\ x \in (-\infty,0),$$

and

$$\int_{-\infty}^0 K(x,y)\frac{|x|^{\alpha-1+\lambda}}{|y|^\alpha}dy = k_2(\alpha), \quad \int_0^\infty K(x,y)\frac{|x|^{\alpha-1+\lambda}}{|y|^\alpha}dy = k_1(\alpha),\ x \in (0,\infty).$$

Proof. We use the substitutions $y = ux$ and $y = -ux$. The proof follows easily from homogeneity of the function $K(x,y)$ and the fact $(-x)^\lambda = x^\lambda$, $\lambda = 2s$. □

Utilizing the inequalities (6) and (7) we obtain the following theorem.

Theorem 2.1. *Let $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and let $\lambda = 2s$, $s \geq 0$. If $K : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is non-negative homogeneous function of degree $-\lambda$, and A_1, A_2 are real parameters such that $k_i(pA_2) < \infty$, $k_i(qA_1) < \infty$, $i = 1,2$, then the inequalities*

$$\int_{-\infty}^\infty\int_{-\infty}^\infty K(x,y)f(x)g(y)dxdy \tag{11}$$
$$\leq L\left(\int_{-\infty}^\infty |x|^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^\infty |y|^{1-\lambda+q(A_2-A_1)}g^q(y)dy\right)^{\frac{1}{q}}$$

and

$$\int_{-\infty}^\infty |y|^{(1-\lambda)(p-1)+p(A_1-A_2)}\left(\int_{-\infty}^\infty K(x,y)f(x)dx\right)^p dy$$
$$\leq L^p\int_{-\infty}^\infty |x|^{1-\lambda+p(A_1-A_2)}f^p(x)dx, \tag{12}$$

where

$$L = \left(k_1(pA_2)+k_2(pA_2)\right)^{\frac{1}{p}}\left(k_1(qA_1)+k_2(qA_1)\right)^{\frac{1}{q}}, \tag{13}$$

hold for all non-negative functions f and g. Moreover, inequalities (11) *and* (12) *are equivalent. Equalities in* (11) *and* (12) *hold if and only if $f = 0$ or $g = 0$ a.e. on $\mathbb{R}$.*

Proof. Rewrite the inequality (6) for the functions $\varphi(x) = |x|^{A_1}$ and $\psi(y) = |y|^{A_2}$. Let the functions $F(x)$ and $G(y)$ are defined by (8). By using Lemma 2.1 we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \varphi^p(x)F(x)f^p(x)dx \\
&\quad = \int_{-\infty}^{0} |x|^{1-\lambda+p(A_1-A_2)} \left(\int_{-\infty}^{\infty} K(x,y)\frac{|x|^{pA_2-1+\lambda}}{|y|^{pA_2}}dy\right) f^p(x)dx \\
&\qquad + \int_{0}^{\infty} |x|^{1-\lambda+p(A_1-A_2)} \left(\int_{-\infty}^{\infty} K(x,y)\frac{|x|^{pA_2-1+\lambda}}{|y|^{pA_2}}dy\right) f^p(x)dx \\
&\quad = (k_1(pA_2)+k_2(pA_2)) \int_{-\infty}^{\infty} |x|^{1-\lambda+p(A_1-A_2)} f^p(x)dx, \qquad (14)
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi^q(y)G(y)g^q(y)dy \\
&\quad = (k_1(qA_1)+k_2(qA_1)) \int_{-\infty}^{\infty} |y|^{1-\lambda+q(A_2-A_1)} g^q(y)dy. \qquad (15)
\end{aligned}$$

Now, from (6), (14) and (15) we get the inequality (11). In the same way the inequalities (12) follows directly from (7). Condition (9) immediately gives that nontrivial case of equality in (11) and (12) leads to the divergent integrals. □

Remark 2.1. *If the homogeneous kernel $K : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a symmetric function, that is when $K(x,y) = K(y,x)$, $\forall\ x, y \in \mathbb{R}$, then the constant L, defined by* (13), *becomes*

$$L = (k_1(pA_2)+k_2(pA_2))^{\frac{1}{p}} \left(k_1(2-\lambda-qA_1)+k_2(2-\lambda-qA_1)\right)^{\frac{1}{q}}.$$

The main idea in obtaining the best possible constant factor is a reduction of constant L, defined by (13), in the form without exponents, by appropriate choice of parameters A_1 and A_2. Thus, it is natural to set the condition

$$pA_2 + qA_1 = 2 - \lambda, \qquad (16)$$

since in that setting relation $k(pA_2) = k(2-\lambda-qA_1)$ holds. In such a way, the constant factor L from Theorem 2.1 becomes

$$L^* = k_1(pA_2) + k_2(pA_2). \qquad (17)$$

Further, under assumption (16), the inequalities (11) and (12) become

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)f(x)g(y)dxdy \qquad (18)\\
&\quad \le L^* \left(\int_{-\infty}^{\infty} |x|^{-1+pqA_1} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{-1+pqA_2} g^q(y)dy\right)^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} |y|^{(p-1)(1-pqA_2)} \left(\int_{-\infty}^{\infty} K(x,y)f(x)dx\right)^p dy\right)^{\frac{1}{p}} \\
&\qquad \le L^* \left(\int_{-\infty}^{\infty} |x|^{-1+pqA_1} f^p(x)dx\right)^{\frac{1}{p}}. \qquad (19)
\end{aligned}$$

Our aim is to show that the constants involved in the right-hand side of inequalities (18) and (19) are the best possible. To obtain our main result we need the following lemma.

Lemma 2.2. *Suppose that p and q are conjugate parameters, $p > 1$. Let parameters λ, A_1, A_2 fulfill conditions as in the statement of Theorem (2.1) and let $pA_2+qA_1 = 2-\lambda$. For $\varepsilon > 0$ define functions $\widetilde{f}$, $\widetilde{g}$ as follows*

$$\widetilde{f}(x) = |x|^{-qA_1-\frac{2\varepsilon}{p}} \cdot \chi_{\mathbb{R}\setminus[-1,1]}, \quad \widetilde{g}(y) = |y|^{-pA_2-\frac{2\varepsilon}{q}} \cdot \chi_{\mathbb{R}\setminus[-1,1]}.$$

Then the relation

$$\varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)\widetilde{f}(x)\widetilde{g}(y) = L^* + o(1), \tag{20}$$

holds for $\varepsilon \to 0^+$, where the constant L^* is defined by (17).

Proof. Let us denote the left-hand side of relation (20) with I_ε. By putting the functions $\widetilde{f}$ and $\widetilde{g}$ in I_ε, we obtain

$$\begin{aligned} I_\varepsilon =&\varepsilon \int_{-\infty}^{-1} (-x)^{-qA_1-\frac{2\varepsilon}{p}} \left(\int_{-\infty}^{\infty} K(x,y)\widetilde{g}(y)dy\right) dx \\ &+ \varepsilon \int_1^{\infty} x^{-qA_1-\frac{2\varepsilon}{p}} \left(\int_{-\infty}^{\infty} K(x,y)\widetilde{g}(y)dy\right) dx \\ =&\varepsilon(I_1+I_2). \end{aligned} \tag{21}$$

Further, we obtain $I_1 = I_{11} + I_{12}$, where

$$I_{11} = \int_{-\infty}^{-1} (-x)^{-qA_1-\frac{2\varepsilon}{p}} \left(\int_{-\infty}^{-1} K(x,y)(-y)^{-pA_2-\frac{2\varepsilon}{q}} dy\right) dx,$$

$$I_{12} = \int_{-\infty}^{-1} (-x)^{-qA_1-\frac{2\varepsilon}{p}} \left(\int_{1}^{\infty} K(x,y)y^{-pA_2-\frac{2\varepsilon}{q}} dy\right) dx.$$

By using substitutions $y = xu$, $y = -xu$, homogeneity of the function $K(x,y)$ and the condition $pA_2 + qA_1 = 2 - \lambda$, we obtain the expressions

$$I_{11} = \int_1^{\infty} x^{-1-2\varepsilon} \left(\int_{1/x}^{\infty} K(1,u)u^{-pA_2-\frac{2\varepsilon}{q}} du\right) dx,$$

$$I_{12} = \int_1^{\infty} x^{-1-2\varepsilon} \left(\int_{1/x}^{\infty} K(1,-u)u^{-pA_2-\frac{2\varepsilon}{q}} du\right) dx.$$

By Fubini theorem we have

$$\begin{aligned} I_{11} =&\int_0^1 \left(\int_{1/u}^{\infty} x^{-1-2\varepsilon}dx\right) K(1,u)u^{-pA_2-\frac{2\varepsilon}{q}} du \\ &\qquad + \left(\int_1^{\infty} x^{-1-2\varepsilon}dx\right) \cdot \int_1^{\infty} K(1,u)u^{-pA_2-\frac{2\varepsilon}{q}} du \\ =&\frac{1}{2\varepsilon}\left(\int_0^1 K(1,u)u^{-pA_2+\frac{2\varepsilon}{p}} du + \int_1^{\infty} K(1,u)u^{-pA_2-\frac{2\varepsilon}{q}} du\right), \end{aligned}$$

and

$$I_{12} = \frac{1}{2\varepsilon}\left(\int_0^1 K(1,-u)u^{-pA_2+\frac{2\varepsilon}{p}} du + \int_1^{\infty} K(1,-u)u^{-pA_2-\frac{2\varepsilon}{q}} du\right).$$

Similarly, we have

$$I_2 = \int_1^{\infty} |x|^{-qA_1 - \frac{2\varepsilon}{p}} \left(\int_{-\infty}^{\infty} K(x,y)\widetilde{g}(y)dy \right) dx = I_{11} + I_{12}.$$

Now, from (21) we get

$$\begin{aligned}
I_\varepsilon =& \varepsilon(I_1 + I_2) \\
=& \varepsilon(2I_{11} + 2I_{12}) \\
=& \int_0^1 K(1,u)u^{-pA_2 + \frac{2\varepsilon}{p}} du + \int_1^{\infty} K(1,u)u^{-pA_2 - \frac{2\varepsilon}{q}} du \\
&+ \int_0^1 K(1,-u)u^{-pA_2 + \frac{2\varepsilon}{p}} du + \int_1^{\infty} K(1,-u)u^{-pA_2 - \frac{2\varepsilon}{q}} du. \qquad (22)
\end{aligned}$$

If $p > 1$, then $q > 1$, so $\frac{2\varepsilon}{p} > 0$ and $\frac{2\varepsilon}{q} > 0$. Hence, relation (22) yields

$$\begin{aligned}
I_\varepsilon \leq& \int_0^1 K(1,u)u^{-pA_2} du + \int_1^{\infty} K(1,u)u^{-pA_2} du \\
&+ \int_0^1 K(1,-u)u^{-pA_2} du + \int_1^{\infty} K(1,-u)u^{-pA_2} du \\
\leq& k_1(pA_2) + k_2(pA_2) = L^*.
\end{aligned}$$

Finally, by the Lebesgue control convergent theorem we obtain (20). □

Now, we are ready to state and prove the main result, concerning the best possible constant factors in inequalities (18) and (19).

Theorem 2.2. *Let p and q be conjugate exponents, $p > 1$, let A_1 and A_2 be real parameters such that $pA_2 + qA_1 = 2 - \lambda$, and let $L^* = k_1(pA_2) + k_2(pA_2) < \infty$. Then the constant factor L^* is the best possible in both inequalities* (18) *and* (19).

Proof. Let the parameter ε and the functions $\widetilde{f}$, $\widetilde{g}$ as in the statement of Lemma 2.2. If $p > 1$, then $q > 1$. Suppose that the constant factor L^* is not the best possible in inequality (18). That means that there exists factor $L_1 < L^*$ such that inequality holds if we replace L^* with L_1. We get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)\widetilde{f}(x)\widetilde{g}(y)dxdy \leq L_1 \left(\int_{-\infty}^{-1} (-x)^{-1-2\varepsilon} dx + \int_1^{\infty} x^{-1-2\varepsilon} dx \right),$$

and further

$$\varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)\widetilde{f}(x)\widetilde{g}(y)dxdy \leq L_1. \qquad (23)$$

The left-hand side of the inequality (23) coincides with the left-hand side of relation (20), so by Lemma 2.2 we have

$$L^* + o(1) \leq L_1.$$

Now by letting $\varepsilon \to O^+$ we obtain that $L^* \leq L_1$ which contradicts with the assumption $L_1 < L^*$. Thus, the constant L^* is the best possible. □

Remark 2.2. *Note that the kernel $K_1(x,y) = |\ln((x^2+2xy\cos\alpha+y^2)/(x^2+2xy\cos\beta+y^2))|$, $0 < \alpha < \beta < \pi$, is homogeneous function of degree* 0. *By putting the kernel $K_1(x,y)$, the parameters $A_1 = (1+r)/q$ and $A_2 = (1-r)/p$ in the inequalities* (18) *and* (19) *we obtain the result of Xie et al. from Introduction (see also* [4]*). Similarly, the kernel $K_2(x,y) = \min_{i\in\{1,2\}}\{1/(x^2 + 2xy\cos\alpha_i + y^2)\}$, $0 < \alpha_1 < \alpha_2 < \pi$, is homogeneous function of degree -2. By putting the kernel $K_2(x,y)$, the parameters*

$A_1 = -\lambda/q$ *and* $A_2 = \lambda/p$ *in the inequalities* (18) *and* (19) *we obtain the result of Xin (see also* [3]*).*

Remark 2.3. *Similarly as in Remark 2.2, in particular*

(i) *for* $K(x,y) = (x^\lambda + y^\lambda)^{-1}$, $(\lambda = 2s,\ s \geq 0,\ pA_2 + qA_1 = 2 - \lambda)$ *in* (18) *and* (19) *we find the best possible constant*

$$L^* = k_1(pA_2) + k_2(pA_2) = 2\int_0^\infty \frac{u^{-pA_2}}{1+u^\lambda}du = \frac{2\pi}{\lambda \cos\frac{\pi(1-pA_2)}{\lambda}},$$

where $A_2 < \frac{1}{p}$ *and* $pA_2 + \lambda > 1$;

(ii) *for* $K(x,y) = (\max\{x^\lambda, y^\lambda\})^{-1}$, $(\lambda = 2s,\ s \geq 0,\ pA_2 + qA_1 = 2 - \lambda)$ *in* (18) *and* (19) *we find the best possible constant*

$$L^* = \int_0^\infty \frac{u^{-pA_2}}{\max\{1, u^\lambda\}}du = \frac{2}{1-pA_2} + \frac{2}{pA_2 + \lambda - 1},$$

where $A_2 < \frac{1}{p}$ *and* $pA_2 + \lambda > 1$;

(iii) *for* $K(x,y) = \left(\frac{|\min\{x,y\}|}{|x-y|}\right)^\beta$, $(0 < \beta < 1,\ pA_2 + qA_1 = 2)$ *in* (18) *and* (19) *we find the best possible constant*

$$\begin{aligned} L^* = &\int_0^\infty \left(\frac{|\min\{1,u\}|}{|1-u|}\right)^\beta u^{-pA_2}du + \int_0^\infty \frac{u^{\beta - pA_2}}{(1+u)^\beta}du \\ =&B(1-\beta, 1-pA_2+\beta) + B(1-\beta, pA_2+\beta-1) \\ &+ B(pA_2 - 1, 1 - pA_2 + \beta), \end{aligned}$$

where $A_2 > \frac{1}{p}$ *and* $pA_2 + \beta > 1,\ pA_2 - \beta < 1$;

Acknowledgement. This research was supported by the Croatian Ministry of Science, Education, and Sports, under Research Grants 058–1170889–1050.

References

[1] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities (Second Edition)*, Cambridge University Press, Cambridge, 1967.

[2] M. Krnić and J. Pečarić, General Hilbert's and Hardy's inequalities, *Math. Inequal. Appl.* **8** (2005), no. 1, 29–51.

[3] Dongmei Xin and Bicheng Yang, A Hilbert-Type Integral Inequality in the Whole Plane with the Homogeneous Kernel of Degree −2, *Jour. Ineq. & Appl.*, Vol. 2011, Art. ID 401428, (2011).

[4] Zheng Zeng and Zitian Xie, On a New Hilbert-Type Integral Inequality with the Integral in Whole Plane, *Jour. Ineq. & Appl.*, Vol. 2010, Art. ID 256796, (2010).

(Predrag Vuković) Faculty of Teacher Education, University of Zagreb, Savska cesta 77, 10000, Zagreb, Croatia

E-mail address: `predrag.vukovic@ufzg.hr`