Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 39(1), 2012, Pages 65–75
ISSN: 1223-6934

Improvements of the Giaccardi and the Petrović inequality and related Stolarsky type means

Josip Pečarić and Jurica Perić

Abstract. Improvements of the Giaccardi and the Petrović inequality are given. The notion of $n-$exponentially convex functions is introduced. An elegant method of producing $n-$exponentially convex and exponentially convex functions is applied using the Giaccardi and the Petrović differences. Cauchy mean value theorems are proved and shown to be useful in studying Stolarsky type means defined by using the Giaccardi and the Petrović differences.

2010 Mathematics Subject Classification. 26D15.
Key words and phrases. Giaccardi inequality, Petrović inequality, Cauchy type mean value theorems, $n-$exponential convexity, exponential convexity, Stolarsky type means.

1. Introduction and preliminaries

The Giaccardi inequality states:

Theorem 1.1. *Let ϕ be a convex function on an interval I, $\mathbf{p}$ a nonnegative n-tuple with $\sum_{i=1}^n p_i = P_n \neq 0$ and $\mathbf{x}$ a real n-tuple. If $\mathbf{x} \in I^n$ and $x_0 \in I$ are such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in I, \tilde{x} \neq x_0$ and*

$$(x_i - x_0)(\tilde{x} - x_i) \geq 0, \quad i = 1, \dots, n,$$

then

$$\sum_{i=1}^n p_i \phi(x_i) \leq A\phi(\tilde{x}) + B\left(\sum_{i=1}^n p_i - 1\right)\phi(x_0),$$

where

$$A = \frac{\sum_{i=1}^n p_i(x_i - x_0)}{\sum_{i=1}^n p_i x_i - x_0}, B = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i - x_0}.$$

A simple consequence of the Giaccardi inequality is the Petrović inequality:

Corollary 1.1. *Let ϕ be a convex function on $[0, a], 0 < a < \infty$. Then for every nonnegative n-tuple $\mathbf{p}$ and every $\mathbf{x} \in [0, a]^n$ such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in (0, a]$ and*

$$\sum_{i=1}^n p_i x_i \geq x_j, \quad j = 1, \dots, n,$$

the following inequality holds

$$\sum_{i=1}^n p_i \phi(x_i) \leq \phi(\tilde{x}) + \left(\sum_{i=1}^n p_i - 1\right)\phi(0).$$

For further details on the Giaccardi and the Petrović inequality see [6].

The main goal of this paper is to improve Theorem 1.1 and Corollary 1.1 using the following lemma.

Received March 27, 2012.

Lemma 1.1. *Let ϕ be a convex function on an interval I, $x, y \in I$ and $p, q \in [0,1]$ such that $p + q = 1$. Then*

$$\min\{p,q\}\left[\phi(x)+\phi(y)-2\phi\left(\frac{x+y}{2}\right)\right] \leq p\phi(x)+q\phi(y)-\phi(px+qy). \tag{1}$$

Lemma 1.1 is a simple consequence of [5, Theorem 1, p.717].

In Section 2 we also prove Cauchy type mean value theorems, which we use in Section 4 in studying Stolarsky type means defined by the Giaccardi and the Petrović differences. In Section 3 we introduce the notion of $n-$ exponentially convex functions and deduce an elegant method of producing $n-$exponentially convex and exponentially convex functions using some known families of functions of the same type.

2. Improvements of the Giaccardi and the Petrović inequality

Next theorem is our main result.

Theorem 2.1. *Let ϕ be a convex function on an interval I, $\mathbf{p}$ a nonnegative n-tuple with $\sum_{i=1}^n p_i = P_n \neq 0$ and $\mathbf{x}$ a real n-tuple. If $\mathbf{x} \in I^n$ and $x_0 \in I$ are such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in I, \tilde{x} \neq x_0$ and*

$$(x_i - x_0)(\tilde{x} - x_i) \geq 0, \quad i = 1, \ldots, n, \tag{2}$$

then

$$\sum_{i=1}^n p_i\phi(x_i) \leq A\phi(\tilde{x}) + B\left(\sum_{i=1}^n p_i - 1\right)\phi(x_0) - \frac{\delta_\phi}{2}P_n + \delta_\phi\sum_{i=1}^n p_i\left|\frac{x_i - \frac{x_0+\tilde{x}}{2}}{\tilde{x}-x_0}\right|, \tag{3}$$

where

$$A = \frac{\sum_{i=1}^n p_i(x_i - x_0)}{\sum_{i=1}^n p_i x_i - x_0}, B = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i - x_0}, \delta_\phi = \phi(x_0) + \phi(\tilde{x}) - 2\phi\left(\frac{x_0+\tilde{x}}{2}\right).$$

Proof. The condition $(x_i - x_0)(\tilde{x} - x_i) \geq 0$, $i = 1, \ldots, n$, means that either $x_0 \leq x_i \leq \tilde{x}$ or $\tilde{x} \leq x_i \leq x_0$, $i = 1, \ldots, n$. Consider the first case (the second is analogous).
Let the functions $p,\ q : [x_0, \tilde{x}] \to [0,1]$ be defined by

$$p(x) = \frac{\tilde{x} - x}{\tilde{x} - x_0}, \quad q(x) = \frac{x - x_0}{\tilde{x} - x_0}.$$

For any $x \in [x_0, \tilde{x}]$ we can write

$$\phi(x) = \phi\left(\frac{\tilde{x}-x}{\tilde{x}-x_0}x_0 + \frac{x-x_0}{\tilde{x}-x_0}\tilde{x}\right) = \phi(p(x)x_0 + q(x)\tilde{x}).$$

By Lemma 1.1 we get for $x \in [x_0, \tilde{x}]$

$$\min\{p(x), q(x)\}\left[\phi(x_0)+\phi(\tilde{x})-2\phi\left(\frac{x_0+\tilde{x}}{2}\right)\right] \leq p(x)\phi(x_0) + q(x)\phi(\tilde{x}) - \phi(p(x)x_0 + q(x)\tilde{x})$$

and then

$$\phi(x) = \phi(p(x)x_0 + q(x)\tilde{x}) \leq \phi(x_0) + q(x)\phi(\tilde{x}) - \min\{p(x), q(x)\}\left[\phi(x_0)+\phi(\tilde{x})-2\phi\left(\frac{x_0+\tilde{x}}{2}\right)\right].$$

Multiplying $\phi(x_i)$ by p_i and summing, we get

$$\begin{aligned}
&\sum_{i=1}^{n} p_i\phi(x_i) \\
&\quad \le \sum_{i=1}^{n} p_i \left[p(x_i)\phi(x_0) + q(x_i)\phi(\tilde{x}) - \min\{p(x_i), q(x_i)\} \left[\phi(x_0) + \phi(\tilde{x}) - 2\phi\left(\frac{x_0+\tilde{x}}{2}\right)\right]\right] \\
&\quad = \phi(\tilde{x}) \sum_{i=1}^{n} p_i \frac{x_i - x_0}{\tilde{x} - x_0} + \phi(x_0) \sum_{i=1}^{n} p_i \frac{\tilde{x} - x_i}{\tilde{x} - x_0} - \delta_\phi \sum_{i=1}^{n} p_i \min\{p(x_i), q(x_i)\} \\
&\quad = A\phi(\tilde{x}) + B(\sum_{i=1}^{n} p_i - 1)\phi(x_0) - \frac{\delta_\phi}{2} P_n + \delta_\phi \sum_{i=1}^{n} p_i \left| \frac{x_i - \frac{x_0+\tilde{x}}{2}}{\tilde{x} - x_0} \right|.
\end{aligned}$$

□

Remark 2.1. *Obviously, Theorem 2.1 is an improvement of Theorem 1.1 since under the required assumptions we have*

$$\delta_\phi \sum_{i=1}^{n} p_i \min\{p(x_i), q(x_i)\} \ge 0.$$

A simple consequence of the Giaccardi inequality is the Petrović inequality, so we give its refinement too.

Corollary 2.1. *Let ϕ be a convex function on $[0,a]$, $0 < a < \infty$. Then for every nonnegative n-tuple $\mathbf{p}$ and every $\mathbf{x} \in [0,a]^n$ such that $\sum_{i=1}^{n} p_i x_i = \tilde{x} \in (0,a]$ and*

$$\sum_{i=1}^{n} p_i x_i \ge x_j, \quad j = 1, \dots, n, \tag{4}$$

the following inequality holds

$$\begin{aligned}
\sum_{i=1}^{n} p_i\phi(x_i) \le \phi(\tilde{x}) + \left(\sum_{i=1}^{n} p_i - 1\right)\phi(0) \\
-\frac{\delta_\phi}{2} P_n + \delta_\phi \sum_{i=1}^{n} p_i \left| \frac{x_i}{\tilde{x}} - \frac{1}{2} \right|,
\end{aligned} \tag{5}$$

where $\delta_\phi = \phi(0) + \phi(\tilde{x}) - 2\phi\left(\frac{\tilde{x}}{2}\right)$.

Proof. This is a special case of Theorem 2.1; choose $x_0 = 0$. □

Motivated by inequalities (3) and (5), we define two functionals:

$$\begin{aligned}
\Phi_1(\mathbf{x}, \mathbf{p}, f) \quad = \quad & Af(\tilde{x}) + B\left(\sum_{i=1}^{n} p_i - 1\right) f(x_0) - \frac{\delta_f}{2} P_n + \\
& + \delta_f \sum_{i=1}^{n} p_i \left| \frac{x_i - \frac{x_0+\tilde{x}}{2}}{\tilde{x} - x_0} \right| - \sum_{i=1}^{n} p_i f(x_i),
\end{aligned} \tag{6}$$

where f is a function on interval I, $\mathbf{p}$ is a nonnegative n-tuple, $\mathbf{x}$ a real n-tuple and $\tilde{x}$, P_n, δ_f, A, B as in Theorem 2.1, and

$$\begin{aligned}\Phi_2(\mathbf{x},\mathbf{p},f) &= f(\tilde{x})+\left(\sum_{i=1}^{n}p_i-1\right)f(0)-\frac{\delta_\phi}{2}P_n \\ &\quad +\delta_f\sum_{i=1}^{n}p_i\left|\frac{x_i}{\tilde{x}}-\frac{1}{2}\right|-\sum_{i=1}^{n}p_i f(x_i),\end{aligned} \tag{7}$$

where f is a function on interval $[0,a]$, $\mathbf{p}$ is a nonnegative n-tuple, $\mathbf{x}$ a real n-tuple and $\tilde{x}$, P_n, δ_f as in Corollary 2.1.
If f is a convex function, then Theorem 2.1 and Corollary 2.1 imply that $\Phi_i(\mathbf{x},\mathbf{p},f)\geq 0, i=1,2$.
Now, we give mean value theorems for the functionals Φ_i, $i=1,2$.

Theorem 2.2. *Let $I=[a,b]$, $\mathbf{p}$ be a nonnegative n-tuple with $\sum_{i=1}^{n}p_i=P_n\neq 0$ and $\mathbf{x}$ a real n-tuple. Let $\mathbf{x}\in I^n$ and $x_0\in I$ be such that $\sum_{i=1}^{n}p_ix_i=\tilde{x}\in I$, $\tilde{x}\neq x_0$ and (2) holds. Let $f\in C^2(I)$. Then there exists $\xi\in I$ such that*

$$\Phi_1(\mathbf{x},\mathbf{p},f)=\frac{f''(\xi)}{2}\Phi_1(x,p,f_0), \tag{8}$$

where $f_0(x)=x^2$.

Proof. Since $f\in C^2(I)$ therefore there exist real numbers $m=\min_{x\in[a,b]}f''(x)$ and $M=\max_{x\in[a,b]}f''(x)$. It is easy to show that the functions $f_1(x), f_2(x)$ defined by

$$\begin{aligned}f_1(x) &= \frac{M}{2}x^2-f(x),\\ f_2(x) &= f(x)-\frac{m}{2}x^2\end{aligned}$$

are convex. Therefore

$$\Phi_1(\mathbf{x},\mathbf{p},f_1)\geq 0$$
$$\Phi_1(\mathbf{x},\mathbf{p},f_2)\geq 0,$$

and we get

$$\Phi_1(\mathbf{x},\mathbf{p},f)\leq\frac{M}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0) \tag{9}$$
$$\Phi_1(\mathbf{x},\mathbf{p},f)\geq\frac{m}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0). \tag{10}$$

From (9) and (10) we get

$$\frac{m}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0)\leq\Phi_1(\mathbf{x},\mathbf{p},f)\leq\frac{M}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0).$$

If $\Phi_1(\mathbf{x},\mathbf{p},x^2)=0$ there is nothing to prove. Suppose $\Phi_1(\mathbf{x},\mathbf{p},x^2)>0$. We have

$$m\leq\frac{2\Phi_1(\mathbf{x},\mathbf{p},f)}{\Phi_1(\mathbf{x},\mathbf{p},x^2)}\leq M.$$

Hence, there exists $\xi\in I$ such that

$$\Phi_1(\mathbf{x},\mathbf{p},f)=\frac{f''(\xi)}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0).$$

□

Theorem 2.3. *Let $I = [0, a]$, $\mathbf{p}$ be a nonnegative n-tuple and $\mathbf{x}$ a real n-tuple. Let $\mathbf{x} \in [0, a]^n$ such that $\sum_{i=1}^{n} p_i x_i = \tilde{x} \in I$ and (4) holds. Let $f \in C^2(I)$. Then there exists $\xi \in I$ such that*

$$\Phi_2(\mathbf{x}, \mathbf{p}, f) = \frac{f''(\xi)}{2} \Phi_2(\mathbf{x}, \mathbf{p}, f_0) \tag{11}$$

where $f_0(x) = x^2$.

Proof. Analogous to the proof of Theorem 2.2 □

Theorem 2.4. *Let $I = [a, b]$, $\mathbf{p}$ be a nonnegative n-tuple with $\sum_{i=1}^{n} p_i = P_n \neq 0$ and $\mathbf{x}$ real n-tuple. Let $\mathbf{x} \in I^n$ and $x_0 \in I$ be such that $\sum_{i=1}^{n} p_i x_i = \tilde{x} \in I$, $\tilde{x} \neq x_0$ and (2) holds. Let $f, g \in C^2(I)$. Then there exists $\xi \in I$ such that*

$$\frac{\Phi_1(\mathbf{x}, \mathbf{p}, f)}{\Phi_1(\mathbf{x}, \mathbf{p}, g)} = \frac{f''(\xi)}{g''(\xi)}, \tag{12}$$

provided that the denominators are non-zero.

Proof. Define $h \in C^2([a, b])$ by

$$h = c_1 f - c_2 g,$$

where

$$c_1 = \Phi_1(\mathbf{x}, \mathbf{p}, g), \quad c_2 = \Phi_1(\mathbf{x}, \mathbf{p}, f).$$

Now using Theorem 2.2 there exists $\xi \in [a, b]$ such that

$$\left(c_1 \frac{f''(\xi)}{2} - c_2 \frac{g''(\xi)}{2} \right) \Phi_1(\mathbf{x}, \mathbf{p}, f_0) = 0.$$

Since $\Phi_1(\mathbf{x}, \mathbf{p}, f_0) \neq 0$ (otherwise we have a contradiction with $\Phi_1(\mathbf{x}, \mathbf{p}, g) \neq 0$ by Theorem 2.2), we get

$$\frac{\Phi_1(\mathbf{x}, \mathbf{p}, f)}{\Phi_1(\mathbf{x}, \mathbf{p}, g)} = \frac{f''(\xi)}{g''(\xi)}.$$

□

Theorem 2.5. *Let $I = [0, a]$, $\mathbf{p}$ be a nonnegative n-tuple and $\mathbf{x}$ a real n-tuple. Let $\mathbf{x} \in [0, a]^n$ be such that $\sum_{i=1}^{n} p_i x_i = \tilde{x} \in I$ and (4) holds. Let $f, g \in C^2(I)$. Then there exists $\xi \in I$ such that*

$$\frac{\Phi_2(\mathbf{x}, \mathbf{p}, f)}{\Phi_2(\mathbf{x}, \mathbf{p}, g)} = \frac{f''(\xi)}{g''(\xi)} \tag{13}$$

provided that the denominators are non zero.

Proof. Analogous to the proof of Theorem 2.4. □

3. $n-$exponential convexity and exponential convexity of the Giaccardi and the Petrović differences

We begin this section by notions which are going to be explored here and some characterizations of these properties.

Definition 3.1. *A function $\psi \colon I \to \mathbb{R}$ is $n-$exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2} \right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$.

A function $\psi\colon I \to \mathbb{R}$ is $n-$exponentially convex if it is $n-$exponentially convex in the Jensen sense and continuous on I.

Remark 3.1. *It is clear from the definition that $1-$exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, $n-$exponentially convex functions in the Jensen sense are $k-$exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.*

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

Proposition 3.1. *If ψ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}, k \leq n$. Particularly,* $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$ *for all $k \in \mathbb{N}$, $k \leq n$.*

Definition 3.2. *A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 3.2. *It is known (and easy to show) that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if*

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is $2-$exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is $2-$exponentially convex.

We will also need the following result (see for example [6]).

Proposition 3.2. *If Ψ is a convex function on an interval I and if $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$, then the following inequality is valid*

$$\frac{\Psi(x_2)-\Psi(x_1)}{x_2-x_1} \leq \frac{\Psi(y_2)-\Psi(y_1)}{y_2-y_1}. \tag{14}$$

If the function Ψ is concave, the inequality reverses.

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

Definition 3.3. *The second order devided difference of a function $f\colon I \to \mathbb{R}$, I an interval in $\mathbb{R}$, at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by*

$$\begin{aligned}
[y_i; f] &= f(y_i), \quad i = 0, 1, 2 \\
[y_i, y_{i+1}; f] &= \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1 \\
[y_0, y_1, y_2; f] &= \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}.
\end{aligned} \tag{15}$$

Remark 3.3. *The value* $[y_0, y_1, y_2; f]$ *is independent of the order of the points* y_0, y_1 *and* y_2*. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit* $y_1 \to y_0$ *in (15), we get*

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, y_2 \neq y_0$$

provided f' *exists, and furthermore, taking the limits* $y_i \to y_0, i = 1, 2$ *in (15), we get*

$$\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2}$$

provided that f'' *exists.*

We use an idea from [3] to give an elegant method of producing an $n-$ exponentially convex functions and exponentially convex functions applying the functionals Φ_1 and Φ_2 on a given family with the same property.

Theorem 3.1. *Let* $\Upsilon = \{f_s : s \in J\}$*, where* J *an interval in* $\mathbb{R}$*, be a family of functions defined on an interval* I *in* $\mathbb{R}$*, such that the function* $s \mapsto [y_0, y_1, y_2; f_s]$ *is* $n-$*exponentially convex in the Jensen sense on* J *for every three mutually different points* $y_0, y_1, y_2 \in I$*. Let* $\Phi_i \quad (i = 1, 2)$ *be linear functionals defined as in (6) and (7). Then* $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ *is an* $n-$*exponentially convex function in the Jensen sense on* J*. If the function* $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ *is continuous on* J*, then it is* $n-$*exponentially convex on* J*.*

Proof. For $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $s_i \in J$, $i = 1, \ldots, n$, we define the function

$$g(y) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i + s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is $n-$exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2; g] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; f_{\frac{s_i + s_j}{2}}] \geq 0,$$

which in turn implies that g is a convex function on I and therefore we have $\Phi_i(\mathbf{x}, \mathbf{p}, g) \geq 0$, $i = 1, 2$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(\mathbf{x}, \mathbf{p}, f_{\frac{s_i + s_j}{2}}) \geq 0.$$

We conclude that the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is $n-$exponentially convex on J in the Jensen sense.

If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is also continuous on J, then $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is $n-$exponentially convex by definition. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.1. *Let* $\Upsilon = \{f_s : s \in J\}$*, where* J *an interval in* $\mathbb{R}$*, be a family of functions defined on an interval* I *in* $\mathbb{R}$*, such that the function* $s \mapsto [y_0, y_1, y_2; f_s]$ *is exponentially convex in the Jensen sense on* J *for every three mutually different points* $y_0, y_1, y_2 \in I$*. Let* $\Phi_i \quad (i = 1, 2)$ *be linear functionals defined as in (6) and (7). Then* $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ *is an exponentially convex function in the Jensen sense on* J*. If the function* $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ *is continuous on* J*, then it is exponentially convex on* J*.*

Corollary 3.2. *Let $\Omega = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is $2-$exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i, $i = 1, 2$, be linear functionals defined as in (6) and (7). Then the following statements hold:*

(*i*) *If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is continuous on J, then it is $2-$exponentially convex function on J, and thus log-convex function.*

(*ii*) *If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is strictly positive and differentiable on J, then for every $s, q, u, v \in J$, such that $s \le u$ and $q \le v$, we have*

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega) \le \mu_{u,v}(\mathbf{x}, \Phi_i, \Omega), \quad i = 1, 2, \tag{16}$$

where

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(\mathbf{x},\mathbf{p},f_s)}{\Phi_i(\mathbf{x},\mathbf{p},f_q)}\right)^{\frac{1}{s-q}} & , \ s \ne q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(\mathbf{x},\mathbf{p},f_s)}{\Phi_i(\mathbf{x},\mathbf{p},f_s)}\right) & , \ s = q. \end{cases} \tag{17}$$

for $f_s, f_q \in \Omega$.

Proof. (*i*) This is an immediate consequence of Theorem 3.1 and Remark 3.2.
(*ii*) Since by (*i*) the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is log-convex on J, that is, the function $s \mapsto \log \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is convex on J. Applying Proposition 3.2 we get

$$\frac{\log \Phi_i(\mathbf{x}, \mathbf{p}, f_s) - \log \Phi_i(\mathbf{x}, \mathbf{p}, f_q)}{s - q} \le \frac{\log \Phi_i(\mathbf{x}, \mathbf{p}, f_u) - \log \Phi_i(\mathbf{x}, \mathbf{p}, f_v)}{u - v} \tag{18}$$

for $s \le u, q \le v, s \ne q, u \ne v$, and therefrom conclude that

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega) \le \mu_{u,v}(\mathbf{x}, \Phi_i, \Omega), \quad i = 1, 2.$$

Cases $s = q$ and $u = v$ follows from (18) as limit cases. □

Remark 3.4. *Note that the results from Theorem 3.1, Corollary 3.1, Corollary 3.2 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is $n-$exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 3.3 and suitable characterization of convexity.*

4. Applications to Stolarsky type means

In this section, we present several families of functions which fulfil the conditions of Theorem 3.1, Corollary 3.1 and Corollary 3.2 (and Remark 3.4). This enable us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [2].

Example 4.1. *Consider a family of functions*

$$\Omega_1 = \{g_s : \mathbb{R} \to [0, \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \ne 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2 g_s}{dx^2}(x) = e^{sx} > 0$ which shows that g_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Using analogous arguing as in *the proof of Theorem 3.1 we also have that* $s \mapsto [y_0, y_1, y_2; g_s]$ *is exponentially convex (and so exponentially convex in the Jensen sense). Using Theorem 3.1 we conclude that* $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, g_s)$, $i = 1, 2$, *are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although mapping* $s \mapsto g_s$ *is not continuous for* $s = 0$*), so they are exponentially convex.*

For this family of functions, $\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega_1)$, $i = 1, 2$, *from (17) become*

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(\mathbf{x}, \boldsymbol{p}, g_s)}{\Phi_i(\mathbf{x}, \boldsymbol{p}, g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(\mathbf{x}, \boldsymbol{p}, id \cdot g_s)}{\Phi_i(\mathbf{x}, \boldsymbol{p}, g_s)} - \frac{2}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(\mathbf{x}, \boldsymbol{p}, id \cdot g_0)}{3\Phi_i(\mathbf{x}, \boldsymbol{p}, g_0)}\right), & s = q = 0, \end{cases}$$

and using (16) they are monotonous functions in parameters s *and* q.

Using Theorems 2.4 and 2.5 it follows that for $i = 1, 2$

$$M_{s,q}(\mathbf{x}, \Phi_i, \Omega_1) = \log \mu_{s,q}(\mathbf{x}, \Phi_i, \Omega_1)$$

satisfy $\min\{x_0, \tilde{x}\} \leq M_{s,q}(\mathbf{x}, \Phi_i, \Omega_1) \leq \max\{x_0, \tilde{x}\}$, *which shows that* $M_{s,q}(\mathbf{x}, \Phi_i, \Omega_1)$ *are means (of* $x_0, x_1, \ldots x_n, \tilde{x}$*). Notice that by (16) they are monotonous means.*

Example 4.2. *Consider a family of functions*

$$\Omega_2 = \{f_s : (0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases}$$

Here, $\frac{d^2 f_s}{dx^2}(x) = x^{s-2} = e^{(s-2)\log x} > 0$ *which shows that* f_s *is convex for* $x > 0$ *and* $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ *is exponentially convex by definition. Arguing as in Example 4.1 we get that the mapping* $s \mapsto \Phi_1(\mathbf{x}, \mathbf{p}, g_s)$ *is exponentially convex. In this case we assume* $x_j > 0$, $j = 0, 1 \ldots, n$. *Notice that the functional* Φ_2 *is not defined in this case (of course it can be defined for* $s \geq 0$*). Functions (17) in this case are equal to:*

$$\mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_2) = \begin{cases} \left(\frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_s)}{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_s f_0)}{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_s)}\right), & s = q \neq 0, 1, \\ \exp\left(1 - \frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_0^2)}{2\Phi_1(\mathbf{x}, \boldsymbol{p}, f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_0 f_1)}{2\Phi_1(\mathbf{x}, \boldsymbol{p}, f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_1 *is positive, then Theorem 2.4 and Theorem 2.5 applied for* $f = f_s \in \Omega_2$ *and* $g = f_q \in \Omega_2$ *yields that there exists* $\xi \in [\min\{x_0, \tilde{x}\}, \max\{x_0, \tilde{x}\}]$ *such that*

$$\xi^{s-q} = \frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_s)}{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ *is invertible for* $s \neq q$, *we then have*

$$\min\{x_0, \tilde{x}\} \leq \left(\frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_s)}{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_q)}\right)^{\frac{1}{s-q}} \leq \max\{x_0, \tilde{x}\}, \tag{19}$$

which together with the fact that $\mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_2)$ *is continuous, symmetric and monotonous (by (16)), shows that* $\mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_2)$ *is a mean. Now, by substitutions* $x_i \to x_i^t$,

$s \to \frac{s}{t}$, $q \to \frac{q}{t}$ $(t \neq 0,\ s \neq q)$ from (19) we get

$$\min\left\{x_0^t, \tilde{x}^t\right\} \leq \left(\frac{\Phi_1(\mathbf{x}^t, \boldsymbol{p}, f_{s/t})}{\Phi_1(\mathbf{x}^t, \boldsymbol{p}, f_{q/t})}\right)^{\frac{t}{s-q}} \leq \max\left\{x_0^t, \tilde{x}^t\right\},$$

where $\mathbf{x}^t = (x_1^t, \ldots, x_n^t)$. We define a new mean as follows

$$\mu_{s,q;t}(\mathbf{x}, \Phi_1, \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{t},\frac{q}{t}}(\mathbf{x}^t, \Phi_1, \Omega_2)\right)^{1/t}, & t \neq 0 \\ \mu_{s,q}(\log \mathbf{x}, \Phi_1, \Omega_1), & t = 0. \end{cases} \tag{20}$$

These new means are also monotonous. More precisely, for $s, q, u, v \in \mathbb{R}$ such that $s \leq u$, $q \leq v$, $s \neq u$, $q \neq v$, we have

$$\mu_{s,q;t}(\mathbf{x}, \Phi_1, \Omega_2) \leq \mu_{u,v;t}(\mathbf{x}, \Phi_1, \Omega_2). \tag{21}$$

We know that

$$\mu_{\frac{s}{t},\frac{q}{t}}(\mathbf{x}, \Phi_1, \Omega_2) = \left(\frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_{s/t})}{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_{q/t})}\right)^{\frac{t}{s-q}} \leq \mu_{\frac{u}{t},\frac{v}{t}}(\mathbf{x}, \Phi_1, \Omega_2) = \left(\frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_{s/t})}{\Phi_1(\mathbf{x}, \boldsymbol{p}, f_{q/t})}\right)^{\frac{t}{s-q}},$$

for $s, q, u, v \in I$ such that $s/t \leq u/t$, $q/t \leq v/t$ and $t \neq 0$, since $\mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_2)$ are monotonous in both parameters, the claim follows. For $t = 0$, we obtain the required result by taking the limit $t \to 0$.

Example 4.3. *Consider a family of functions*

$$\Omega_3 = \{h_s : (0, \infty) \to (0, \infty) : s \in (0, \infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\log^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

Since $s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [7]*) it is exponentially convex. Obviously h_s are convex functions for every $s > 0$.*
For this family of functions, $\mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_3)$, in this case for $x_j > 0$, $j = 0, 1, \ldots, n$, from (17) becomes

$$\mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_3) = \begin{cases} \left(\frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, h_s)}{\Phi_1(\mathbf{x}, \boldsymbol{p}, h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, id \cdot h_s)}{s\Phi_1(\mathbf{x}, \boldsymbol{p}, h_s)} - \frac{2}{s \log s}\right), & s = q \neq 1, \\ \exp\left(-\frac{\Phi_1(\mathbf{x}, \boldsymbol{p}, id \cdot h_1)}{3\Phi_1(\mathbf{x}, \boldsymbol{p}, h_1)}\right), & s = q = 1, \end{cases}$$

and it is monotonous in parameters s and q by (16).
Using Theorem 2.4, it follows that

$$M_{s,q}(\mathbf{x}, \Phi_1, \Omega_3) = -L(s, q) \log \mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_3),$$

satisfies $\min\{x_0, \tilde{x}\} \leq M_{s,q}(\mathbf{x}, \Phi_1, \Omega_3) \leq \max\{x_0, \tilde{x}\}$, which shows that $M_{s,q}(\mathbf{x}, \Phi_1, \Omega_3)$ is a mean (of $x_0, x_1, \ldots x_n, \tilde{x}$). $L(s, q)$ is the logarithmic mean defined by $L(s, q) = \frac{s-q}{\log s - \log q}$, $s \neq q$, $L(s, s) = s$.

Example 4.4. *Consider a family of functions*

$$\Omega_4 = \{k_s : (0, \infty) \to (0, \infty) : s \in (0, \infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}$$

Since $s \mapsto \frac{d^2 k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ *is the Laplace transform of a non-negative function (see* [7]*) it is exponentially convex. Obviously* k_s *are convex functions for every* $s > 0$*.*
For this family of functions, $\mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_4)$*, in this case for* $x_j > 0$*,* $j = 0, 1, \ldots, n$*, from (17) becomes*

$$\mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_4) \;=\; \begin{cases} \left(\frac{\Phi_1(\mathbf{x},\boldsymbol{p},k_s)}{\Phi_1(\mathbf{x},\boldsymbol{p},k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_1(\mathbf{x},\boldsymbol{p},id\cdot k_s)}{2\sqrt{s}\Phi(\mathbf{x},\boldsymbol{p},k_s)} - \frac{1}{s}\right), & s = q, \end{cases}$$

and it is monotonous function in parameters s *and* q *by (16).*
Using Theorem 2.4, it follows that

$$M_{s,q}(\mathbf{x}, \Phi_1, \Omega_4) \;=-\left(\sqrt{s} + \sqrt{q}\right) \log \mu_{s,q}(\mathbf{x}, \Phi_1, \Omega_4)$$

satisfies $\min\{x_0, \tilde{x}\} \le M_{s,q}(\mathbf{x}, \Phi_1, \Omega_4) \le \max\{x_0, \tilde{x}\}$*, which shows that* $M_{s,q}(\mathbf{x}, \Phi_1, \Omega_4)$ *is a mean (of* $x_0, x_1, \ldots x_n, \tilde{x}$*).*

References

[1] M. Anwar, J. Jakšetić, J. Pečarić and Atiq ur Rehman, Exponential convexity, positive semi-definite matrices and fundamental inequalities, *J. Math. Inequal.* **4** (2) (2010), 171–189.

[2] W. Ehm, M. G. Genton, T. Gneiting, Stationary covariances associated with exponentially convex functions, *Bernoulli* **9**(4) (2003), 607–615.

[3] J. Jakšetić and J. Pečarić, *Exponential Convexity Method*, submitted.

[4] Khuram A. Khan, J. Pečarić, I. Perić, Differences of weighted mixed symmetric means and related results, *J. Inequal. Appl.* **2010** (2010), Article ID 289730, 16 pages.

[5] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and new inequalities in analysis. Mathematics and its Applications (East European Series)*, **61**, Kluwer Academic Publishers Group, Dordrecht, 1993.

[6] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press Inc, 1992.

[7] D. V. Widder, *The Laplace transform*, Princeton Univ. Press, New Jersey, 1941.

(Josip Pečarić) Faculty of Textile Technology, University of Zagreb, Zagreb, Croatia
E-mail address: pecaric@element.hr

(Jurica Perić) Faculty of Science, Departments of Mathematics, University of Split, Teslina 12, 21000 Split, Croatia
E-mail address: jperic@pmfst.hr