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Sharp integral inequalities based on general three-point formula via a generalization of Montgomery identity

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Abstract. In this paper we establish the families of general three-point quadrature formulae, by using the generalization of the weighted Montgomery identity via Taylor's formula. The results are applied to prove a number of inequalities which give error estimates for the general three-point formula and for three-point Gauss-Chebyshev formulae of the first and of the second kind.

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1. Introduction

The most elementary quadrature rules in three nodes are the Simpson's rule, based on the Simpson's formula

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \tag{1}$$

where $\xi \in [a,b]$, and the dual Simpson's rule based on the following three point formula

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] + \frac{7(b-a)^5}{23040} f^{(4)}(\eta), \tag{2}$$

where $\eta \in [a,b]$. These formulae are valid for any function f with continuous forth derivative $f^{(4)}$ on $[a,b]$.

Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $f' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Then the Montgomery identity holds (see [3])

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P(x,t)\, f'(t)\,dt, \tag{3}$$

where the Peano kernel is

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b. \end{cases}$$

In [4] J. Pečarić has proved the following weighted Montgomery identity

$$f(x) = \int_a^b w(t)\, f(t)\,dt + \int_a^b P_w(x,t)\, f'(t)\,dt, \tag{4}$$

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where $w : [a,b] \to [0,\infty\rangle$ is some probability density function, that is integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$ and $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases} \tag{5}$$

Now, let us suppose I is an open interval in $\mathbb{R}$, $[a,b] \subset I$, $f : I \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$, $w : [a,b] \to [0,\infty\rangle$ a probability density function. Then the following generalization of the weighted Montgomery identity via Taylor's formula states (given by A. Aglić Aljinović and J. Pečarić in [1])

$$\begin{aligned} f(x) &= \int_a^b w(t) f(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}\,ds \\ &+ \frac{1}{(n-1)!} \int_a^b T_{w,n}(x,s) f^{(n)}(s)\,ds, \end{aligned} \tag{6}$$

where $x \in [a,b]$ and

$$T_{w,n}(x,s) = \begin{cases} \int_a^s w(u)(u-s)^{n-1}\,du, & a \le s \le x, \\ -\int_s^b w(u)(u-s)^{n-1}\,du, & x < s \le b. \end{cases} \tag{7}$$

If we take $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, the equality (6) reduces to

$$\begin{aligned} f(x) &= \frac{1}{b-a} \int_a^b f(t)\,dt - \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \\ &+ \frac{1}{(n-1)!} \int_a^b T_n(x,s) f^{(n)}(s)\,ds, \end{aligned} \tag{8}$$

where $x \in [a,b]$ and

$$T_n(x,s) = \begin{cases} \frac{-(a-s)^n}{n(b-a)}, & a \le s \le x, \\ \frac{-(b-s)^n}{n(b-a)}, & x < s \le b. \end{cases}$$

For $n = 1$ (8) reduces to Montgomery identity (3).

In this paper we continue work which has been started in [2]. Namely, we use the identities (6) and (8) to establish for each number $x \in \left[a, \frac{a+b}{2}\right)$ the general weighted three-point quadrature formula

$$\int_a^b w(t) f(t)\,dt = A(x)\left[f(x) + f(a+b-x)\right] + (1 - 2A(x))\, f\left(\frac{a+b}{2}\right) + E(f,w;x) \tag{9}$$

where $E(f,w;x)$ is the reminder and $A : \left[a, \frac{a+b}{2}\right) \to \mathbb{R}$ a real function. Obtained formula is used to prove a number of inequalities which give error estimates for the general three-point formula for functions whose derivatives are from the L_p- spaces. These inequalities are generally sharp. Also, we obtain three-point Gauss-Chebyshev formulae of the first and of the second kind as special cases of the general weighted three-point quadrature formula and prove some sharp inequalities. As special cases of general non-weighted three-point quadrature formula, we obtain generalizations of the well-known Simpson's (1) and dual Simpson's formula (2).

2. General weighted three-point formula

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right\rangle$

$$D(x) = A(x)\left[f(x) + f(a+b-x)\right] + (1-2A(x))\, f\left(\frac{a+b}{2}\right),$$

$$\begin{aligned} t_{w,n}(x) = A(x) & \left[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1} \mathrm{d}s \right. \\ & \left. + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a+b-x)}{(i+1)!} \int_a^b w(s)(s-a-b+x)^{i+1} \mathrm{d}s \right] \\ & + (1-2A(x)) \sum_{i=0}^{n-2} \frac{f^{(i+1)}\left(\frac{a+b}{2}\right)}{(i+1)!} \int_a^b w(s) \left(s - \frac{a+b}{2}\right)^{i+1} \mathrm{d}s \end{aligned} \tag{10}$$

and

$$\widehat{T}_{w,n}(x,s) = -A(x)\left[T_{w,n}(x,s) + T_{w,n}(a+b-x,s)\right] - (1-2A(x))\, T_{w,n}\left(\frac{a+b}{2}, s\right),$$

where $T_{w,n}(x,s)$ is defined as (7).

The following is a general weighted three-point formula.

Theorem 2.1. *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $w : [a,b] \to [0,\infty\rangle$ be some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right\rangle$ the following identity holds*

$$\int_a^b w(t) f(t) \mathrm{d}t = D(x) + t_{w,n}(x) + \frac{1}{(n-1)!} \int_a^b \widehat{T}_{w,n}(x,s) f^{(n)}(s) \mathrm{d}s \tag{11}$$

Proof. We put $x \equiv x, x \equiv \frac{a+b}{2}$ and $x \equiv a+b-x$ in (6) to obtain three new formulae. After multiplying these three formulae by $A(x), 1-2A(x)$ and $A(x)$ and adding we get (11). □

Remark 2.1. *Identity (11) holds true in the case $n = 1$. In this special case we have*

$$\int_a^b w(t)\, f(t)\, dt = D(x) + \int_a^b \widehat{T}_{w,1}(x,s)\, f'(s)\, ds \tag{12}$$

where

$$\begin{aligned} \widehat{T}_{w,1}(x,s) &= -A(x)\left[T_{w,1}(x,s) + T_{w,1}(a+b-x,s)\right] - (1-2A(x))\, T_{w,1}\left(\frac{a+b}{2}, s\right) \\ &= -A(x)\left[P_w(x,s) + P_w(a+b-x,s)\right] - (1-2A(x))\, P_w\left(\frac{a+b}{2}, s\right) \end{aligned}$$

There follows an error estimate for general formula (11).

Theorem 2.2. *Suppose that all the assumptions of Theorem 2.1 hold. Additionally, assume that* (p,q) *is a pair of conjugate exponents, that is* $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *let* $f^{(n)} \in L^p[a,b]$ *for some* $n \ge 1$. *Then for each* $x \in \left[a, \frac{a+b}{2}\right)$ *we have*

$$\left| \int_a^b w(t) f(t) \mathrm{d}t - D(x) - t_{w,n}(x) \right| \le \frac{1}{(n-1)!} \left\| \widehat{T}_{w,n}(x,\cdot) \right\|_q \left\| f^{(n)} \right\|_p . \tag{13}$$

Inequality (13) *is sharp for* $1 < p \le \infty$.

Proof. Applying the Hölder inequality we have

$$\left| \frac{1}{(n-1)!} \int_a^b \widehat{T}_{w,n}(x,s) f^{(n)}(s) \mathrm{d}s \right| \le \frac{1}{(n-1)!} \left\| \widehat{T}_{w,n}(x,\cdot) \right\|_q \left\| f^{(n)} \right\|_p .$$

Using the above inequality from (11) we get estimate (13). Let us denote $U_n^x(s) = \widehat{T}_{w,n}(x,s)$. For the proof of sharpness, we will find a function f such that

$$\left| \int_a^b U_n^x(s) f^{(n)}(s) \mathrm{d}s \right| = \| U_n^x \|_q \left\| f^{(n)} \right\|_p .$$

For $1 < p < \infty$, take f to be such that

$$f^{(n)}(s) = \mathrm{sign} U_n^x(s) \cdot |U_n^x(s)|^{\frac{1}{p-1}},$$

and for $p = \infty$, take

$$f^{(n)}(s) = \mathrm{sign} U_n^x(s).$$

□

3. Applications to Gaussian quadrature formulae

Let us recall that Gaussian quadrature formulae are formulae of the following type

$$\int_a^b \varpi(t) f(t) \mathrm{d}t = \sum_{i=1}^{k} A_i f(x_i) + E_k(f), \tag{14}$$

where $k \in \mathbf{N}$. Without loss of generality, we may restrict ourselves to $[a,b] = [-1,1]$.

3.1. $\varpi(t) = \frac{1}{\sqrt{1-t^2}}$, $\quad t \in (-1,1)$**.** In this case we have Gauss-Chebyshev formula of the first kind

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t) \mathrm{d}t = \pi \sum_{i=1}^{k} A_i f(x_i) + E_k(f) \tag{15}$$

where

$$A_i = \frac{1}{k}, \quad i = 1, \dots, k$$

and x_i are zeros of the Chebyshev polynomials of the first kind defined as

$$T_k(x) = \cos(k \arccos(x)).$$

$T_k(x)$ has exactly k distinct zeros

$$x_i = \cos \frac{(2i-1)\pi}{2k},$$

all of which lie in the interval $(-1,1)$ (see [6]).

Error of the approximation formula (15) is given by

$$E_k(f) = \frac{\pi}{2^{2k-1}(2k)!} f^{(2k)}(\xi), \quad \xi \in (-1,1).$$

In case $k=3$ (15) reduces to

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,dt=\frac{\pi}{3}\left[f\left(-\tfrac{\sqrt{3}}{2}\right)+f(0)+f\left(\tfrac{\sqrt{3}}{2}\right)\right]+\frac{\pi}{23040}f^{(6)}(\xi),$$

where $\xi\in(-1,1)$.

Remark 3.1. *If we apply (12) with $[a,b]=[-1,1]$, $x=-\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right)=\frac{1}{3}$ and $w(t)=\frac{1}{\pi\sqrt{1-t^2}}$, $t\in(-1,1)$, we get*

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t=\frac{\pi}{3}\left[f\left(-\tfrac{\sqrt{3}}{2}\right)+f(0)+f\left(\tfrac{\sqrt{3}}{2}\right)\right]+\pi\int_{-1}^{1}Q_1(s)\,f'(s)\,\mathrm{d}s$$

where

$$Q_1(s)=\begin{cases}-\frac{1}{2}-\frac{1}{\pi}\arcsin s, & -1\le s\le-\frac{\sqrt{3}}{2},\\ -\frac{1}{6}-\frac{1}{\pi}\arcsin s, & -\frac{\sqrt{3}}{2}<s\le0,\\ \frac{1}{6}-\frac{1}{\pi}\arcsin s, & 0<s\le\frac{\sqrt{3}}{2},\\ \frac{1}{2}-\frac{1}{\pi}\arcsin s, & \frac{\sqrt{3}}{2}<s\le1.\end{cases}$$

Corollary 3.1. *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$. Let $f:I\to\mathbb{R}$ be an absolutely continuous function and $f'\in L^p[-1,1]$. Then we have*

$$\left|\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t-\frac{\pi}{3}\left[f\left(-\tfrac{\sqrt{3}}{2}\right)+f(0)+f\left(\tfrac{\sqrt{3}}{2}\right)\right]\right|\le\pi\left\|Q_1\right\|_q\left\|f'\right\|_p. \tag{16}$$

Inequality (16) is sharp for $1<p\le\infty$.

Proof. This is a special case of Theorem 2.2 for $[a,b]=[-1,1]$, $x=-\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right)=\frac{1}{3}$ and $w(t)=\frac{1}{\pi\sqrt{1-t^2}}$, $t\in(-1,1)$. □

Corollary 3.2. *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and $f:I\to\mathbb{R}$ absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t-\frac{\pi}{3}\left[f\left(-\tfrac{\sqrt{3}}{2}\right)+f(0)+f\left(\tfrac{\sqrt{3}}{2}\right)\right]\right|$$
$$\le\begin{cases}\left(4-2\sqrt{3}\right)\left\|f'\right\|_\infty, & f'\in L^\infty[-1,1],\\ 2\sqrt{\frac{1}{3}\pi-1}\left\|f'\right\|_2, & f'\in L^2[-1,1],\\ \frac{1}{6}\pi\left\|f'\right\|_1, & f'\in L^1[-1,1].\end{cases}$$

The first and the second inequality are sharp.

Proof. Applying (16) with $p=\infty$, we have

$$\int_{-1}^{1}\left|Q_1(s)\right|\mathrm{d}s=\int_{-1}^{-\frac{\sqrt{3}}{2}}\left|-\tfrac{1}{2}-\tfrac{1}{\pi}\arcsin s\right|\mathrm{d}s+\int_{-\frac{\sqrt{3}}{2}}^{0}\left|-\tfrac{1}{6}-\tfrac{1}{\pi}\arcsin s\right|\mathrm{d}s$$
$$+\int_{0}^{\frac{\sqrt{3}}{2}}\left|\tfrac{1}{6}-\tfrac{1}{\pi}\arcsin s\right|\mathrm{d}s+\int_{\frac{\sqrt{3}}{2}}^{1}\left|\tfrac{1}{2}-\tfrac{1}{\pi}\arcsin s\right|\mathrm{d}s=\tfrac{4-2\sqrt{3}}{\pi}$$

and the first inequality is obtained. To prove the second inequality we take $p=2$

$$\int_{-1}^{1}\left|Q_1(s)\right|^2\mathrm{d}s=\int_{-1}^{-\frac{\sqrt{3}}{2}}\left|-\tfrac{1}{2}-\tfrac{1}{\pi}\arcsin s\right|^2\mathrm{d}s+\int_{-\frac{\sqrt{3}}{2}}^{0}\left|-\tfrac{1}{6}-\tfrac{1}{\pi}\arcsin s\right|^2\mathrm{d}s$$
$$+\int_{0}^{\frac{\sqrt{3}}{2}}\left|\tfrac{1}{6}-\tfrac{1}{\pi}\arcsin s\right|^2\mathrm{d}s+\int_{\frac{\sqrt{3}}{2}}^{1}\left|\tfrac{1}{2}-\tfrac{1}{\pi}\arcsin s\right|^2\mathrm{d}s=\tfrac{4\pi-12}{3\pi^2}$$

Finally, for $p=1$, we have

$$\sup_{s\in[-1,1]}\left|Q_1(s)\right|=\max\left\{\sup_{s\in\left[-1,-\frac{\sqrt{3}}{2}\right]}\left|-\tfrac{1}{2}-\tfrac{1}{\pi}\arcsin s\right|,\ \sup_{s\in\left[-\frac{\sqrt{3}}{2},0\right]}\left|-\tfrac{1}{6}-\tfrac{1}{\pi}\arcsin s\right|,\right.$$
$$\left.\sup_{s\in\left[0,\frac{\sqrt{3}}{2},\right]}\left|\tfrac{1}{6}-\tfrac{1}{\pi}\arcsin s\right|,\ \sup_{s\in\left[\frac{\sqrt{3}}{2},1\right]}\left|\tfrac{1}{2}-\tfrac{1}{\pi}\arcsin s\right|\right\}.$$

Now, by elementary calculation we get

$$\sup_{s\in\left[-1,-\frac{\sqrt{3}}{2}\right]}\left|-\tfrac{1}{2}-\tfrac{1}{\pi}\arcsin s\right|=\tfrac{1}{6},\qquad \sup_{s\in\left[-\frac{\sqrt{3}}{2},0\right]}\left|-\tfrac{1}{6}-\tfrac{1}{\pi}\arcsin s\right|=\tfrac{1}{6},$$
$$\sup_{s\in\left[0,\frac{\sqrt{3}}{2},\right]}\left|\tfrac{1}{6}-\tfrac{1}{\pi}\arcsin s\right|=\tfrac{1}{6},\qquad \sup_{s\in\left[\frac{\sqrt{3}}{2},1\right]}\left|\tfrac{1}{2}-\tfrac{1}{\pi}\arcsin s\right|=\tfrac{1}{6},$$

and the third inequality is proved. □

Remark 3.2. *Inequalities from the last Corollary are proved by J. Pečarić et al. in* [5].

Remark 3.3. *If we apply Theorem 2.1 with* $n=2$, $[a,b]=[-1,1]$, $x=-\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right)=\frac{1}{3}$ *and* $w(t)=\frac{1}{\pi\sqrt{1-t^2}}$, $t\in(-1,1)$, *we get*

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t=\frac{\pi}{3}\left[f\left(-\tfrac{\sqrt{3}}{2}\right)+f(0)+f\left(\tfrac{\sqrt{3}}{2}\right)\right]$$
$$+\frac{\pi\sqrt{3}}{6}\left[f'\left(-\tfrac{\sqrt{3}}{2}\right)-f'\left(\tfrac{\sqrt{3}}{2}\right)\right]+\pi\int_{-1}^{1}Q_2(s)\,f''(s)\,\mathrm{d}s$$

where

$$Q_2(s)=\begin{cases}\frac{1}{2}s+\frac{1}{\pi}\left(s\arcsin s+\sqrt{1-s^2}\right), & -1\le s\le-\frac{\sqrt{3}}{2},\\ \frac{1}{6}s+\frac{1}{\pi}\left(s\arcsin s+\sqrt{1-s^2}\right), & -\frac{\sqrt{3}}{2}<s\le 0,\\ -\frac{1}{6}s+\frac{1}{\pi}\left(s\arcsin s+\sqrt{1-s^2}\right), & 0<s\le\frac{\sqrt{3}}{2},\\ -\frac{1}{2}s+\frac{1}{\pi}\left(s\arcsin s+\sqrt{1-s^2}\right), & \frac{\sqrt{3}}{2}<s\le 1.\end{cases}$$

Corollary 3.3. *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and let $f:I\to\mathbb{R}$ be such that f' is an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1}\frac{f(t)\,\mathrm{d}t}{\sqrt{1-t^2}}-\frac{\pi}{3}\left[f\left(-\tfrac{\sqrt{3}}{2}\right)+f(0)+f\left(\tfrac{\sqrt{3}}{2}\right)+\tfrac{\sqrt{3}}{2}f'\left(-\tfrac{\sqrt{3}}{2}\right)-\tfrac{\sqrt{3}}{2}f'\left(\tfrac{\sqrt{3}}{2}\right)\right]\right|$$
$$\le\begin{cases}\frac{1}{2}\pi\left\|f''\right\|_\infty, & f''\in L^\infty[-1,1],\\ \frac{1}{3}\sqrt{\frac{32+2\pi}{3}}\left\|f''\right\|_2, & f''\in L^2[-1,1],\\ \left\|f''\right\|_1, & f''\in L^1[-1,1].\end{cases}$$

The first and the second inequality are sharp.

Proof. Applying (13) with $n = 2$, $[a,b] = [-1,1]$, $x = -\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$, $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in (-1,1)$ and $p = \infty$, $p = 2$, $p = 1$ and carrying out the same analysis as in Corollary 3.2 we obtain the above inequalities ☐

Remark 3.4. *If we apply Theorem 2.1 with* $n = 3$, $[a,b] = [-1,1]$, $x = -\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$ *and* $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in (-1,1)$, *we get*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t = \frac{\pi}{3}\left[f\left(-\tfrac{\sqrt{3}}{2}\right) + f(0) + f\left(\tfrac{\sqrt{3}}{2}\right)\right] + \frac{\pi\sqrt{3}}{6}\left[f'\left(-\tfrac{\sqrt{3}}{2}\right) - f'\left(\tfrac{\sqrt{3}}{2}\right)\right]$$
$$+ \frac{\pi}{12}\left[\tfrac{5}{2}f''\left(-\tfrac{\sqrt{3}}{2}\right) + f''(0) + \tfrac{5}{2}f''\left(\tfrac{\sqrt{3}}{2}\right)\right] + \frac{\pi}{2}\int_{-1}^{1} Q_3(s)\, f'''(s)\,\mathrm{d}s$$

where

$$Q_3(s) = \begin{cases} -\frac{1}{2}\left(\frac{1}{2}+s^2\right) - \frac{3}{2\pi}s\sqrt{1-s^2} - \frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & -1 \le s \le -\frac{\sqrt{3}}{2}, \\ -\frac{1}{6}\left(\frac{1}{2}+s^2\right) - \frac{3}{2\pi}s\sqrt{1-s^2} - \frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & -\frac{\sqrt{3}}{2} < s \le 0, \\ \frac{1}{6}\left(\frac{1}{2}+s^2\right) - \frac{3}{2\pi}s\sqrt{1-s^2} - \frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & 0 < s \le \frac{\sqrt{3}}{2}, \\ \frac{1}{2}\left(\frac{1}{2}+s^2\right) - \frac{3}{2\pi}s\sqrt{1-s^2} - \frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & \frac{\sqrt{3}}{2} < s \le 1. \end{cases}$$

Corollary 3.4. *Let* I *be an open interval in* $\mathbb{R}$, $[-1,1] \subset I$, (p,q) *a pair of conjugate exponents,* $1 \le p,q \le \infty$ *and let* $f : I \to \mathbb{R}$ *be such that* f'' *is an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1} \frac{f(t)\,\mathrm{d}t}{\sqrt{1-t^2}} - \frac{\pi}{3}\left[f\left(-\tfrac{\sqrt{3}}{2}\right) + f(0) + f\left(\tfrac{\sqrt{3}}{2}\right) + \tfrac{\sqrt{3}}{2}f'\left(-\tfrac{\sqrt{3}}{2}\right)\right.\right.$$
$$\left.\left. -\tfrac{\sqrt{3}}{2}f'\left(\tfrac{\sqrt{3}}{2}\right) + \tfrac{5}{8}f''\left(-\tfrac{\sqrt{3}}{2}\right) + \tfrac{1}{4}f''(0) + \tfrac{5}{8}f''\left(\tfrac{\sqrt{3}}{2}\right)\right]\right|$$
$$\le \begin{cases} 0.493373\,\|f'''\|_{\infty}, & f''' \in L^{\infty}[-1,1], \\ 0.45485\,\|f'''\|_{2}, & f''' \in L^{2}[-1,1], \\ \frac{1}{48}\left(9\sqrt{3}+5\pi\right)\|f'''\|_{1}, & f''' \in L^{1}[-1,1]. \end{cases}$$

The first and the second inequality are sharp.

Proof. Applying (13) with $n = 3$, $[a,b] = [-1,1]$, $x = -\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$, $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in (-1,1)$ and $p = \infty$, $p = 2$, $p = 1$, respectively, and carrying out the same analysis as in Corollary 3.2 we get the above inequalities. ☐

3.2. $\varpi(t) = \sqrt{1-t^2}$, $t \in [-1,1]$. In this case we have Gauss-Chebyshev formula of the second kind

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t = \frac{\pi}{2}\sum_{i=1}^{k} A_i f(x_i) + E_k(f) \tag{17}$$

where

$$A_i = \frac{2}{k+1}\sin^2\frac{i\pi}{k+1}, \quad i = 1,\dots,k$$

and x_i are zeros of the Chebyshev polynomials of the second kind defined as

$$C_k(x) = \frac{\sin\left[(k+1)\arccos(x)\right]}{\sin\left[\arccos(x)\right]}.$$

$C_k(x)$ has exactly k distinct zeros

$$x_i = \cos\frac{i\pi}{k+1},$$

all of which lie in the interval $[-1,1]$ (see [6]).

Error of the approximation formula (17) is

$$E_k(f) = \frac{\pi}{2^{2k+1}(2k)!} f^{(2k)}(\xi), \quad \xi \in (-1,1).$$

In case $k=3$ the following identity holds

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t = \frac{\pi}{8}\left[f\left(-\tfrac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\tfrac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{92160} f^{(6)}(\xi),$$

where $\xi \in (-1,1)$.

Remark 3.5. *If we apply the (12) with* $[a,b]=[-1,1]$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ *and* $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$, *we get*

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t = \frac{\pi}{8}\left[f\left(-\tfrac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\tfrac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{2}\int_{-1}^{1} R_1(s) f'(s)\,\mathrm{d}s$$

where

$$R_1(s) = \begin{cases} -\frac{1}{2} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right), & -1 \le s \le -\frac{\sqrt{2}}{2}, \\ -\frac{1}{4} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right), & -\frac{\sqrt{2}}{2} < s \le 0, \\ \frac{1}{4} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right), & 0 < s \le \frac{\sqrt{2}}{2}, \\ \frac{1}{2} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right), & \frac{\sqrt{2}}{2} < s \le 1. \end{cases}$$

Corollary 3.5. *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$. Let $f: I\to\mathbb{R}$ be an absolutely continuous function and $f'\in L^p[-1,1]$. Then we have*

$$\left|\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t - \frac{\pi}{8}\left[f\left(-\tfrac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\tfrac{\sqrt{2}}{2}\right)\right]\right| \le \frac{\pi}{2}\|R_1\|_q \|f'\|_p. \quad (18)$$

Inequality (18) *is sharp for* $1<p\le\infty$.

Proof. This is a special case of Theorem 2.2 for $[a,b]=[-1,1]$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ and $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$. □

Corollary 3.6. *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and $f: I\to\mathbb{R}$ an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t - \frac{\pi}{8}\left[f\left(-\tfrac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\tfrac{\sqrt{2}}{2}\right)\right]\right|$$
$$\le \begin{cases} 0.26917\,\|f'\|_\infty, & f'\in L^\infty[-1,1], \\ 0.239162\,\|f'\|_2, & f'\in L^2[-1,1], \\ \frac{1}{8}\pi\,\|f'\|_1, & f'\in L^1[-1,1]. \end{cases}$$

The first and the second inequality are sharp.

Proof. Applying (18) with $p = \infty$, we get

$$\int_{-1}^{1} |R_1(s)|\,\mathrm{d}s = \int_{-1}^{-\frac{\sqrt{2}}{2}} \left|-\tfrac{1}{2} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right| \mathrm{d}s$$

$$+ \int_{-\frac{\sqrt{2}}{2}}^{0} \left|-\tfrac{1}{4} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right| \mathrm{d}s + \int_{0}^{\frac{\sqrt{2}}{2}} \left|\tfrac{1}{4} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right| \mathrm{d}s$$

$$+ \int_{\frac{\sqrt{2}}{2}}^{1} \left|\tfrac{1}{2} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right| \mathrm{d}s \approx 0.171359$$

and the first inequality is obtained. To prove the second inequality we take $p = 2$

$$\int_{-1}^{1} |R_1(s)|^2\,\mathrm{d}s = \int_{-1}^{-\frac{\sqrt{2}}{2}} \left|-\tfrac{1}{2} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right|^2 \mathrm{d}s$$

$$+ \int_{-\frac{\sqrt{2}}{2}}^{0} \left|-\tfrac{1}{4} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right|^2 \mathrm{d}s + \int_{0}^{\frac{\sqrt{2}}{2}} \left|\tfrac{1}{4} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right|^2 \mathrm{d}s$$

$$+ \int_{\frac{\sqrt{2}}{2}}^{1} \left|\tfrac{1}{2} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right|^2 \mathrm{d}s \approx 0.0231817$$

If $p = 1$, we have

$$\sup_{s\in\left[-1,-\frac{\sqrt{2}}{2}\right]} \left|-\tfrac{1}{2} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right| = \tfrac{1}{4} - \tfrac{1}{2\pi},$$

$$\sup_{s\in\left[-\frac{\sqrt{2}}{2},0\right]} \left|-\tfrac{1}{4} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right| = \tfrac{1}{4},$$

$$\sup_{s\in\left[0,\frac{\sqrt{2}}{2}\right]} \left|\tfrac{1}{4} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right| = \tfrac{1}{4},$$

$$\sup_{s\in\left[\frac{\sqrt{2}}{2},1\right]} \left|\tfrac{1}{2} - \tfrac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right| = \tfrac{1}{4} - \tfrac{1}{2\pi}$$

so

$$\sup_{s\in[-1,1]} |R_1(s)| = \max\left\{\tfrac{1}{4} - \tfrac{1}{2\pi}, \tfrac{1}{4}\right\} = \tfrac{1}{4}$$

and the third inequality is proved. □

Remark 3.6. *Inequalities from the last Corollary are proved by J. Pečarić et al. in* [5].

Remark 3.7. *If we apply Theorem 2.1 with $n = 2$, $[a,b] = [-1,1]$, $x = -\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$ and $w(t) = \frac{2\sqrt{1-t^2}}{\pi}$, $t \in [-1,1]$, we get*

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t = \frac{\pi}{8}\left[f\left(-\tfrac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\tfrac{\sqrt{2}}{2}\right)\right]$$
$$+ \frac{\pi\sqrt{2}}{16}\left[f'\left(-\tfrac{\sqrt{2}}{2}\right) - f'\left(\tfrac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{2}\int_{-1}^{1} R_2(s) f''(s)\,\mathrm{d}s$$

where

$$R_2(s)=\begin{cases} \frac{s}{2}+\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & -1\le s\le -\frac{\sqrt{2}}{2},\\ \frac{s}{4}+\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & -\frac{\sqrt{2}}{2}< s\le 0,\\ -\frac{s}{4}+\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & 0< s\le \frac{\sqrt{2}}{2},\\ -\frac{s}{2}+\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & \frac{\sqrt{2}}{2}< s\le 1.\end{cases}$$

Corollary 3.7. *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and let $f:I\to\mathbb{R}$ be such that f' is an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{8}\left[f\left(-\tfrac{\sqrt{2}}{2}\right)+2f(0)+f\left(\tfrac{\sqrt{2}}{2}\right)\right.\right.$$
$$\left.\left.+\ \tfrac{\sqrt{2}}{2}f'\left(-\tfrac{\sqrt{2}}{2}\right)-\tfrac{\sqrt{2}}{2}f'\left(\tfrac{\sqrt{2}}{2}\right)\right]\right|$$
$$\le\begin{cases} \frac{1}{8}\pi\left\|f''\right\|_\infty, & f''\in L^\infty[-1,1],\\ 0.3287364\left\|f''\right\|_2, & f''\in L^2[-1,1],\\ \frac{1}{3}\left\|f''\right\|_1, & f''\in L^1[-1,1].\end{cases}$$

The first and the second inequality are sharp.

Proof. Applying (13) with $n=2$, $[a,b]=[-1,1]$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$, $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$ and $p=\infty$, $p=2$, $p=1$, respectively, and carrying out the same analysis as in Corollary 3.6 we get the above inequalities. □

Remark 3.8. *If we apply Theorem 2.1 with $n=3$, $[a,b]=[-1,1]$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ and $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$, we get*

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t=\frac{\pi}{8}\left[f\left(-\tfrac{\sqrt{2}}{2}\right)+2f(0)+f\left(\tfrac{\sqrt{2}}{2}\right)\right]+\frac{\pi\sqrt{2}}{16}\left[f'\left(-\tfrac{\sqrt{2}}{2}\right)-f'\left(\tfrac{\sqrt{2}}{2}\right)\right]$$
$$+\frac{\pi}{64}\left[3f''\left(-\tfrac{\sqrt{2}}{2}\right)+2f''(0)+3f''\left(\tfrac{\sqrt{2}}{2}\right)\right]+\frac{\pi}{4}\int_{-1}^{1}R_3(s)f'''(s)\,\mathrm{d}s$$

where

$$R_3(s)=\begin{cases} -\frac{1}{8}\left(1+4s^2\right)-\frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2} & \\ \quad-\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & -1\le s\le -\frac{\sqrt{2}}{2},\\ -\frac{1}{16}\left(1+4s^2\right)-\frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2} & \\ \quad-\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & -\frac{\sqrt{2}}{2}< s\le 0,\\ \frac{1}{16}\left(1+4s^2\right)-\frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2} & \\ \quad-\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & 0< s\le \frac{\sqrt{2}}{2},\\ \frac{1}{8}\left(1+4s^2\right)-\frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2} & \\ \quad-\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & \frac{\sqrt{2}}{2}< s\le 1.\end{cases}$$

Corollary 3.8. *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and let $f:I\to\mathbb{R}$ be such that f'' is an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{8}\left[f\left(-\tfrac{\sqrt{2}}{2}\right)+2f(0)+f\left(\tfrac{\sqrt{2}}{2}\right)+\tfrac{\sqrt{2}}{2}f'\left(-\tfrac{\sqrt{2}}{2}\right)\right.\right.$$
$$\left.\left.-\tfrac{\sqrt{2}}{2}f'\left(\tfrac{\sqrt{2}}{2}\right)+\tfrac{3}{8}f''\left(-\tfrac{\sqrt{2}}{2}\right)+\tfrac{1}{4}f''(0)+\tfrac{3}{8}f''\left(\tfrac{\sqrt{2}}{2}\right)\right]\right|$$
$$\le\begin{cases} 0.0869419\left\|f'''\right\|_\infty, & f'''\in L^\infty[-1,1],\\ 0.0885601\left\|f'''\right\|_2, & f'''\in L^2[-1,1],\\ \frac{7}{48}\left\|f'''\right\|_1, & f'''\in L^1[-1,1].\end{cases}$$

The first and the second inequality are sharp.

Proof. Applying (13) with $n=3$, $[a,b]=[-1,1]$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$, $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$ and $p=\infty$, $p=2$, $p=1$, respectively, and carrying out the same analysis as in Corollary 3.6 we obtain the above inequalities. □

4. Non-weighted three-point formula and applications

We define

$$\begin{aligned}\widehat{t}_n(x) &= A(x)\sum_{i=0}^{n-2}\left[f^{(i+1)}(x)+(-1)^{i+1}f^{(i+1)}(a+b-x)\right]\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\\ &+ (1-2A(x))\sum_{i=0}^{n-2}f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1-(-1)^i\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}\end{aligned}$$

and

$$\widehat{T}_n(x,s)=-nA(x)\left[T_n(x,s)+T_n(a+b-x,s)\right]-n(1-2A(x))\,T_n\left(\frac{a+b}{2},s\right).$$

Theorem 4.1. *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t=D(x)+\widehat{t}_n(x)+\frac{1}{n!}\int_a^b\widehat{T}_n(x,s)\,f^{(n)}(s)\,\mathrm{d}s \tag{19}$$

Proof. This is a special case of Theorem 2.1 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Remark 4.1. *Identity* (19) *holds true in the case $n=1$.*

Theorem 4.2. *Suppose that all the assumptions of Theorem 4.1 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1\leq p,q\leq\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $f^{(n)}\in L^p[a,b]$ for some $n\geq 1$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D(x)-\widehat{t}_n(x)\right|\leq\frac{1}{n!}\left\|\widehat{T}_n(x,\cdot)\right\|_q\left\|f^{(n)}\right\|_p. \tag{20}$$

Inequality (20) *is sharp for $1<p\leq\infty$.*

Proof. This is a special case of Theorem 2.2 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Now, we set

$$A(x)=\frac{(b-a)^2}{6(a+b-2x)^2},\ x\in\left[a,\frac{a+b}{2}\right).$$

This special choice of the function A enables us to consider generalizations of the well-known Simpson's formula (1) and dual Simpson's formula (2).

4.1. $x = a$**.** Suppose that all the assumptions of Theorem 4.1 hold, then the generalization of the Simpson's formula reads

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D(a) + \widehat{t}_n(a) + \frac{1}{n!}\int_a^b \widehat{T}_n(a,s) f^{(n)}(s)\,\mathrm{d}s, \tag{21}$$

where

$$D(a) = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

$$\begin{aligned}\widehat{t}_n(a) &= \frac{1}{6}\sum_{i=0}^{n-2}\left[f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b)\right]\frac{(b-a)^{i+1}}{(i+2)!}\\ &+ \frac{2}{3}\sum_{i=1}^{n-2} f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1-(-1)^i\right)(b-a)^{i+1}}{2^{i+2}(i+2)!},\end{aligned}$$

and

$$\begin{aligned}\widehat{T}_n(a,s) &= -\frac{n}{6}\left[T_n(a,s) + 4T_n\left(\frac{a+b}{2},s\right) + T_n(b,s)\right]\\ &= \begin{cases} \frac{5(a-s)^n+(b-s)^n}{6(b-a)}, & a \le s \le \frac{a+b}{2},\\ \frac{(a-s)^n+5(b-s)^n}{6(b-a)}, & \frac{a+b}{2} < s \le b.\end{cases}\end{aligned}$$

In the next corollaries we will use the Beta function and the incomplete Beta function of Euler type defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t, \qquad B_r(x,y) = \int_0^r t^{x-1}(1-t)^{y-1}\,\mathrm{d}t, \qquad x,y>0.$$

Corollary 4.1. *Suppose that all the assumptions of Theorem 4.1 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$.*

(a) *If $f^{(n)} \in L^\infty[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D(a) - \widehat{t}_n(a)\right|\\ &\le \frac{1}{(n+1)!}\cdot\left(\frac{\left[5-(-1)^n+2^{n+1}\right](b-a)^n}{3\cdot 2^{n+1}} - \frac{5\left(1-(-1)^n\right)(b-a)^n}{3\left(1+\sqrt[n]{5}\right)^n}\right)\left\|f^{(n)}\right\|_\infty.\end{aligned}$$

(b) *If $f^{(n)} \in L^2[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D(a) - \widehat{t}_n(a)\right|\\ &\le \frac{1}{n!}\left(\frac{\left(2^{2n-2}+3\right)(b-a)^{2n-1}}{9\cdot 2^{2n-1}(2n+1)} + \frac{5(-1)^n(b-a)^{2n-1}}{18}B(n+1,n+1)\right)^{\frac{1}{2}}\left\|f^{(n)}\right\|_2.\end{aligned}$$

(c) *If $f^{(n)} \in L^1[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D(a) - \widehat{t}_n(a)\right|\\ &\le \frac{1}{n!}\max\left\{\frac{(b-a)^{n-1}}{6}, \frac{\left((-1)^n+5\right)(b-a)^{n-1}}{3\cdot 2^{n+1}}\right\}\left\|f^{(n)}\right\|_1.\end{aligned}$$

The first and the second inequality are sharp.

Proof. We apply (20) with $x = a$ and $p = \infty$

$$\int_a^b \left|\widehat{T}_n(a,s)\right| ds = \int_a^{\frac{a+b}{2}} \left|\frac{5(a-s)^n + (b-s)^n}{6(b-a)}\right| \mathrm{d}s + \int_{\frac{a+b}{2}}^b \left|\frac{(a-s)^n + 5(b-s)^n}{6(b-a)}\right| \mathrm{d}s$$
$$= 2 \cdot \frac{\left[5 - (-1)^n + 2^{n+1}\right](b-a)^n}{6 \cdot 2^{n+1}(n+1)} - 2 \cdot \frac{5\left(1 - (-1)^n\right)(b-a)^n}{6\left(1 + \sqrt[n]{5}\right)^n (n+1)}$$

and the first inequality is obtained. If $p = 2$, we have

$$\int_a^b \left|\widehat{T}_n(a,s)\right|^2 \mathrm{d}s = \int_a^{\frac{a+b}{2}} \left|\frac{5(a-s)^n + (b-s)^n}{6(b-a)}\right|^2 \mathrm{d}s + \int_{\frac{a+b}{2}}^b \left|\frac{(a-s)^n + 5(b-s)^n}{6(b-a)}\right|^2 \mathrm{d}s$$
$$= \frac{(b-a)^{2n-1}}{36}\left[\frac{24 + 2^{2n+1}}{2^{2n+1}(2n+1)} + 10 \cdot (-1)^n B_{\frac{1}{2}}(n+1, n+1)\right]$$
$$+ \frac{(b-a)^{2n-1}}{36}\left[\frac{24 + 2^{2n+1}}{2^{2n+1}(2n+1)} + 10 \cdot (-1)^n \Big(B(n+1, n+1) - B_{\frac{1}{2}}(n+1, n+1)\Big)\right].$$

and the second inequality is proved.

To obtain the third inequality we take $p = 1$

$$\sup_{s \in [a,b]} \left|\widehat{T}_n(a,s)\right|$$
$$= \max\left\{\sup_{s \in \left[a, \frac{a+b}{2}\right]} \left|\frac{5(a-s)^n + (b-s)^n}{6(b-a)}\right|, \sup_{s \in \left[\frac{a+b}{2}, b\right]} \left|\frac{(a-s)^n + 5(b-s)^n}{6(b-a)}\right|\right\}.$$

The function $y : [a,b] \to \mathbb{R}$, $y(x) = 5(a-x)^n + (b-x)^n$, is decreasing on (a, x_0) and increasing on (x_0, b) if n is even, where $x_0 = \frac{\sqrt[n-1]{5}a + b}{1 + \sqrt[n-1]{5}}$, and decreasing on (a,b) if n is odd. By an elementary calculation we get

$$\sup_{s \in \left[a, \frac{a+b}{2}\right]} \left|\frac{5(a-s)^n + (b-s)^n}{6(b-a)}\right| = \max\left\{\frac{(b-a)^{n-1}}{2^n}, \frac{(b-a)^{n-1}}{6}\right\},$$

for even n, and

$$\sup_{s \in \left[a, \frac{a+b}{2}\right]} \left|\frac{5(a-s)^n + (b-s)^n}{6(b-a)}\right| = \max\left\{\frac{(b-a)^{n-1}}{2^{n-1} \cdot 3}, \frac{(b-a)^{n-1}}{6}\right\},$$

for odd n. Also

$$\sup_{s \in \left[\frac{a+b}{2}, b\right]} \left|\frac{(a-s)^n + 5(b-s)^n}{6(b-a)}\right| = \max\left\{\frac{(b-a)^{n-1}}{2^n}, \frac{(b-a)^{n-1}}{6}\right\},$$

if n is even, and

$$\sup_{s \in \left[\frac{a+b}{2}, b\right]} \left|\frac{(a-s)^n + 5(b-s)^n}{6(b-a)}\right| = \max\left\{\frac{(b-a)^{n-1}}{2^{n-1} \cdot 3}, \frac{(b-a)^{n-1}}{6}\right\},$$

if n is odd. □

4.2. $x=\dfrac{3a+b}{4}$**.** Suppose that all the assumptions of Theorem 4.1 hold, then the generalization of the famous dual Simpson's formula reads

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D\left(\frac{3a+b}{4}\right)+\widehat{t}_n\left(\frac{3a+b}{4}\right)+\frac{1}{n!}\int_a^b \widehat{T}_n\left(\frac{3a+b}{4},s\right)f^{(n)}(s)\,\mathrm{d}s, \tag{22}$$

where

$$D\left(\frac{3a+b}{4}\right)=\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right),$$

$$\begin{aligned}\widehat{t}_n\left(\frac{3a+b}{4}\right)&=\frac{2}{3}\sum_{i=0}^{n-2}\left[f^{(i+1)}\left(\frac{3a+b}{4}\right)+(-1)^{i+1}f^{(i+1)}\left(\frac{a+3b}{4}\right)\right]\\&\quad\times\frac{\left[3^{i+2}-(-1)^{i+2}\right](b-a)^{i+1}}{4^{i+2}(i+2)!}-\frac{1}{3}\sum_{i=1}^{n-2}f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1-(-1)^i\right)(b-a)^{i+1}}{2^{i+2}(i+2)!},\end{aligned}$$

and

$$\begin{aligned}\widehat{T}_n\left(\frac{3a+b}{4},s\right)&=-\frac{n}{3}\left[2T_n\left(\frac{3a+b}{4},s\right)-T_n\left(\frac{a+b}{2},s\right)+2T_n\left(\frac{a+3b}{4},s\right)\right]\\&=\begin{cases}\frac{(a-s)^n}{b-a}, & a\le s\le\frac{3a+b}{4},\\ \frac{(a-s)^n+2(b-s)^n}{3(b-a)}, & \frac{3a+b}{4}<s\le\frac{a+b}{2},\\ \frac{2(a-s)^n+(b-s)^n}{3(b-a)}, & \frac{a+b}{2}<s\le\frac{a+3b}{4},\\ \frac{(b-s)^n}{b-a}, & \frac{a+3b}{4}<s\le b.\end{cases}\end{aligned}$$

Corollary 4.2. *Suppose that all the assumptions of Theorem 4.1 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1\le p,q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$.*

(a) *If $f^{(n)}\in L^\infty[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D\left(\frac{3a+b}{4}\right)-\widehat{t}_n\left(\frac{3a+b}{4}\right)\right|\\&\le\frac{1}{(n+1)!}\left(\frac{\left[3\left(2\cdot3^n+1\right)-2^{n+1}\left(2+(-1)^{n+1}\right)+(-1)^{n+1}\right](b-a)^n}{3\cdot2^{2n+1}}\right)\left\|f^{(n)}\right\|_\infty.\end{aligned}$$

(b) *If $f^{(n)}\in L^2[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D\left(\frac{3a+b}{4}\right)-\widehat{t}_n\left(\frac{3a+b}{4}\right)\right|\\&\le\frac{1}{n!}\left(\frac{\left[9\left(4\cdot3^{2n-1}+1\right)-3\cdot2^{2n+1}-1\right](b-a)^{2n-1}}{9\cdot2^{4n+1}(2n+1)}\right.\\&\qquad\left.+\frac{4(-1)^n(b-a)^{2n-1}}{9}\left[B_{\frac{3}{4}}(n+1,n+1)-B_{\frac{1}{4}}(n+1,n+1)\right]\right)^{\frac{1}{2}}\left\|f^{(n)}\right\|_2.\end{aligned}$$

(c) *If* $f^{(n)} \in L^1[a,b]$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D\left(\frac{3a+b}{4}\right) - \widehat{t}_n\left(\frac{3a+b}{4}\right)\right|$$
$$\leq \frac{(b-a)^{n-1}}{n!}\max\left\{\frac{1}{2^{2n}}, \frac{2\cdot 3^n+(-1)^n}{3\cdot 2^{2n}}, \frac{2^n+(1+(-1)^n)\left(3^n-2^{n-1}+\frac{1}{2}\right)}{3\cdot 2^{2n}}\right\}\left\|f^{(n)}\right\|_1.$$

The first and the second inequality are sharp.

Proof. We apply (20) with $x=\frac{3a+b}{4}$ and $p=\infty$

$$\int_a^b\left|\widehat{T}_n\left(\frac{3a+b}{4},s\right)\right|ds = \int_a^{\frac{3a+b}{4}}\left|\frac{(a-s)^n}{b-a}\right|\mathrm{d}s + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left|\frac{(a-s)^n+2(b-s)^n}{3(b-a)}\right|\mathrm{d}s$$
$$+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left|\frac{2(a-s)^n+(b-s)^n}{3(b-a)}\right|\mathrm{d}s + \int_{\frac{a+3b}{4}}^{b}\left|\frac{(b-s)^n}{b-a}\right|\mathrm{d}s$$
$$=2\cdot\frac{(b-a)^n}{2^{2n+2}(n+1)} + 2\cdot\frac{(b-a)^n\left[2\cdot 3^{n+1}+(-1)^{n+1}-2^{n+1}\left(2+(-1)^{n+1}\right)\right]}{3\cdot 2^{2n+2}(n+1)}$$
$$=\frac{\left[3\left(2\cdot 3^n+1\right)-2^{n+1}\left(2+(-1)^{n+1}\right)+(-1)^{n+1}\right](b-a)^n}{3\cdot 2^{2n+1}(n+1)}$$

and the first inequality is obtained. To prove the second inequality we take $p=2$

$$\int_a^b\left|\widehat{T}_n\left(\frac{3a+b}{4},s\right)\right|^2\mathrm{d}s = \int_a^{\frac{3a+b}{4}}\left|\frac{(a-s)^n}{b-a}\right|^2\mathrm{d}s + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left|\frac{(a-s)^n+2(b-s)^n}{3(b-a)}\right|^2\mathrm{d}s$$
$$+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left|\frac{2(a-s)^n+(b-s)^n}{3(b-a)}\right|^2\mathrm{d}s + \int_{\frac{a+3b}{4}}^{b}\left|\frac{(b-s)^n}{b-a}\right|^2\mathrm{d}s$$
$$=2\cdot\frac{(b-a)^{2n-1}}{4^{2n+1}(2n+1)} + 2\cdot\frac{\left[4\cdot 3^{2n+1}-3\cdot 2^{2n+1}-1\right](b-a)^{2n-1}}{9\cdot 4^{2n+1}(2n+1)}$$
$$+\frac{4(-1)^n(b-a)^{2n-1}}{9}\left[B_{\frac{3}{4}}(n+1,n+1)-B_{\frac{1}{4}}(n+1,n+1)\right].$$

If $p=1$, we have

$$\sup_{s\in[a,b]}\left|\widehat{T}_n\left(\frac{3a+b}{4},s\right)\right| = \max\left\{\sup_{s\in\left[a,\frac{3a+b}{4}\right]}\left|\frac{(a-s)^n}{b-a}\right|, \sup_{s\in\left[\frac{3a+b}{4},\frac{a+b}{2}\right]}\left|\frac{(a-s)^n+2(b-s)^n}{3(b-a)}\right|,\right.$$
$$\left.\sup_{s\in\left[\frac{a+b}{2},\frac{a+3b}{4}\right]}\left|\frac{2(a-s)^n+(b-s)^n}{3(b-a)}\right|, \sup_{s\in\left[\frac{a+3b}{4},b\right]}\left|\frac{(b-s)^n}{b-a}\right|\right\}.$$

By an elementary calculation we get

$$\sup_{s\in\left[a,\frac{3a+b}{4}\right]}\left|\frac{(a-s)^n}{b-a}\right| = \frac{(b-a)^{n-1}}{2^{2n}}, \qquad \sup_{s\in\left[\frac{a+3b}{4},b\right]}\left|\frac{(b-s)^n}{b-a}\right| = \frac{(b-a)^{n-1}}{2^{2n}},$$

$$\sup_{s\in\left[\frac{3a+b}{4},\frac{a+b}{2}\right]}\left|\frac{(a-s)^n+2(b-s)^n}{3(b-a)}\right| = \frac{\left(2\cdot 3^n+(-1)^n\right)(b-a)^{n-1}}{3\cdot 2^{2n}}.$$

Also

$$\sup_{s\in\left[\frac{a+b}{2},\frac{a+3b}{4}\right]}\left|\frac{2\left(a-s\right)^{n}+\left(b-s\right)^{n}}{3\left(b-a\right)}\right|=\frac{\left(2\cdot 3^{n}+1\right)\left(b-a\right)^{n-1}}{3\cdot 2^{2n}},$$

if n is even, and

$$\sup_{s\in\left[\frac{a+b}{2},\frac{a+3b}{4}\right]}\left|\frac{2\left(a-s\right)^{n}+\left(b-s\right)^{n}}{3\left(b-a\right)}\right|=\max\left\{\frac{\left(b-a\right)^{n-1}}{3\cdot 2^{n}},\frac{\left(2\cdot 3^{n}-1\right)\left(b-a\right)^{n-1}}{3\cdot 2^{2n}}\right\},$$

if n is odd. Thus

$$\sup_{s\in[a,b]}\left|\widehat{T}_{n}\left(\frac{3a+b}{4},s\right)\right|=\max\left\{\frac{\left(b-a\right)^{n-1}}{2^{2n}},\frac{\left(2\cdot 3^{n}+\left(-1\right)^{n}\right)\left(b-a\right)^{n-1}}{3\cdot 2^{2n}},\right.$$
$$\left.\frac{\left[2^{n}+\left(1+\left(-1\right)^{n}\right)\left(3^{n}-2^{n-1}+\frac{1}{2}\right)\right]\left(b-a\right)^{n-1}}{3\cdot 2^{2n}}\right\}$$

and the third inequality is proved.

□

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