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Prime Ideals in BCI and BCK-Algebras

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Abstract. In this paper, we introduce a new definition of prime ideal in BCI-algebras and show that it is equivalent to the last definition of prime ideal in lower BCK-semilattice. Then we attempt to generalize some useful theorems about prime ideals, in BCI-algebras, instead of lower BCK-semilattices.

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1. Introduction

The notions of BCK and BCI-algebras were introduced by Imai and Iseki [3, 4] in 1966. They are two important classes of logical algebras. Most of the algebras related to the t-norm based logic, such as MTL-algebras, BL-algebras and residuated lattices are extensions of BCK-algebras. These algebras have been extensively studied since their introduction. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. The concept of ideal in these algebra follows from the concepts of deductive system and ideal in logical algebras such as BL-algebras and residuated lattices.

Iseki [5], introduced the concept of prime ideal in commutative BCK-algebras and Palasinski [10], generalized this definition for any lower BCK-semilattices. Then many authors have studied the properties of this ideal in lower BCK-semilattices (see [1, 2, 5, 9, 10]). They showed that this ideal is one of the most important ideals in lower BCK-semilattices. Any ideal F of a lower BCK-semilattices contained in a prime ideal, has prime and minimal prime decomposition. But prime ideal and irreducible ideal are the same in lower BCK-semilattice. In this paper, we generalize the concept of prime ideals for BCI-algebras and attempt to generalize the properties of prime ideals in BCI-algebras. We show that prime ideals are irreducible in any BCI-algebras, but the converse may not true in general. Then we verify some useful properties of this ideals in BCI and BCK-algebra such as relation between prime ideals and maximal ideals.

2. Preliminaries

Definition 2.1. [3, 4] A *BCI-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions: for all $x, y, z \in X$

($BCI1$) $((x * y) * (x * z)) * (z * y) = 0$

($BCI2$) $x * 0 = x$

($BCI3$) $x * y = 0$ and $y * x = 0$ imply $y = x$

Let X be a BCI-algebra and $x * y^n = (...((x * y) * y) * ...) * y$, where y occurs n times and $x, y \in X$. Then for all $x, y, z \in X$ and $k \in \mathbb{N}$, the following hold: (see [11])

$(BCI4)$ $x * x = 0$

$(BCI5)$ $(x * y) * z = (x * z) * y$

$(BCI6)$ $x * (x * (x * y))^k = x * y^k$

$(BCI7)$ $0 * (x * y)^k = (0 * x^k) * (0 * y^k)$

$(BCI8)$ $0 * (0 * x)^k = 0 * (0 * x^k)$.

A nonempty subset S of BCI-algebra $(X, *, 0)$ is called a *subalgebra* of X if $x*y \in S$, for any $x, y \in S$.

For any BCI-algebra X, the relation $x \leq y \Leftrightarrow x * y = 0$ is a partial order relation [4]. It is called BCI*-ordering* of X. The set $P = \{x \in X \ \ |0 * (0 * x) = x\}$ is called P*-semisimple* part of BCI-algebra X and X is called a P*-semisimple* BCI-algebra if $P = X$ (see [8, 11]). The set $\{x \in X \ \ |0 * x = 0\}$ is called BCK*-part* of BCI-algebra X and is denoted by $BCK(X)$. If $X = BCK(X)$, then we say X is a BCK*-algebra*. A lower BCK*-semilattice* is a BCK-algebra $(X, *, 0)$, such that it with respect to it's BCI-ordering formes a lower semilattice. Moreover, a BCI-algebra X is called *associative* if $(x*y)*z = x*(y*z)$, for any $x, y, z \in X$. In any associative BCI-algebra, $x * y = y * x$ and $0 * x = x$, for any $x, y \in X$ (see [7]).

Definition 2.2. [3, 4] Let I be a nonempty subset of BCI-algebra X containing 0. I is called an *ideal* of X if $y * x \in I$ and $x \in I$ imply $y \in I$, for any $x, y \in X$. Clearly, $\{0\}$ is an ideal of X and we write 0 is an ideal of X, for convenience. An ideal I is called *proper*, if $I \neq X$ and is called *closed*, if $x*y \in I$, for all $x, y \in I$. The BCK-part of X is a closed ideal of X. Let S be a nonempty subset of X. We call the least ideal of X containing S , the *generated* ideal of X by S and is denoted by $\langle S\rangle$.

If A and B are two subalgebras of X, then we usually denote $A + B$ for $\langle A \cup B\rangle$. Moreover, $A + B$ is a closed ideal of X [see [11], Proposition 1.4.15]. If X is a BCI-algebra, then BCK-part of X is a closed ideal of X and P-semisimple part of X is a subalgebra of X. If X is a lower BCK-semilattice, then for any $x, y \in X$, we have

(P1) $\langle x\rangle \cap \langle y\rangle = \langle x \wedge y\rangle$ (see [11], Proposition 1.4.16).

Let A be an ideal of a BCI-algebra X. Then the relation θ defined by $(x, y) \in \theta \Leftrightarrow x * y, y * x \in A$ is a congruence relation on X. We usually denote A_x for $[x] = \{y \in X \ \ |(x, y) \in \theta\}$. Moreover, A_0 is a closed ideal of BCI-algebra X. In fact, it is the greatest closed ideal contained in A. Assume that $X/A = \{A_x \ \ |x \in X\}$. Then $(X/A, *, A_0)$ is a BCI-algebra, where $A_x * A_y = A_{x*y}$, for all $x, y \in X$.

Let X and Y be two BCI-algebras. A map $f : X \to Y$ is called a BCI*-homomorphism*, if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. If $f : X \to Y$ is a BCI-homomorphism, then the set $ker(f) = f^{-1}(0)$ is a closed ideal of X. A homomorphism is one to one if and only if $ker(f) = \{0\}$ (see [11]). The homomorphism f is called an *epimorphism* if f is onto. Moreover, an *isomorphism* is a homomorphism, which is both one to one and onto. Note that, if $f : X \to Y$ is a BCI-homomorphism, then $f(0) = 0$. An element x of BCI-algebra X is called *nilpotent* if $0 * x^n = 0$, for some $n \in \mathbb{N}$. A BCI-algebra is called *nilpotent* if any element of X is nilpotent (see [6]).

Theorem 2.1. [11] *BCI-algebra X is nilpotent if and only if every ideal of X is closed.*

Theorem 2.2. [11] *Let S be a nonempty subset of a BCI-algebra X and*

$$A = \{x \in X \ \ |(...((x*a_1)*a_2)*...)*a_n = 0, \text{ for some } n \in \mathbb{N} \text{ and some } a_1, ..., a_n \in S\}.$$

Then $\langle S\rangle = A\cup\{0\}$*. Especially, if* S *contains a nilpotent element of* X*, then* $\langle S\rangle = A$*. Moreover, if* I *is an ideal of* X*, then*

$$\langle A\cup S\rangle = \{x \in X \ \ |(...((x*a_1)*a_2)*...)*a_n \in A, \text{ for some } n \in \mathbb{N} \text{ and } a_1, ..., a_n \in S\}.$$

Definition 2.3. [10] A proper ideal I of BCI-algebra X is called an *irreducible* ideal if $A \cap B = I$ implies $A = I$ or $B = I$, for any ideals A and B of X.

Definition 2.4. [10] Let X be a BCI-algebra. A proper ideal M of X is called a *maximal* ideal if $\langle M \cup \{x\}\rangle = X$ for any $x \in X\backslash M$, where $\langle M \cup \{x\}\rangle$ is an ideal generated by $M\cup\{x\}$. Note that, M is a maximal ideal of X if and only if $M \subseteq A \subseteq X$ implies that $M = A$ or $A = X$, for any ideal A of X.

Theorem 2.3. [10] *Let* X *and* Y *be two* BCI*-algebras and* $f : X \to Y$ *be a* BCI*-epimorphism. If* $A = ker(f)$*, then* $\alpha : X/A \to Y$ *which is defined by* $\alpha(A_x) = f(x)$ *is a* BCI*-isomorphism.*

Lemma 2.4. [11] *Let* I *and* J *be two ideals of* BCI*-algebra* X *such that* $I \subseteq J$*. Denote* $J/I = \{I_x \in X/I \ \ |x \in J\}$*. Then*

(i) $x \in J$ *if and only if* $I_x \in J/I$*, for any* $x \in X$*.*

(ii) $J/I = \{I_x \in X/I \ \ |x \in J\}$ *is an ideal of* X/I*.*

(iii) Let I *be a closed ideal of* X*. If* S *and* T *are the sets of all ideals of* X *and* X/I*, respectively, then the map* $g : S \to T$ *defined by* $g(J) = J/I$*, is a bijective map. The inverse of* g *is the map* $f : T \to S$*, is defined by* $f(J) = \cup\{I_x \ \ |I_x \in J\}$*.*

Definition 2.5. [5] A proper ideal I of lower BCK-semilattice X is called *prime* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

Let $\{X_i\}_{i\in I}$ be a family of BCI-algebras. Then $\prod_{i\in I} X_i$ is a BCI-algebra and the map $\pi_j : \prod_{i\in I} X_i \to X_j$, defined by $\pi_j((x_i)_{i\in I}) = x_j$ is called $j-th$ *natural projection map.*

Definition 2.6. [11] A BCI-algebra X is called a *subdirect product* of BCI-algebras family $\{X_i\}_{i\in I}$ if there is an one to one BCI-homomorphism $f : X \to \prod_{i\in I} X_i$ such that $\pi_i(f(X)) = X_i$, where $\pi_i : \prod_{i\in I} X_i \to X_i$ is the $i-th$ natural projection map, for all $i \in I$. Moreover, the map f is called *subdirect embedding.*

3. Prime ideals in BCI and BCK-algebras

In this section, we introduce the concept of prime ideals in BCI-algebras and we prove that this concept and the last definition of prime ideal in a lower BCK-semilattice are equivalent. Then we generalize some useful theorems about the prime ideals on BCI and BCK-algebras. Finally, we discuss some relations between BCK-part and prime ideals in BCI and BCK-algebras.

Throughout this section, X is a BCI-algebra, B is BCK-part of X and P is P-semisimple part of X, unless otherwise stated.

Definition 3.1. A proper ideal I of BCI-algebra X is called *prime* if $A \cap B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$, for all ideals A and B of X.

Example 3.1. Let "-" be the subtraction of integers. Then $X = (\mathbb{Z}, -, 0)$ is a BCI-algebra. Clearly, $M_1 = \mathbb{N} \cup \{0\}$ and $M_2 = \{-n \ \ |n \in \mathbb{N}\} \cup \{0\}$ are two maximal ideals of X (see [11], Example 5.3.2). Let $I \cap J \subseteq \mathbb{N}$. If $I \nsubseteq \mathbb{N}$ and $J \nsubseteq \mathbb{N}$ then there exist $m, n \in \mathbb{N}$ such that $-n \in I$ and $-m \in J$. By Theorem 2.3, we conclude that $-mn \in I \cap J \subseteq \mathbb{N} \cup \{0\}$, which is impossible. Hence $\mathbb{N} \cup \{0\}$ is a prime ideal of X. By the similar way, M_2 is a prime ideal of X.

Theorem 3.1. *(i) Let I be an ideal of X. Then I is a prime ideal of X if and only if $\langle x\rangle \cap \langle y\rangle \subseteq I$ implies $x \in I$ or $y \in I$, for any $x, y \in X$.*
(ii) If X is a lower BCK-semilattice, then Definition 3.1 and Definition 2.5 are equivalent.

Proof. (i) Let I be an ideal of X, such that $\langle x\rangle \cap \langle y\rangle \subseteq I$ implies $x \in I$ or $y \in I$. If A and B are two ideals of X, such that $A \cap B \subseteq I$, then there is no harm in assuming $A \nsubseteq I$. Hence there exists $a \in A$ such that $a \notin I$. For any $b \in B$, since $\langle a\rangle \cap \langle b\rangle \subseteq A \cap B \subseteq I$ and $a \notin I$, the primeness of I implies $b \in I$. Therefore, $B \subseteq I$. Conversely, let I be a prime ideal of X. Clearly, $\langle x\rangle \cap \langle y\rangle \subseteq I$ implies $x \in I$ or $y \in I$, for any $x, y \in X$.
(ii) Since by $(P1)$, $\langle x\rangle \cap \langle y\rangle = \langle x \wedge y\rangle$, for any $x, y \in X$ so Definition 3.1 and Definition 2.5 are equivalent. □

Clearly, any prime ideal of X is an irreducible ideal. Moreover, if $\{0\}$ is an irreducible ideal of X, then $\{0\}$ is a prime ideal.

Definition 3.2. A nonempty subset F of X is called a *finite $\cap$-structure*, if $(\langle x\rangle \cap \langle y\rangle) \cap F \neq \emptyset$, for all $x, y \in F$, and X is called a *finite $\cap$-structure* if $X\backslash\{0\}$ is a finite $\cap$-structure.

Proposition 3.2. *Let Y be a BCI-algebra and $f : X \to Y$ be an onto BCI-homomorphism. Then the following assertions hold:*
(i) An ideal I of X is prime if and only if $F = X - I$ is a finite $\cap$-structure.
(ii) Let I be a closed ideal of X and J be an ideal of X containing I. If J is a prime ideal of X, then J/I is a prime ideal of X/I.
(iii) Let I be a prime ideal of X and $ker(f) \subseteq I$. Then $f(I)$ is a prime ideal of Y.
(iv) Let $ID(X)$ be the set of all ideals of X. Then $ID(X)$ is a chain if and only if every proper ideal of X is prime.

Proof. (i) Let I be a prime ideal of X and $x, y \in F$. If $(\langle x\rangle \cap \langle y\rangle) \cap F = \emptyset$, then $\langle x\rangle \cap \langle y\rangle \subseteq I$. Since I is a prime ideal of X, we have $x \in I$ or $y \in I$, which is impossible. Hence $(\langle x\rangle \cap \langle y\rangle) \cap F \neq \emptyset$. Conversely, let F be a finite $\cap$-structure and $x, y \in X$ such that $\langle x\rangle \cap \langle y\rangle \subseteq I$. If $x \notin I$ and $y \notin I$, then $x, y \in F$ and so $(\langle x\rangle \cap \langle y\rangle) \cap F \neq \emptyset$. Hence, $\langle x\rangle \cap \langle y\rangle \nsubseteq I$, which is impossible. Therefore, $x \in I$ or $y \in I$ and so by Theorem 3.1(i), I is a prime ideal of X.
(ii) Let J be a prime ideal of X. By Lemma 2.4(ii), J/I is an ideal of X/I. Let A and B be two ideals of X/I such that, $A \cap B \subseteq J/I$. By Lemma 2.4(iii), there are two ideals E and F of X, such that $A = E/I$ and $B = F/I$. Then $(E \cap F)/I = E/I \cap F/I = A \cap B \subseteq J/I$. Therefore, $E \cap F \subseteq J$ and so $E \subseteq J$ or $F \subseteq J$. Hence $E/I \subseteq J/I$ or $F/I \subseteq J/I$. Thus J/I is a prime ideal of X/I.
(iii) Since $ker(f)$ is a closed ideal of X, then by Theorem 2.3 and (ii), $X/ker(f) \cong Y$ and $I/ker(f)$ is a prime ideal of $X/kerf$. Moreover, $f(I) \cong I/ker(f)$. Hence $f(I)$ is a prime ideal of Y.
(iv) Let $ID(X)$ be a chain and I be a proper ideal of X. Clearly, $\langle a\rangle \cap \langle b\rangle \subseteq I$ implies $a \in I$ or $b \in I$. Hence, I is a prime ideal of X. Conversely, let any proper ideal of X

be prime. Let I and J be two proper ideals of X. Since $I \cap J$ is a proper ideal of X, then $I \subseteq I \cap J$ or $J \subseteq I \cap J$ and so $I \subseteq J$ or $J \subseteq I$. Therefore, $ID(X)$ is a chain. □

Corollary 3.3. *Let $x \in X - \{0\}$, such that $x * y = x$, for all $y \in X - \{x\}$. Then there exists a prime ideal Q of X, such that $x \notin Q$.*

Proof. Let $Q = X - \{x\}$. Then $0 \in Q$. If $a * b, b \in Q$, then $a \neq x$ and so $a \in Q$. Hence Q is an ideal of X. Clearly, $X - Q$ is a finite $\cap$-structure. By Proposition 3.2(i), Q is a prime ideal of X. Therefore, there exists a prime ideal Q of X such that $x \notin Q$. □

Example 3.2. Let $X = \{0, 1, 2, a\}$. Define the binary operation "*" on X by the following table:

Table 1

*	0	1	2	a
0	0	0	0	a
1	1	0	0	a
2	2	1	0	a
a	a	a	a	0

It is easy to prove that $(X, *, 0)$ is a BCI-algebra. Since $a*y = a$, for any $y \in X - \{a\}$, then by Corollary 3.3, $Q = X - \{a\}$ is a prime ideal of X, such that $a \notin Q$.

Proposition 3.4. *Let I be an ideal of X.*
(i) If I is a prime ideal of X, then I/I_0 is a prime ideal of X/I_0.
(ii) If I is a closed prime ideal of X, then I_0 is a closed prime ideal of X/I.
(iii) If I_0 is a prime ideal of X/I and $I \subseteq B$, then I is a prime ideal of X.

Proof. (i) Since I_0 is a closed ideal of X, then by Lemma 2.4, I/I_0 is an ideal of X/I_0. Let A' and B' be two ideals of X/I_0 such that $A' \cap B' \subseteq I/I_0$. Then by Lemma 2.4(iii), there are ideals A and B of X containing I_0 such that $A' = A/I_0$ and $B' = B/I_0$ and so $(A \cap B)/I_0 = A' \cap B' \subseteq I/I_0$. Hence by Lemma 2.4(i),(ii), $A \cap B \subseteq I$ and so $A \subseteq I$ or $B \subseteq I$ and so $A' \subseteq I/I_0$ or $B' \subseteq I/I_0$. Therefore, I/I_0 is a prime ideal of X/I_0.
(ii) If I is closed, then $I = I_0$ and so $X/I = X/I_0$ and $I/I_0 = I_0$. Hence the proof of this part is straightforward consequent of (i).
(iii) Let $I \subseteq B$ and I_0 be a prime ideal of X/I and $\langle x\rangle \cap \langle y\rangle \subseteq I$, for some $x, y \in X$. If $I_u \in \langle I_x\rangle \cap \langle I_y\rangle$, then by Theorem 2.2, there exist $n, m \in \mathbb{N}$ such that $I_u * (I_x)^n = I_0$ and $I_u * (I_y)^m = I_0$ and so by definition of $*$ on X/I we get $I_{u*x^n} = I_u * I_{x^n} = I_0$ and $I_{u*y^m} = I_u * I_{y^m} = I_0$. It follows from $(BCI2)$ that, $u * x^n \in I$ and $u * y^m \in I$ and so $u * x^m = a, u * y^m = b$, for some $a, b \in I$. Since $I \subseteq B$, then by Theorem 2.2, we obtained $(u * a) * b \in \langle x\rangle \cap \langle y\rangle$ and so $(u * a) * b \in I$. Moreover, I is an ideal and $a, b \in I$. Hence $u, 0 * u \in I$ and so $I_u = I_0$. Thus, $\langle I_x\rangle \cap \langle I_y\rangle \subseteq I_0$. Since I_0 is a prime ideal of X/I, then we have $I_x = I_0$ or $I_y = I_0$ and so $x \in I$ or $y \in I$. Hence by Theorem 3.1(i), I is a prime ideal of X. □

By definition of prime and irreducible ideals, any prime ideal is an irreducible ideal in any BCI-algebra. But the converse is false. In next example, we will show that there exists an irreducible ideal which is not prime.

Example 3.3. (i) Let $X = \{0, a, b, c\}$. Define the binary operation "*" on X by the following table:

Table 2

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X, *, 0)$ is a BCI-algebra (see [11]) and $\{\{0\}, \{0, a\}, \{0, b\}, \{0, c\}\}$ is the set of all proper ideals of X. Clearly, $\{0, a\}, \{0, b\}$ and $\{0, c\}$ are irreducible ideals of X. We have $\{0, a\} \cap \{0, b\} \subseteq \{0, c\}$. Hence $\{0, c\}$ is not a prime ideal of X. By the similar way, $\{0\}$, $\{0, a\}$ and $\{0, b\}$ are not prime ideals of X. Therefore, X has not any prime ideal.

(ii) Let $(X, *, 0)$ be the BCI-algebra in Example 3.1. Then $I = \{0, a\}$ is an irreducible ideal of X. Now, we have $b, c \in X - I$ and $\langle b\rangle \cap \langle c\rangle = \{0, b\} \cap \{0, c\} = \{0\}$ and so $(\langle b\rangle \cap \langle c\rangle) \cap (X - I) = \emptyset$. Therefore, $X - I$ is not a finite-$\cap$ structure.

(iii) Let $X = \{0, 1, a, b, c\}$. Define the binary operation "*" on X by the following table:

Table 3

*	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Then $(X, *, 0)$ is a BCI-algebra and $\{\{0\}, \{0, 1, a\}, \{0, 1, b\}, \{0, 1, c\}\}$ is the set of all proper ideals of X and $\{0, 1, b\} \cap \{0, 1, c\} \subseteq \{0, 1, a\}$ and so $I = \{0, 1, a\}$ is not a prime ideal of X. But, $\{\{I_0\}, \{I_0, I_c\}\}$ is the set of all ideals of X/I. Hence I_0 is a prime ideal of X/I. Therefore, the converse of Proposition 3.4(iii), is not true in general.

Theorem 3.5. *Let A be an ideal of X such that $A \subseteq B$. Then $I \cap J \subseteq A$ if and only if $\langle A \cup I\rangle \cap \langle A \cup J\rangle = A$, for any ideals I and J of X.*

Proof. Let $\langle A \cup I\rangle \cap \langle A \cup J\rangle = A$. Since $I \cap J \subseteq (\langle A \cup I\rangle \cap \langle A \cup J\rangle)$, we obtain $I \cap J \subseteq A$. Conversely, assume that $I \cap J \subseteq A$. Clearly, $A \subseteq \langle A \cup I\rangle \cap \langle A \cup J\rangle$. Let $u \in \langle A \cup I\rangle \cap \langle A \cup J\rangle$. Since A is an ideal of X, then by Theorem 2.2, we get $((...(u * x_1)*...) * x_n) \in A$, for some $n \in \mathbb{N}$ and $x_1, ..., x_n \in I$. It follows that, there exists $m_1 \in A$ such that $((...(u * x_1)*...) * x_n) = m_1$. By the similar way, we have $((...(u * y_1)*...) * y_m) = m_2$, for some $m \in \mathbb{N}$, y_1,..., $y_m \in J$ and $m_2 \in A$. Hence by ($BCI4$), and ($BCI5$), we get

$$(((...(u * m_1) * ...) * x_n)) * x_1 = (((...(u * x_1) * ...) * x_n)) * m_1 = 0.$$

Since I is an ideal of X and $x_1, ..., x_n \in I$, then $u * m_1 \in I$. By the similar way, we can show that $u * m_2 \in J$. Since $m_1, m_2 \in B$, we conclude that $(u * m_1) * m_2 \leq u * m_1$ and $(u * m_1) * m_2 \leq u * m_2$, and so $(u * m_1) * m_2 \in I \cap J \subseteq A$. Hence, $m \in A$ and so $\langle A \cup I\rangle \cap \langle A \cup J\rangle \subseteq A$. Therefore, $\langle A \cup I\rangle \cap \langle A \cup J\rangle = A$. $\square$

Example 3.4. Let $(X, *, 0)$ be the BCI-algebra in Example 3.3(i). Then $I = \{0, a\}, J = \{0, b\}$ and $K = \{0, c\}$ are three ideals of X and $J \cap K \subseteq I$, but $\langle I \cup J\rangle = X = \langle I \cup K\rangle$. Hence, if A is not contained in B then Theorem 3.5, may not true, in general.

Remark 3.6. We know that, if M is a maximal ideal of lower BCK-semilattice X, then M is a prime ideal [see [10], Corollary 4]. In Theorem 3.7, we will show that, any maximal ideal is a prime ideal in any BCK-algebra.

Theorem 3.7. *If M is a maximal ideal of BCK-algebra X, then M is a prime ideal of X.*

Proof. Let $\langle x\rangle\cap\langle y\rangle \subseteq M$, for some $x, y \in X$. If $x \notin M$ and $y \notin M$, then $\langle M\cup\{x\}\rangle = X$ and $\langle M \cup \{y\}\rangle = X$ and so $\langle M \cup \{x\}\rangle \cap \langle M \cup \{y\}\rangle = X$. Now, by Theorem 3.5, $\langle x\rangle \cap \langle y\rangle \nsubseteq M$, which is impossible. Hence by Theorem 3.1(i), M is a prime ideal of X. □

Example 3.5. Let X be the BCI-algebra in Example 3.3(i). Clearly, $M = \{0, b\}$ is a maximal ideal of X. Since $\{0, a\} \cap \{0, c\} = \{0\} \subseteq M$, $\{0, a\} \nsubseteq M$ and $\{0, c\} \nsubseteq M$, then M is not a prime ideal of X. Hence Theorem 3.7, may not true in general.

It has been known, if X is a lower BCK-semilattice and A is an ideal of X such that $A \cap F = \emptyset$, where F is $\wedge-$closed subset of X. Then there is a prime ideal Q of X such that $A \subseteq Q$ and $Q \cap F = \emptyset$ [see [11], Proposition 1.4.19]. We generalize this theorem for BCK-algebra.

Theorem 3.8. *Let X be a BCK-algebra and F be a nonempty subset of X such that F is closed under "o", where $xoy := x * (x * y)$, for any $x, y \in F$. If A is an ideal of X such that $A \cap F = \emptyset$, then there exist a prime ideal Q of X such that $A \subseteq Q$ and $Q \cap F = \emptyset$.*

Proof. Let $S = \{I | I \lhd X, A \subseteq I \text{ and } F \cap I = \emptyset\}$. Then S with respect to the inclusion relation "$\subseteq$" formes a poset. Clearly, every chain on S has an upper bound (union of its elements). Hence Zorn's Lemma implies that, S has a maximal element, say Q. Obviously, Q is an ideal of X such that $P \cap A = \emptyset$. We claim that Q is a prime ideal, otherwise there are ideals I, J of X, such that $I \cap J \subseteq Q$, $I \nsubseteq Q$ and $J \nsubseteq Q$. By maximality of Q we have $\langle Q\cup I\rangle\cap F \neq \emptyset$ and $\langle Q\cup J\rangle\cap F \neq \emptyset$. Let $a \in \langle Q\cup I\rangle\cap F$ and $b \in \langle Q\cup J\rangle\cap F$. Since $(aob)*a = 0$ and $(aob)*b = 0$, we have $aob \in (\langle Q\cup I\rangle\cap\langle Q\cup J\rangle)$. On the other hand, $a, b \in F$ and F is $o-$closed and so $aob \in F$. Hence

$$aob \in (\langle Q \cup I\rangle \cap \langle Q \cup J\rangle) \cap F.$$

Comparison of last relation with $Q \cap F = \emptyset$ gives $Q \neq \langle Q \cup I\rangle \cap \langle Q \cup J\rangle$. Hence Theorem 3.5, implies $I \cap J \nsubseteq Q$. Therefore, Q is a prime ideal. □

Corollary 3.9. *Let X be a BCK-algebra. Then the following assertions hold:*

(i) For any $x \in X\backslash\{0\}$, there exists a prime ideal Q such that $x \notin P$.

(ii) $\cap\{Q \mid Q$ is a prime ideal of $X\} = \{0\}$.

(iii) Any proper ideal A of X can be expressed as the intersection of all prime ideals of X containing A.

(iv) Let Y be a BCI-algebra and $f : X \to Y$ be a BCI-homomorphism, such that $f(X)$ is an ideal of Y. If I is a prime ideal of Y and $f^{-1}(I) \neq X$, then $f^{-1}(I)$ is a prime ideal of X.

Proof. (i) Let $x \in X\backslash\{0\}$. Then we set $A = \{0\}$ and $F = \{x\}$. Clearly, F is $o-$closed and $A \cap F = \emptyset$. Hence by Theorem 3.8, there exists a prime ideal Q such that Q is not contain x.

(ii) The proof is straightforward.

(iii) Let $a \in (X - A)$ and $F = \{a\}$. Then by $(BCI2)$, $x * (x * y) \in F$, for all $x, y \in F$. By Theorem 3.8, there exists a prime ideal Q_a of X such that $a \notin Q_a$ and $A \subseteq Q_a$.

Therefore, $A \subseteq \bigcap_{a\in X-A} Q_a$. On the other hand $b \notin \bigcap_{a\in X-A} Q_a$, for any $b \in X - A$. Hence $\bigcap_{a\in X-A} Q_a \subseteq A$ and so $A = \bigcap_{a\in X-A} Q_a$.
(iv) Let $\langle x\rangle \cap \langle y\rangle \subseteq f^{-1}(I)$, for some $x, y \in X$. If $\langle f(x)\rangle \cap \langle f(y)\rangle = 0$, then $\langle f(x)\rangle \cap \langle f(y)\rangle \subseteq I$. Let $u \in \langle f(x)\rangle \cap \langle f(y)\rangle - \{0\}$. Then there exist $m, n \in \mathbb{N}$ such that $u * f(x)^n = 0$ and $u * f(y)^m = 0$. Since $f(X)$ is an ideal of Y and $\langle f(x)\rangle \subseteq f(X)$, $\langle f(y)\rangle \subseteq f(X)$, then $u = f(a)$, for some $a \in X$. Moreover, f is a BCI-homomorphism and so $f(a * x^n) = 0 = f(a * y^m)$. Hence, $a * x^n \in f^{-1}(I)$ and $a * y^m \in f^{-1}(I)$ and so $a \in \langle f^{-1}(I) \cup \{x\}\rangle \cap \langle f^{-1}(I) \cup \{y\}\rangle$. Since $\langle x\rangle \cap \langle y\rangle \subseteq f^{-1}(I)$, then by Theorem 3.5, $a \in f^{-1}(I)$ and so $u = f(a) \in I$. Hence $\langle f(x)\rangle \cap \langle f(y)\rangle \subseteq I$. Now, since I is a prime ideal of Y we have $f(x) \in I$ or $f(y) \in I$ and so $x \in f^{-1}(I)$ or $y \in f^{-1}(I)$. Therefore, by Theorem 3.1(i), $f^{-1}(I)$ is a prime ideal of X. □

Corollary 3.10. *Let A be an ideal of X generated by P. If I is a proper ideal of X containing P, then*

$$I = \cap\{\cup\{A_x \ \ |A_x \in J\} \ \ |J \text{ is a prime ideal of } X/A\ \}.$$

Proof. Clearly, X/A is a BCK-algebra. By Corollary 3.9(iii), we have

$$I/A = \cap\{J \ \ |\text{J is a prime ideal of } X/A\}.$$

Let J be a prime ideal of X/A. Since $A = \langle P\rangle = \langle P \cup P\rangle = P + P$, then A is a closed ideal of X and so by Lemma 2.4(iii), $J = F_J/A$, where $F_J = \cup\{A_x \ \ |A_x \in J\}$. Therefore,

$$I/A = \cap\{F_J/A \ \ |\text{J is a prime ideal of } X/A\} = (\cap\{F_J \ \ |\text{J is a prime ideal of } X/A\})/A$$

Now, by Lemma 2.4(ii), we conclude that $I = \cap\{F_J \ \ |\text{J is a prime ideal of } X/A\}$. □

Let X be a lower BCK-semilattice and I be an ideal of X. If X/I is a BCK-chain, then I is a prime ideal of X. In next theorem, we generalize this theorem. Note that, if X has not any prime ideal we say the intersection of all prime ideals of X is X.

Theorem 3.11. *Let X be a BCI-algebra and I be a prime ideal of X.*
(i) If $I \subseteq B$ and $ID(X/I)$ is a chain, then I is a prime ideal of X.
(ii) Let M_1,..., M_n and M be maximal ideals of X such that $\bigcap_{i=1}^{n} M_i \subseteq M$. Then there exists $j \in \{1, 2, ..., n\}$, such that $M_j = M$.
(iii) Let X be a non zero nilpotent BCI-algebra and $S = \{P_\alpha \ \ |\alpha \in J\}$ be the set of all prime ideals of X. Then $\bigcap_{\alpha\in J} P_\alpha = \{0\}$ if and only if X is subdirect product of special family $\{X_i\}_{i\in I}$, such that X_i is a finite $\cap$-structure, for any $i \in I$.

Proof. (i) Let $x, y \in X$ such that $\langle x\rangle \cap \langle y\rangle \subseteq I$. Since $ID(X/I)$ is a chain, then $\langle I_x\rangle \subseteq \langle I_y\rangle$ or $\langle I_y\rangle \subseteq \langle I_x\rangle$. Let $\langle I_x\rangle \subseteq \langle I_y\rangle$. Then by Theorem 2.2, there exists $n \in \mathbb{N}$ such that $I_{x*y^n} = I_x * I_{y^n} = I_x * (I_y)^n = I_0$ and so $x * y^n \in I$. Since $I \subseteq B$, then by Theorem 2.2, we have $x * (x * y^n) \in \langle x\rangle \cap \langle y\rangle$ and so $x \in I$. By the similar way, we get $y \in I$, when $I_y \subseteq I_x$. Therefore, I is a prime ideal of X.
(ii) By Theorem 3.7, M is a prime ideal of X. Hence there exists $j \in \{1, ..., n\}$ such that $M_j \subseteq M$. Since M_j is a maximal ideal of X we obtain that $M_j = M$.
(iii) Clearly, the map $\varphi : X \to \prod_{\alpha\in J} X/P_\alpha$ defined by $\varphi(x) = ((P_\alpha)_x)_{\alpha\in J}$, for all $x \in X$, is a homomorphism and $ker(\varphi) = \bigcap_{\alpha\in J} P_\alpha = \{0\}$. Thus φ is a one to one

homomorphism and so it is a subdirect embedding. Now, let $\alpha \in J$. Since X is nilpotent, then I is closed and so by Proposition 3.4(ii) $(P_\alpha)_0$ is a prime ideal of X/P. Hence by Proposition 3.2(i), $X/P_\alpha - \{P_0\}$ is a finite $\cap$-structure and so by definition X/P_α is a finite $\cap$-structure. Conversely, let X be subdirect product of family $\{X_i\}_{i\in I}$, such that X_i is a finite $\cap$-structure, for any $i \in I$. Then there is an one to one BCI-homomorphism $\varphi : X \to \prod_{j\in J} X_j$ such that $(\pi_i o\varphi) : X \to X_i$ is an onto BCI-homomorphism and so $X/B_i \cong X_i$, for any $i \in J$, where $B_i = (\pi_i o\varphi)^{-1}(\{0\})$. Let $i \in J$. Since X_i is a finite $\cap$-structure, then X/B_i is finite $\cap$-structure and so by Proposition 3.2(i), B_i is a prime ideal of X. Clearly, $\bigcap_{j\in J} B_j = ker(\varphi) = \{0\}$. Therefore, the intersection of all prime ideals of X is $\{0\}$. □

Corollary 3.12. *Every non zero BCK-algebra is subdirect product of a family of finite $\cap$-structure BCI-algebras.*

Proof. It is straight consequent of Corollary 3.9(ii) and Theorem 3.11(iii). □

Example 3.6. Let $X = \{0, 1, 2, a, b\}$. Define the binary operation "*" on X by the following table:

Table 4

*	0	1	2	a	b
0	0	0	0	b	a
1	1	0	1	b	a
2	2	2	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

Then $(X, *, 0)$ is a BCI-algebra (see [11] Appendix B Example 8). Let $I = \{0, 1\}$. Then $I \subseteq B$ and $\{\{I_0\}, \{I_0, I_2\}, X/I\}$ is the set of all ideals of X/I. Therefore, the set of ideals of X/I is a chain. By Theorem 3.11(i), we conclude that I is a prime ideal of X.

Note 3.13. [11] Let X be a P-semisimple BCI-algebra. Then $(X, ., 0)$ is an Abelian group, where $x.y = x * (0 * y)$, for all $x, y \in X$. Moreover, any closed ideal of X is a subgroup of $(X, ., 0)$.

Theorem 3.14. *Let X be an associative BCI-algebras and I be an ideal of X.*

(i) If there exist distinct elements x, y of X such that $x, y \notin I$. Then I is not a prime ideal.

(ii) If X is of order $n > 2$, then there is not any prime ideal on X.

Proof. (i) Since X is an associative BCI-algebra, we have $\langle x\rangle = \{x, 0\}$ and $\langle y\rangle = \{0, y\}$ and so $\langle x\rangle \cap \langle y\rangle = 0$. Therefore, I is not a prime ideal of X.
(ii) Let I be a proper ideal of X. Since X is finite, then I is a closed ideal. Hence by Note 3.13, I is a subgroup of $(X, ., 0)$ and so there exists $t \in \mathbb{N} - \{1\}$ such that $n = t|I|$, where $|I|$ is the number of elements of I. Hence $|I| \leq n - 2$. Now, by (i), I is not a prime ideal of X and so X has not any prime ideals. □

Theorem 3.15. *Let M be a maximal ideal of X containing P. If $I = \langle P\rangle$, then M/I is a prime ideal of X/I.*

Proof. Since, $I = \langle P \rangle = \langle P \cup P \rangle = P + P$, then I is a closed ideal of X. Since $P \subseteq M$, we have $I \subseteq M$ and so M/I is a maximal ideal of X/I. Also X/I is a BCK-algebra. Hence by Theorem 3.7, M/I is a prime ideal of X/I. $\square$

Example 3.7. Let $X = \{0, 1, a, b\}$. Define the operation "*" on X by

Table 5

*	0	1	a	b
0	0	0	a	a
1	1	0	b	a
a	a	a	0	0
b	b	a	1	0

Clearly, $(X, *, 0)$ is a BCI-algebra (see [11] Appendix B Example 4) and {{0}, {0,a}, {0,1}} is the set of all proper ideals of X. Hence $M = \{0, a\}$ is a maximal ideal of X. It is obvious that $P = \{0, a\}$ is the P-semisimple part of X. By Theorem 3.15, M/I is a prime ideal of X/I, where $I = \langle P \rangle$.

Lemma 3.16. *Let X be a nilpotent BCI-algebra. Then for any $b \in B \backslash \{0\}$, there exists a prime ideal Q such that $b \notin Q$.*

Proof. Let $b \in B \backslash \{0\}$. By Corollary 3.9(i), there exists a prime ideal I of B such that $b \notin I$. Let P be P-semisimple part of X. We claim that $b \notin I + P$. Otherwise, $b \in I + P$. Then by Theorem 2.2, there exist $a_1, ..., a_n \in I$ such that $(...(b * a_1) * ...) * a_n \in P$. Since B is a closed ideal of X we have $(...(b * a_1) * ...) * a_n \in B$. Hence $(...(b * a_1) * ...) * a_n \in B \cap P = \{0\}$. Therefore,

$$(...(b * a_1) * ...) * a_n = 0 \in I.$$

Since I is an ideal of X containing $a_1, ..., a_n$, we conclude that $b \in I$, which is a contradiction. Hence $b \notin I + P$. It remains to show that $I + P$ is a prime ideal of X. Let J and K be two ideals of X such that $J \cap K \subseteq I + P$. Then

$$(J \cap B) \cap (K \cap B) = (J \cap K) \cap B \subseteq (I + P) \cap B. \tag{3.1}$$

Now, we show that $(I + P) \cap B = I$. Clearly, $I \subseteq (I + P) \cap B$. Let $x \in (I + P) \cap B$. Then there exist $a_1, ..., a_n \in I$ such that $(...(x*a_1)*...)*a_n \in P$. Since $x, a_1, ..., a_n \in B$ we have

$$(...(x * a_1) * ...) * a_n \in P \cap B = \{0\}. \tag{3.2}$$

Moreover, since $a_1, ..., a_n \in I$ we obtain $x \in I$. Hence $(I + P) \cap B \subseteq I$. Therefore, $(I + P) \cap B = I$. By (3.1) and (3.2) we have $(J \cap B) \cap (K \cap B) \subseteq I$. Since I is a prime ideal of B we have $J \cap B \subseteq I$ or $K \cap B \subseteq I$. Assume that $J \cap B \subseteq I$. Since $x * (0 * (0 * x)) \in B$ and X is nilpotent, then by Theorem 2.1, $x * (0 * (0 * x)) \in B \cap J$ and $0 * (0 * x) \in P$, for all $x \in J$. Since $J \cap B \subseteq I$, we have $x \in I + P$. Therefore,

$$J \subseteq I + P, \tag{3.3}$$

If $K \cap B \subseteq I$, then by the similar way, we obtain

$$K \subseteq I + P. \tag{3.4}$$

Putting (3.3) and (3.4) together, we obtain that $J \subseteq I + P$ or $K \subseteq I + P$. Hence $I + P$ is a prime ideal of X. $\square$

Corollary 3.17. *If X is a nilpotent BCI-algebra such that $B \neq \{0\}$, then X has a prime ideal.*

Theorem 3.18. *Let X be a nilpotent BCI-algebra.*
(i) For any $x \in X - P$, there exists a prime ideal Q of X, such that $x \notin Q$.
(ii) $\cap\{Q \ |Q$ is a prime ideal of $X\} \subseteq P$.

Proof. (i) Let $x \in X - P$. Then by $(BCI6)$ and $(BCI7)$, we conclude that $x * (0 * (0 * x)) \in B - \{0\}$. Hence by Lemma 3.16, there is a prime ideal Q of X such that $x * (0 * (0 * x)) \notin Q$. Therefore, $x \notin Q$. Since if $x \in Q$, then by $(BCI4)$, $(BCI5)$ and $(BCI6)$, we get $(x * (0 * (0 * x))) * x = 0 * x \in Q$ (since Q is closed) and so $x * (0 * (0 * x)) \in Q$, which is impossible.
(ii) It is straight consequent of (i). □

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