Prime Ideals in $BCI$ and $BCK$-Algebras

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Abstract. In this paper, we introduce a new definition of prime ideal in $BCI$-algebras and show that it is equivalent to the last definition of prime ideal in lower $BCK$-semilattice. Then we attempt to generalize some useful theorems about prime ideals, in $BCI$-algebras, instead of lower $BCK$-semilattices.

2010 Mathematics Subject Classification. 06F35; 03G25.

Key words and phrases. $BCI$-algebra, $BCK$-algebra, ideal, prime ideal, maximal ideal.

1. Introduction

The notions of $BCK$ and $BCI$-algebras were introduced by Imai and Iseki [3, 4] in 1966. They are two important classes of logical algebras. Most of the algebras related to the t-norm based logic, such as $MTL$-algebras, $BL$-algebras and residuated lattices are extensions of $BCK$-algebras. These algebras have been extensively studied since their introduction. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. The concept of ideal in these algebra follows from the concepts of deductive system and ideal in logical algebras such as $BL$-algebras and residuated lattices.

Iseki [5], introduced the concept of prime ideal in commutative $BCK$-algebras and Palasinski [10], generalized this definition for any lower $BCK$-semilattices. Then many authors have studied the properties of this ideal in lower $BCK$-semilattices (see [1, 2, 5, 9, 10]). They showed that this ideal is one of the most important ideals in lower $BCK$-semilattices. Any ideal $F$ of a lower $BCK$-semilattices contained in a prime ideal, has prime and minimal prime decomposition. But prime ideal and irreducible ideal are the same in lower $BCK$-semilattice. In this paper, we generalize the concept of prime ideals for $BCI$-algebras and attempt to generalize the properties of prime ideals in $BCI$-algebras. We show that prime ideals are irreducible in any $BCI$-algebras, but the converse may not true in general. Then we verify some useful properties of this ideals in $BCI$ and $BCK$-algebra such as relation between prime ideals and maximal ideals.

2. Preliminaries

Definition 2.1. [3, 4] A $BCI$-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions: for all $x, y, z \in X$

$(BC11) \ ((x * y) * (x * z)) * (z * y) = 0$
$(BC12) x * 0 = x$
$(BC13) x * y = 0$ and $y * x = 0$ imply $y = x$
Let $X$ be a $BCI$-algebra and $x \ast y^n = \cdots ((x \ast y) \ast y) \ast \cdots \ast y$, where $y$ occurs $n$ times and $x, y \in X$. Then for all $x, y, z \in X$ and $k \in \mathbb{N}$, the following hold: (see [11])

$(BCIA)\ x \ast x = 0$

$(BC15)\ (x \ast y) \ast z = (x \ast z) \ast y$

$(BC16)\ x \ast (x \ast (x \ast y))^k = x \ast y^k$

$(BC17)\ 0 \ast (x \ast y)^k = (0 \ast x^k) \ast (0 \ast y^k)$

$(BC18)\ 0 \ast (0 \ast x^k) = 0 \ast (0 \ast y^k)$

A nonempty subset $S$ of $BCI$-algebra $(X, \ast, 0)$ is called a subalgebra of $X$ if $x \ast y \in S$, for any $x, y \in S$.

For any $BCI$-algebra $X$, the relation $x \leq y \iff x \ast y = 0$ is a partial order relation [4]. It is called $BCI$-ordering of $X$. The set $P = \{x \in X\ | 0 \ast (0 \ast x) = x\}$ is called $P$-semisimple part of $BCI$-algebra $X$ and is called a $P$-semisimple $BCI$-algebra if $P = X$ (see [8, 11]). The set $\{x \in X\ | 0 \ast x = 0\}$ is called $BCI$-part of $BCI$-algebra $X$ and is denoted by $BCI(X)$. If $X = BCI(X)$, then we say $X$ is a $BCI$-algebra. A lower $BCI$-semilattice is a $BCI$-algebra $(X, \ast, 0)$, such that it with respect to its $BCI$-ordering forms a lower semilattice. Moreover, a $BCI$-algebra $X$ is called associative if $(x \ast y) \ast z = x \ast (y \ast z)$, for any $x, y, z \in X$. In any associative $BCI$-algebra, $x \ast y = y \ast x$ and $0 \ast x = x$, for any $x, y \in X$ (see [7]).

Definition 2.2. [3, 4] Let $I$ be a nonempty subset of $BCI$-algebra $X$ containing 0. $I$ is called an ideal of $X$ if $y \ast x \in I$ and $x \in I$ imply $y \in I$, for any $x, y \in X$. Clearly, $\{0\}$ is an ideal of $X$ and we write 0 is an ideal of $X$, for convenience. An ideal $I$ is called proper, if $I \neq X$ and is called closed, if $x, y \in I$, for all $x, y \in I$. The $BCI$-part of $X$ is a closed ideal of $X$. Let $S$ be a nonempty subset of $X$. We call the least ideal of $X$ containing $S$, the generated ideal of $X$ by $S$ and is denoted by $(S)$.

If $A$ and $B$ are two subalgebras of $X$, then we usually denote $A \cup B$ for $(A \cup B)$.

Moreover, $A + B$ is a closed ideal of $X$ (see [11], Proposition 1.4.15). If $X$ is a $BCI$-algebra, then $BCI$-part of $X$ is a closed ideal of $X$ and $P$-semisimple part of $X$ is a subalgebra of $X$. If $X$ is a lower $BCI$-semilattice, then for any $x, y \in X$, we have

$(P1)\ (x) \cap (y) = (x \land y)$ (see [11], Proposition 1.4.16).

Let $A$ be an ideal of a $BCI$-algebra $X$. Then the relation $\theta$ defined by $(x, y) \in \theta \iff x \ast y = 0 \ast x \in A$ is a congruence relation on $X$. We usually denote $A_x$ for $[x] = \{y \in X\ | (x, y) \in \theta\}$. Moreover, $A_0$ is a closed ideal of $BCI$-algebra $X$. In fact, it is the greatest closed ideal contained in $A$. Assume that $X/A = \{A_x\ | x \in X\}$. Then $(X/A, \ast, A_0)$ is a $BCI$-algebra, where $A_x \ast A_y = A_{x \ast y}$, for all $x, y \in X$.

Let $X$ and $Y$ be two $BCI$-algebras. A map $f : X \to Y$ is called a $BCI$-homomorphism, if $f(x \ast y) = f(x) \ast f(y)$, for all $x, y \in X$. If $f : X \to Y$ is a $BCI$-homomorphism, then the set $\ker(f) = f^{-1}(0)$ is a closed ideal of $X$. A homomorphism is one to one if and only if $\ker(f) = \{0\}$ (see [11]). The homomorphism $f$ is called an epimorphism if $f$ is onto. Moreover, an isomorphism is a homomorphism, which is both one to one and onto. Note that, if $f : X \to Y$ is a $BCI$-homomorphism, then $f(0) = 0$. An element $x$ of $BCI$-algebra $X$ is called nilpotent if $0 \ast x^n = 0$, for some $n \in \mathbb{N}$. A $BCI$-algebra is called nilpotent if any element of $X$ is nilpotent (see [6]).

Theorem 2.1. [11] $BCI$-algebra $X$ is nilpotent if and only if every ideal of $X$ is closed.

Theorem 2.2. [11] Let $S$ be a nonempty subset of a $BCI$-algebra $X$ and

$A = \{x \in X\ | ((x \ast a_1) \ast a_2) \ast \cdots \ast a_n = 0, \text{ for some } n \in \mathbb{N} \text{ and some } a_1, \ldots, a_n \in S\}.$
Then \( \langle S \rangle = A \cup \{0\} \). Especially, if \( S \) contains a nilpotent element of \( X \), then \( \langle S \rangle = A \). Moreover, if \( I \) is an ideal of \( X \), then
\[
(A \cup S) = \{x \in X \mid (\cdots((x*a_1)*a_2)*\cdots)*a_n \in A, \text{ for some } n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in S\}.
\]

**Definition 2.3.** [10] A proper ideal \( I \) of BCI-algebra \( X \) is called an irreducible ideal if \( A \cap B = I \) implies \( A = I \) or \( B = I \), for any ideals \( A \) and \( B \) of \( X \).

**Definition 2.4.** [10] Let \( X \) be a BCI-algebra. A proper ideal \( M \) of \( X \) is called a maximal ideal if \( \langle M \cup \{x\} \rangle = X \) for any \( x \in X \setminus M \), where \( \langle M \cup \{x\} \rangle \) is an ideal generated by \( M \cup \{x\} \). Note that, \( M \) is a maximal ideal of \( X \) if and only if \( M \subseteq A \subseteq X \) implies that \( M = A \) or \( A = X \), for any ideal \( A \) of \( X \).

**Theorem 2.3.** [10] Let \( X \) and \( Y \) be two BCI-algebras and \( f : X \to Y \) be a BCI-epimorphism. If \( A = \ker(f) \), then \( \alpha : X/A \to Y \) which is defined by \( \alpha(A_x) = f(x) \) is a BCI-isomorphism.

**Lemma 2.4.** [11] Let \( I \) and \( J \) be two ideals of BCI-algebra \( X \) such that \( I \subseteq J \). Denote \( J/I = \{I_x \in X/I \mid x \in J\} \). Then
\[
(i) \quad \text{if and only if } I \subseteq J/I, \text{ for any } x \in X.
\]
\[
(ii) \quad J/I = \{I_x \in X/I \mid x \in J\} \text{ is an ideal of } X/I.
\]
\[
(iii) \quad \text{Let } I \text{ be a closed ideal of } X. \text{ If } S \text{ and } T \text{ are the sets of all ideals of } X \text{ and } X/I, \text{ respectively, then the map } g : S \to T \text{ defined by } g(J) = J/I, \text{ is a bijective map.}
\]
\[
\text{The inverse of } g \text{ is the map } f : T \to S, \text{ is defined by } f(J) = \cup\{I_x \mid I_x \in J\}.
\]

**Definition 2.5.** [5] A proper ideal \( I \) of lower BCK-semilattice \( X \) is called prime if \( x \land y \in I \) implies \( x \in I \) or \( y \in I \).

Let \( \{X_i\}_{i \in I} \) be a family of BCI-algebras. Then \( \prod_{i \in I} X_i \) is a BCI-algebra and the map \( \pi_j : \prod_{i \in I} X_i \to X_j \), defined by \( \pi_j((x_i)_{i \in I}) = x_j \) is called the \( j \)-th natural projection map.

**Definition 2.6.** [11] A BCI-algebra \( X \) is called a subdirect product of BCI-algebras family \( \{X_i\}_{i \in I} \) if there is an one to one BCI-homomorphism \( f : X \to \prod_{i \in I} X_i \) such that \( \pi_i(f(X)) = X_i \), where \( \pi_i : \prod_{i \in I} X_i \to X_i \) is the \( i \)-th natural projection map, for all \( i \in I \). Moreover, the map \( f \) is called subdirect embedding.

3. **Prime ideals in BCI and BCK-algebras**

In this section, we introduce the concept of prime ideals in BCI-algebras and we prove that this concept and the last definition of prime ideal in a lower BCK-semilattice are equivalent. Then we generalize some useful theorems about the prime ideals on BCI and BCK-algebras. Finally, we discuss some relations between BCK-part and prime ideals in BCI and BCK-algebras.

Throughout this section, \( X \) is a BCI-algebra, \( B \) is BCK-part of \( X \) and \( P \) is \( P \)-semisimple part of \( X \), unless otherwise stated.

**Definition 3.1.** A proper ideal \( I \) of BCI-algebra \( X \) is called prime if \( A \cap B \subseteq I \) implies \( A \subseteq I \) or \( B \subseteq I \), for all ideals \( A \) and \( B \) of \( X \).
Example 3.1. Let "-" be the subtraction of integers. Then \( X = (\mathbb{Z}, -) \) is a BCI-algebra. Clearly, \( M_1 = \mathbb{N} \cup \{0\} \) and \( M_2 = \{-n \mid n \in \mathbb{N}\} \cup \{0\} \) are two maximal ideals of \( X \) (see [11], Example 5.3.2). Let \( I \cap J \subseteq \mathbb{N} \). If \( I \not\subseteq \mathbb{N} \) and \( J \not\subseteq \mathbb{N} \) then there exist \( m,n \in \mathbb{N} \) such that \( -n \in I \) and \( -m \in J \). By Theorem 2.3, we conclude that \(-mn \in I \cap J \subseteq \mathbb{N} \cup \{0\} \), which is impossible. Hence \( \mathbb{N} \cup \{0\} \) is a prime ideal of \( X \).

By the similar way, \( M_2 \) is a prime ideal of \( X \).

Theorem 3.1. (i) Let \( I \neq \emptyset \) be an ideal of \( X \). Then \( I \) is a prime ideal of \( X \) if and only if \( \langle x \rangle \cap \langle y \rangle \subseteq I \) implies \( x \in I \) or \( y \in I \), for any \( x, y \in X \).

(ii) If \( X \) is a lower BCK-semilattice, then Definition 3.1 and Definition 2.5 are equivalent.

Proof. (i) Let \( I \) be an ideal of \( X \), such that \( \langle x \rangle \cap \langle y \rangle \subseteq I \) implies \( x \in I \) or \( y \in I \). If \( A \) and \( B \) are two ideals of \( X \), such that \( A \cap B \subseteq I \), then there is no harm in assuming \( A \nsubseteq I \). Hence there exists \( a \in A \) such that \( a \notin I \). For any \( b \in B \), since \( \langle a \rangle \cap \langle b \rangle \subseteq A \cap B \subseteq I \) and \( a \notin I \), the primeness of \( I \) implies \( b \in I \). Therefore, \( B \subseteq I \). Conversely, let \( I \) be a prime ideal of \( X \). Clearly, \( \langle x \rangle \cap \langle y \rangle \subseteq I \) implies \( x \in I \) or \( y \in I \), for any \( x, y \in X \).

(ii) Since by (P1), \( \langle x \rangle \cap \langle y \rangle = \langle xy \rangle \), for any \( x, y \in X \) so Definition 3.1 and Definition 2.5 are equivalent.

Clearly, any prime ideal of \( X \) is an irreducible ideal. Moreover, if \( \{0\} \) is an irreducible ideal of \( X \), then \( \{0\} \) is a prime ideal.

Definition 3.2. A nonempty subset \( F \) of \( X \) is called a finite \(-\)-structure, if \( \langle x \rangle \cap \langle y \rangle \neq \emptyset \), for all \( x, y \in F \), and \( X \) is called a finite \(-\)-structure if \( X \setminus \{0\} \) is a finite \(-\)-structure.

Proposition 3.2. Let \( Y \) be a BCI-algebra and \( f : X \to Y \) be an onto BCI-homomorphism. Then the following assertions hold:

(i) An ideal \( I \) of \( X \) is prime if and only if \( F = X - I \) is a finite \(-\)-structure.

(ii) Let \( I \) be a closed ideal of \( X \) and \( J \) be an ideal of \( X \) containing \( I \). If \( J \) is a prime ideal of \( X \), then \( J/I \) is a prime ideal of \( X/I \).

(iii) Let \( I \) be a prime ideal of \( X \) and \( \ker f \subseteq I \). Then \( f(I) \) is a prime ideal of \( Y \).

(iv) Let \( ID(X) \) be the set of all ideals of \( X \). Then \( ID(X) \) is a chain if and only if every proper ideal of \( X \) is prime.

Proof. (i) Let \( I \) be a prime ideal of \( X \) and \( x, y \in F \). If \( \langle x \rangle \cap \langle y \rangle \cap F = \emptyset \), then \( \langle x \rangle \cap \langle y \rangle \subseteq I \). Since \( I \) is a prime ideal of \( X \), we have \( x \in I \) or \( y \in I \), which is impossible. Hence \( \langle x \rangle \cap \langle y \rangle \cap F \neq \emptyset \). Conversely, let \( F \) be a finite \(-\)-structure and \( x, y \in X \) such that \( \langle x \rangle \cap \langle y \rangle \subseteq I \). If \( x \notin I \) and \( y \notin I \), then \( x, y \in F \) and so \( \langle x \rangle \cap \langle y \rangle \cap F \neq \emptyset \). Hence, \( \langle x \rangle \cap \langle y \rangle \not\subseteq I \), which is impossible. Therefore, \( x \in I \) or \( y \in I \) and so by Theorem 3.1(i), \( I \) is a prime ideal of \( X \).

(ii) Let \( J \) be a prime ideal of \( X \). By Lemma 2.4(ii), \( J/I \) is an ideal of \( X/I \). Let \( A \) and \( B \) be two ideals of \( X/I \) such that, \( A \cap B \subseteq J/I \). By Lemma 2.4(iii), there are two ideals \( E \) and \( F \) of \( X \), such that \( A = E/I \) and \( B = F/I \). Then \( (E \cap F)/I = E/I \cap F/I = A \cap B \subseteq J/I \). Therefore, \( E \cap F \subseteq J \) and so \( E \subseteq J \) or \( F \subseteq J \). Hence \( E/I \subseteq J/I \) or \( F/I \subseteq J/I \). Thus \( J/I \) is a prime ideal of \( X/I \).

(iii) Since \( \ker f \) is a closed ideal of \( X \), then by Theorem 2.3 and (ii), \( X/\ker f \equiv Y \) and \( J/\ker f \) is a prime ideal of \( X/\ker f \). Moreover, \( f(I) \cong I/\ker f \). Hence \( f(I) \) is a prime ideal of \( Y \).

(iv) Let \( ID(X) \) be a chain and \( I \) be a proper ideal of \( X \). Clearly, \( \langle a \rangle \cap \langle b \rangle \subseteq I \) implies \( a \in I \) or \( b \in I \). Hence, \( I \) is a prime ideal of \( X \). Conversely, let any proper ideal of \( X \)
be prime. Let $I$ and $J$ be two proper ideals of $X$. Since $I \cap J$ is a proper ideal of $X$, then $I \subseteq I \cap J$ or $J \subseteq I \cap J$ and so $I \subseteq J$ or $J \subseteq I$. Therefore, $ID(X)$ is a chain. □

**Corollary 3.3.** Let $x \in X - \{0\}$, such that $x * y = x$, for all $y \in X - \{x\}$. Then there exists a prime ideal $Q$ of $X$, such that $x \notin Q$.

**Proof.** Let $Q = X - \{x\}$. Then $0 \in Q$. If $a * b, b \in Q$, then $a \neq x$ and so $a \in Q$. Hence $Q$ is an ideal of $X$. Clearly, $X - Q$ is a finite $\cap$-structure. By Proposition 3.2(i), $Q$ is a prime ideal of $X$. Therefore, there exists a prime ideal $Q$ of $X$ such that $x \notin Q$. □

**Example 3.2.** Let $X = \{0, 1, 2, a\}$. Define the binary operation “$*$” on $X$ by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

It is easy to prove that $(X, *, 0)$ is a BCI-algebra. Since $a * y = a$, for any $y \in X - \{a\}$, then by Corollary 3.3, $Q = X - \{a\}$ is a prime ideal of $X$, such that $a \notin Q$.

**Proposition 3.4.** Let $I$ be an ideal of $X$.

(i) If $I$ is a prime ideal of $X$, then $I/I_0$ is a prime ideal of $X/I_0$.

(ii) If $I$ is a closed prime ideal of $X$, then $I_0$ is a closed prime ideal of $X/I$.

(iii) If $I_0$ is a prime ideal of $X/I$ and $I \subseteq B$, then $I$ is a prime ideal of $X$.

**Proof.** (i) Since $I_0$ is a closed ideal of $X$, then by Lemma 2.4, $I/I_0$ is an ideal of $X/I_0$. Let $A'$ and $B'$ be two ideals of $X/I_0$ such that $A' \cap B' \subseteq I/I_0$. Then by Lemma 2.4(iii), there are ideals $A$ and $B$ of $X$ containing $I_0$ such that $A' = A/I_0$ and $B' = B/I_0$ and so $(A \cap B)/I_0 = A' \cap B' \subseteq I/I_0$. Hence by Lemma 2.4(i),(ii), $A \cap B \subseteq I$ and so $A \subseteq I$ or $B \subseteq I$ and so $A' \subseteq I/I_0$ or $B' \subseteq I/I_0$. Therefore, $I/I_0$ is a prime ideal of $X/I_0$.

(ii) If $I$ is closed, then $I = I_0$ and so $X/I = X/I_0$ and $I/I_0 = I_0$. Hence the proof of this part is straightforward consequent of (i).

(iii) Let $I \subseteq B$ and $I_0$ be a prime ideal of $X/I$ and $(x) \cap (y) \subseteq I$, for some $x, y \in X$. If $I_u \in (I_x) \cap (I_y)$, then by Theorem 2.2, there exist $n, m \in \mathbb{N}$ such that $I_u * (I_x)_n = I_0$ and $I_u * (I_y)_m = I_0$ and so by definition of $*$ on $X/I$ we get $I_u * x^n = I_u * I_x = I_0$ and $I_u * y^m = I_u * I_y = I_0$. It follows from (BCI2) that, $u * x^n \in I$ and $u * y^m \in I$ and so $u * x^n = a, u * y^m = b$, for some $a, b \in I$. Since $I \subseteq B$, then by Theorem 2.2, we obtained $(u * a) * b \in (x) \cap (y)$ and so $(u * a) * b \in I$. Moreover, $I$ is an ideal and $a, b \in I$. Hence $u, 0 * u \in I$ and so $I_u = I_0$. Thus, $(I_u) \cap (I_y) \subseteq I_0$. Since $I_0$ is a prime ideal of $X/I$, then we have $I_x = I_0$ or $I_y = I_0$ and so $x \in I$ or $y \in I$. Hence by Theorem 3.1(i), $I$ is a prime ideal of $X$.

By definition of prime and irreducible ideals, any prime ideal is an irreducible ideal in any BCI-algebra. But the converse is false. In next example, we will show that there exists an irreducible ideal which is not prime.
Example 3.3. (i) Let $X = \{0, a, b, c\}$. Define the binary operation $*^m$ on $X$ by the following table:

$$
\begin{align*}
\begin{array}{ccc}
* & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0
\end{array}
\end{align*}
$$

Then $(X, *, 0)$ is a BCI-algebra (see [11]) and $\emptyset, \{0\}, \{0, a\}, \{0, b\}, \{0, c\}$ is the set of all proper ideals of $X$. Clearly, $\emptyset, \{0, a\}, \{0, b\}$ and $\{0, c\}$ are irreducible ideals of $X$. We have $\emptyset \cap \{0, b\} \subseteq \{0, c\}$. Hence $\{0, c\}$ is not a prime ideal of $X$. By the similar way, $\emptyset, \{0, a\}$ and $\{0, b\}$ are not prime ideals of $X$. Therefore, $X$ has not any prime ideal.

(ii) Let $(X, *, 0)$ be the BCI-algebra in Example 3.1. Then $I = \{0, a\}$ is an irreducible ideal of $X$. Now, we have $b, c \in X - I$ and $(b) \cap (c) = \{0, b\} \cap \{0, c\} = \{0\}$ and so $((b) \cap (c)) \cap (X - I) = \emptyset$. Therefore, $X - I$ is not a finite-∩ structure.

(iii) Let $X = \{0, 1, a, b, c\}$. Define the binary operation $*^m$ on $X$ by the following table:

$$
\begin{align*}
\begin{array}{ccc}
* & 0 & 1 & a & b & c \\
0 & 0 & 0 & a & b & c \\
1 & i & 0 & a & b & c \\
a & a & a & 0 & c & b \\
b & b & b & c & 0 & a \\
c & c & c & b & a & 0
\end{array}
\end{align*}
$$

Then $(X, *, 0)$ is a BCI-algebra and $\emptyset, \{0\}, \{0, 1\}, \{0, 1, b\}, \{0, 1, c\}$ is the set of all proper ideals of $X$ and $\emptyset, \{0, 1\} \cap \{0, 1, c\} \subseteq \{0, 1\}$ and so $I = \{0, 1, a\}$ is not a prime ideal of $X$. But, $\{I_0\}, \{I_0, I_1\}$ is the set of all ideals of $X/I$. Hence $I_0$ is a prime ideal of $X/I$. Therefore, the converse of Proposition 3.4(iii), is not true in general.

Theorem 3.5. Let $A$ be an ideal of $X$ such that $A \subseteq B$. Then $I \cap J \subseteq A$ if and only if $(A \cup I) \cap (A \cup J) = A$, for any ideals $I$ and $J$ of $X$.

Proof. Let $(A \cup I) \cap (A \cup J) = A$. Since $I \cap J \subseteq ((A \cup I) \cap (A \cup J))$, we obtain $I \cap J \subseteq A$. Conversely, assume that $I \cap J \subseteq A$. Clearly, $A \subseteq (A \cup I) \cap (A \cup J)$. Let $u \in (A \cup I) \cap (A \cup J)$. Since $A$ is an ideal of $X$, then by Theorem 2.2, we get $((...\langle u \times x_1 \rangle ... \times x_n)) \in A$, for some $n \in \mathbb{N}$ and $x_1, ..., x_n \in I$. It follows that, there exists $m_1 \in A$ such that $((...\langle u \times x_1 \rangle ... \times x_n) = m_1$. By the similar way, we have $((...\langle u \times y_1 \rangle ... \times y_m) = m_2$, for some $m \in \mathbb{N}$, $y_1, ..., y_m \in J$ and $m_2 \in A$. Hence by (BCI4), and (BCI5), we get

$$
((...\langle u \times m_1 \rangle ... \times x_n)) \times x_1 = ((...\langle u \times x_1 \rangle ... \times x_n) \times m_1 = 0.
$$

Since $I$ is an ideal of $X$ and $x_1, ..., x_n \in I$, then $u \times m_1 \in I$. By the similar way, we can show that $u \times m_2 \in J$. Since $m_1, m_2 \in B$, we conclude that $u \times m_1 \times m_2 \leq u \times m_1$ and $(u \times m_1) \times m_2 \leq u \times m_2$, and so $(u \times m_1) \times m_2 \in I \cap J \subseteq A$. Hence, $m \in A$ and so $(A \cup I) \cap (A \cup J) = A$. Therefore, $(A \cup I) \cap (A \cup J) = A$. □

Example 3.4. Let $(X, *, 0)$ be the BCI-algebra in Example 3.3(i). Then $I = \{0, a\}, J = \{0, b\}$ and $K = \{0, c\}$ are three ideals of $X$ and $J \cap K \subseteq I$, but $\langle I \cup J \rangle = X = \langle I \cup K \rangle$. Hence, if $A$ is not contained in $B$ then Theorem 3.5, may not true, in general.
Remark 3.6. We know that, if $M$ is a maximal ideal of lower BCK-semilattice $X$, then $M$ is a prime ideal [see [10], Corollary 4]. In Theorem 3.7, we will show that, any maximal ideal is a prime ideal in any BCK-algebra.

Theorem 3.7. If $M$ is a maximal ideal of BCK-algebra $X$, then $M$ is a prime ideal of $X$.

Proof. Let $(x) \cap (y) \subseteq M$, for some $x, y \in X$. If $x \notin M$ and $y \notin M$, then $(M \cup \{x\}) = X$ and $(M \cup \{y\}) = X$ and so $(M \cup \{x\}) \cap (M \cup \{y\}) = X$. Now, by Theorem 3.5, $(x) \cap (y) \not\subseteq M$, which is impossible. Hence by Theorem 3.1(i), $M$ is a prime ideal of $X$. \hfill $\Box$

Example 3.5. Let $X$ be the BCI-algebra in Example 3.3(i). Clearly, $M = \{0, b\}$ is a maximal ideal of $X$. Since $\{0, a\} \cap \{0, c\} = \{0\} \subseteq M$, $\{0, a\} \not\subseteq M$ and $\{0, c\} \not\subseteq M$, then $M$ is not a prime ideal of $X$. Hence Theorem 3.7, may not true in general.

It has been known, if $X$ is a lower BCK-semilattice and $A$ is an ideal of $X$ such that $A \cap F = \emptyset$, where $F$ is $\land$-closed subset of $X$. Then there is a prime ideal $Q$ of $X$ such that $A \subseteq Q$ and $Q \cap F = \emptyset$ [see [11], Proposition 1.4.19]. We generalize this theorem for BCK-algebra.

Theorem 3.8. Let $X$ be a BCK-algebra and $F$ be a nonempty subset of $X$ such that $F$ is closed under “$*$”, where $x * y := x + (x * y)$, for any $x, y \in F$. If $A$ is an ideal of $X$ such that $A \cap F = \emptyset$, then there exist a prime ideal $Q$ of $X$ such that $A \subseteq Q$ and $Q \cap F = \emptyset$.

Proof. Let $S = \{I | I \triangleleft X, A \subseteq I$ and $F \cap I = \emptyset\}$. Then $S$ with respect to the inclusion relation “$\subseteq$” forms a poset. Clearly, every chain on $S$ has an upper bound (union of its elements). Hence Zorn’s Lemma implies that, $S$ has a maximal element, say $Q$. Obviously, $Q$ is an ideal of $X$ such that $P \cap A = \emptyset$. We claim that $Q$ is a prime ideal, otherwise there are ideals $I, J$ of $X$, such that $I \cap J \subseteq Q$, $I \not\subseteq Q$ and $J \not\subseteq Q$. By maximality of $Q$ we have $(Q \cup I) \cap F \neq \emptyset$ and $(Q \cup J) \cap F \neq \emptyset$. Let $a \in (Q \cup I) \cap F$ and $b \in (Q \cup J) \cap F$. Since $(aob) * a = 0$ and $(aob) * b = 0$, we have $aob \in ((Q \cup I) \cap (Q \cup J))$. On the other hand, $a, b \in F$ and $F$ is $o$-closed and so $aob \in F$. Hence $aob \in ((Q \cup I) \cap (Q \cup J)) \cap F$.

Comparison of last relation with $Q \cap F = \emptyset$ gives $Q \neq (Q \cup I) \cap (Q \cup J)$. Hence Theorem 3.5, implies $I \cap J \not\subseteq Q$. Therefore, $Q$ is a prime ideal. \hfill $\Box$

Corollary 3.9. Let $X$ be a BCK-algebra. Then the following assertions hold:

(i) For any $x \in X \setminus \{0\}$, there exists a prime ideal $Q$ of $X$ such that $x \notin P$.

(ii) $\cap \{Q \mid Q \text{ is a prime ideal of } X \} = \{0\}$.

(iii) Any proper ideal $A$ of $X$ can be expressed as the intersection of all prime ideals of $X$ containing $A$.

(iv) Let $Y$ be a BCI-algebra and $f : X \rightarrow Y$ be a BCI-homomorphism, such that $f(X)$ is an ideal of $Y$. If $I$ is a prime ideal of $Y$ and $f^{-1}(I) \neq X$, then $f^{-1}(I)$ is a prime ideal of $X$.

Proof. (i) Let $x \in X \setminus \{0\}$. Then we set $A = \{0\}$ and $F = \{x\}$. Clearly, $F$ is $o$-closed and $A \cap F = \emptyset$. Hence by Theorem 3.8, there exists a prime ideal $Q$ such that $Q$ does not contain $x$.

(ii) The proof is straightforward.

(iii) Let $a \in (X - A)$ and $F = \{a\}$. Then by $(BCI2)$, $x * (x * y) \in F$, for all $x, y \in F$. By Theorem 3.8, there exists a prime ideal $Q$ of $X$ such that $a \notin Q$ and $A \subseteq Q$. 


Therefore, \( A \subseteq \bigcap_{a \in X - A} Q_a \). On the other hand \( b \not\in \bigcap_{a \in X - A} Q_a \), for any \( b \in X - A \).

Hence \( \bigcap_{a \in X - A} Q_a \subseteq A \) and so \( A = \bigcap_{a \in X - A} Q_a \).

(iv) Let \( (x) \cap (y) \subseteq f^{-1}(I) \), for some \( x, y \in X \). If \( \langle f(x) \rangle \cap \langle f(y) \rangle = 0 \), then \( \langle f(x) \rangle \cap \langle f(y) \rangle \subseteq I \). Let \( u \in \langle f(x) \rangle \cap \langle f(y) \rangle - \{0\} \). Then there exist \( m, n \in \mathbb{N} \) such that \( u * f(x)^n = 0 \) and \( u * f(y)^m = 0 \). Since \( f(X) \) is an ideal of \( Y \) and \( \langle f(x) \rangle \subseteq f(X) \), \( \langle f(y) \rangle \subseteq f(X) \), then \( u = f(a) \), for some \( a \in X \). Moreover, \( f \) is a BCI-homomorphism and so \( f(a * x^n) = 0 = f(a * y^m) \). Hence, \( a * x^n \in f^{-1}(I) \) and \( a * y^m \in f^{-1}(I) \) and so \( a \in \langle f^{-1}(I) \cup \{x\} \rangle \cap \langle f^{-1}(I) \cup \{y\} \rangle \). Since \( \langle x \rangle \cap \langle y \rangle \subseteq f^{-1}(I) \), then by Theorem 3.5, \( a \in f^{-1}(I) \) and so \( u = f(a) \in I \). Hence \( \langle f(x) \rangle \cap \langle f(y) \rangle \subseteq I \). Now, since \( I \) is a prime ideal of \( Y \) we have \( f(x) \in I \) or \( f(y) \in I \) and so \( x \in f^{-1}(I) \) or \( y \in f^{-1}(I) \). Therefore, by Theorem 3.1(i), \( f^{-1}(I) \) is a prime ideal of \( X \). \( \square 

Corollary 3.10. Let \( A \) be an ideal of \( X \) generated by \( P \). If \( I \) is a proper ideal of \( X \) containing \( P \), then

\[ I = \bigcap \{ \bigcup \{ A_x \mid A_x \in J \} \mid J \text{ is a prime ideal of } X/A \}. \]

Proof. Clearly, \( X/A \) is a BCK-algebra. By Corollary 3.9(iii), we have

\[ I/A = \bigcap \{ J \mid J \text{ is a prime ideal of } X/A \}. \]

Let \( J \) be a prime ideal of \( X/A \). Since \( A = (P) = (P \cup P) = P + P \), then \( A \) is an closed ideal of \( X \) and so by Lemma 2.4(iii), \( J = F_j/A \), where \( F_j = \bigcup \{ A_x \mid A_x \in J \} \).

Therefore, \( I/A = \bigcap \{ F_j/A \mid J \text{ is a prime ideal of } X/A \} = (\bigcap \{ F_j \mid J \text{ is a prime ideal of } X/A \})/A \).

Now, by Lemma 2.4(ii), we conclude that \( I = \bigcap \{ F_j \mid J \text{ is a prime ideal of } X/A \} \). \( \square 

Let \( X \) be a lower BCK-semilattice and \( I \) be an ideal of \( X \). If \( X/I \) is a BCK-chain, then \( I \) is a prime ideal of \( X \). In next theorem, we generalize this theorem. Note that, if \( X \) has not any prime ideal we say the intersection of all prime ideals of \( X \) is \( X \).

Theorem 3.11. Let \( X \) be a BCI-algebra and \( I \) be a prime ideal of \( X \).

(i) If \( I \subseteq B \) and \( ID(X/I) \) is a chain, then \( I \) is a prime ideal of \( X \).

(ii) Let \( M_1, \ldots, M_n \) and \( M \) be maximal ideals of \( X \) such that \( \bigcap_{i=1}^{n} M_i \subseteq M \). Then there exists \( j \in \{1, 2, \ldots, n\} \), such that \( M_j = M \).

(iii) Let \( X \) be a non zero nilpotent BCI-algebra and \( S = \{ P_{\alpha} \mid \alpha \in J \} \) be the set of all prime ideals of \( X \). Then \( \bigcap_{\alpha \in J} P_{\alpha} = \{0\} \) if and only if \( X \) is subdirect product of special family \( \{ X_i \}_{i \in I} \), such that \( X_i \) is a finite \( \cap \)-structure, for any \( i \in I \).

Proof. (i) Let \( x, y \in X \) such that \( (x) \cap (y) \subseteq I \). Since \( ID(X/I) \) is a chain, then \( \langle I_x \rangle \subseteq \langle I_y \rangle \) or \( \langle I_y \rangle \subseteq \langle I_x \rangle \). Let \( \langle I_z \rangle \subseteq \langle I_y \rangle \). Then by Theorem 2.2, there exists \( n \in \mathbb{N} \) such that \( I_{x \cdot y^n} = I_z \cdot I_{y^n} = I_z \cdot (I_y)^n = I_0 \) and so \( x \cdot y^n \in I \). Since \( I \subseteq B \), then by Theorem 2.2, we have \( x \cdot (x \cdot y^n) \in (x) \cap (y) \) and so \( x \in I \). By the similar way, we get \( y \in I \), when \( I_y \subseteq I_z \). Therefore, \( I \) is a prime ideal of \( X \).

(ii) By Theorem 3.7, \( M \) is a prime ideal of \( X \). Hence there exists \( j \in \{1, \ldots, n\} \) such that \( M_j \subseteq M \). Since \( M_j \) is a maximal ideal of \( X \) we obtain that \( M_j = M \).

(iii) Clearly, the map \( \varphi : X \to \prod_{\alpha \in J} X/P_{\alpha} \) defined by \( \varphi(x) = ((P_{\alpha})_x)_{\alpha \in J} \), for all \( x \in X \), is a homomorphism and \( ker(\varphi) = \bigcap_{\alpha \in J} P_{\alpha} = \{0\} \). Thus \( \varphi \) is a one to one
homomorphism and so it is a subdirect embedding. Now, let α ∈ J. Since X is nilpotent, then I is closed and so by Proposition 3.4(ii) \((P_α)_0\) is a prime ideal of \(X/P\). Hence by Proposition 3.2(i), \(X/P_α = \{P_0\}\) is a finite \(∩\)-structure and so by definition \(X/P_α\) is a finite \(∩\)-structure. Conversely, let X be subdirect product of family \(\{X_i\}_{i ∈ I}\), such that \(X_i\) is a finite \(∩\)-structure, for any \(i ∈ I\). Then there is an one to one \(BCI\)-homomorphism \(φ : X → \prod_{j ∈ J} X_j\) such that \((π_iφ) : X → X_i\) is an onto \(BCI\)-homomorphism and so \(X/B_i \cong X_i\), for any \(i ∈ J\), where \(B_i = (π_iφ)^{-1}(\{0\})\).

Let \(i ∈ J\). Since \(X_i\) is a finite \(∩\)-structure, then \(X/B_i\) is finite \(∩\)-structure and so by Proposition 3.2(i), \(B_i\) is a prime ideal of \(X\). Clearly, \(\bigcap_{j ∈ J} B_j = ker(φ) = \{0\}\).

Therefore, the intersection of all prime ideals of \(X\) is \(\{0\}\). \(\square\)

**Corollary 3.12.** Every non zero \(BCK\)-algebra is subdirect product of a family of finite \(∩\)-structure \(BCI\)-algebras.

**Proof.** It is straight consequent of Corollary 3.9(ii) and Theorem 3.11(iii). \(\square\)

**Example 3.6.** Let \(X = \{0, 1, 2, a, b\}\). Define the binary operation \(*^\ast\) on \(X\) by the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>b</td>
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</tr>
<tr>
<td>a</td>
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<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X, *, 0)\) is a \(BCI\)-algebra (see [11] Appendix B Example 8). Let \(I = \{0, 1\}\). Then \(I ⊆ B\) and \(\{I_0\}, \{I_0, I_2\}, X/I\) is the set of all ideals of \(X/I\). Therefore, the set of ideals of \(X/I\) is a chain. By Theorem 3.11(i), we conclude that \(I\) is a prime ideal of \(X\).

**Note 3.13.** [11] Let \(X\) be a \(P\)-semisimple \(BCI\)-algebra. Then \((X, *, 0)\) is an Abelian group, where \(x, y = x * (0 * y)\), for all \(x, y ∈ X\). Moreover, any closed ideal of \(X\) is a subgroup of \((X, *, 0)\).

**Theorem 3.14.** Let \(X\) be an associative \(BCI\)-algebras and \(I\) be an ideal of \(X\).

(i) If there exist distinct elements \(x, y\) of \(X\) such that \(x, y ∉ I\). Then \(I\) is not a prime ideal.

(ii) If \(I\) is of order \(n > 2\), then there is not any prime ideal on \(X\).

**Proof.** (i) Since \(X\) is an associative \(BCI\)-algebra, we have \(\langle x \rangle = \{x, 0\}\) and \(\langle y \rangle = \{0, y\}\) and so \(\langle x \rangle \cap \langle y \rangle = 0\). Therefore, \(I\) is not a prime ideal of \(X\).

(ii) Let \(I\) be a proper ideal of \(X\). Since \(X\) is finite, then \(I\) is a closed ideal. Hence by Note 3.13, \(I\) is a subgroup of \((X, *, 0)\) and so there exists \(t ∈ \mathbb{N} \setminus \{1\}\) such that \(n = t|I|\), where \(|I|\) is the number of elements of \(I\). Hence \(|I| ≤ n - 2\). Now, by (i), \(I\) is not a prime ideal of \(X\) and so \(X\) has not any prime ideals. \(\square\)

**Theorem 3.15.** Let \(M\) be a maximal ideal of \(X\) containing \(P\). If \(I = \langle P \rangle\), then \(M/I\) is a prime ideal of \(X/I\).
Proof. Since, \(I = \langle P \rangle = \langle P \cup P \rangle = P + P\), then \(I\) is a closed ideal of \(X\). Since \(P \subseteq M\), we have \(I \subseteq M\) and so \(M/I\) is a maximal ideal of \(X/I\). Also \(X/I\) is a \(BCK\)-algebra. Hence by Theorem 3.7, \(M/I\) is a prime ideal of \(X/I\). \(\square\)

Example 3.7. Let \(X = \{0,1,a,b\}\). Define the operation "\(*\)" on \(X\) by

<table>
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<tr>
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<th>0</th>
<th>a</th>
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<tr>
<td>a</td>
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<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
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</table>

Clearly, \((X, *, 0)\) is a \(BCI\)-algebra (see [11] Appendix B Example 4) and \(\{\{0\}, \{0,a\}, \{0,1\}\}\) is the set of all proper ideals of \(X\). It is obvious that \(P = \{0,a\}\) is the \(P\)-semisimple part of \(X\). By Theorem 3.15, \(M/I\) is a prime ideal of \(X/I\), where \(I = \langle P \rangle\).

Lemma 3.16. Let \(X\) be a nilpotent \(BCI\)-algebra. Then for any \(b \in B\setminus\{0\}\), there exists a prime ideal \(Q\) such that \(b \notin Q\).

Proof. Let \(b \in B\setminus\{0\}\). By Corollary 3.9(i), there exists a prime ideal \(I\) of \(B\) such that \(b \notin I\). Let \(P\) be \(P\)-semisimple part of \(X\). We claim that \(b \notin I + P\). Otherwise, \(b \in I + P\). Then by Theorem 2.2, there exist \(a_1,\ldots,a_n \in I\) such that \((\ldots(b * a_1) * \ldots) * a_n \in P\). Since \(B\) is a closed ideal of \(X\) we have \((\ldots(b * a_1) * \ldots) * a_n \in B\). Hence \((\ldots(b * a_1) * \ldots) * a_n \in B \cap P = \{0\}\). Therefore,

\[\ldots(b * a_1) * \ldots * a_n = 0 \in I.\]

Since \(I\) is an ideal of \(X\) containing \(a_1,\ldots,a_n\), we conclude that \(b \notin I\), which is a contradiction. Hence \(b \notin I + P\). It remains to show that \(I + P\) is a prime ideal of \(X\). Let \(J\) and \(K\) be two ideals of \(X\) such that \(J \cap K \subseteq I + P\). Then

\[(J \cap B) \cap (K \cap B) = (J \cap K) \cap B \subseteq (I + P) \cap B.\]  \hspace{1cm} (3.1)

Now, we show that \((I + P) \cap B = \{0\}\). Clearly, \(I \subseteq (I + P) \cap B\). Let \(x \in (I + P) \cap B\). Then there exist \(a_1,\ldots,a_n \in I\) such that \((\ldots(x * a_1) * \ldots) * a_n \in P\). Since \(x, a_1,\ldots,a_n \in B\) we have

\[(\ldots(x * a_1) * \ldots) * a_n \in P \cap B = \{0\}.\]  \hspace{1cm} (3.2)

Moreover, since \(a_1,\ldots,a_n \in I\) we obtain \(x \in I\). Hence \((I + P) \cap B \subseteq I\). Therefore, \((I + P) \cap B = \{0\}\). By (3.1) and (3.2) we have \((J \cap B) \cap (K \cap B) \subseteq I\). Since \(I\) is a prime ideal of \(B\) we have \(J \cap B \subseteq I\) or \(K \cap B \subseteq I\). Assume that \(J \cap B \subseteq I\). Since \(x * (0 * (0 * x)) \in B\) and \(X\) is nilpotent, then by Theorem 2.1, \(x * (0 * (0 * x)) \in B \cap J\) and \(0 * (0 * x) \in P\), for all \(x \in J\). Since \(J \cap B \subseteq I\), we have \(x \in I + P\). Therefore,

\[J \subseteq I + P.\]  \hspace{1cm} (3.3)

If \(K \cap B \subseteq I\), then by the similar way, we obtain

\[K \subseteq I + P.\]  \hspace{1cm} (3.4)

Putting (3.3) and (3.4) together, we obtain that \(J \subseteq I + P\) or \(K \subseteq I + P\). Hence \(I + P\) is a prime ideal of \(X\). \(\square\)

Corollary 3.17. If \(X\) is a nilpotent \(BCI\)-algebra such that \(B \neq \{0\}\), then \(X\) has a prime ideal.
Theorem 3.18. Let $X$ be a nilpotent $BCI$-algebra.

(i) For any $x \in X - P$, there exists a prime ideal $Q$ of $X$, such that $x \notin Q$.

(ii) $\cap\{Q \mid Q \text{ is a prime ideal of } X\} \subseteq P$.

Proof. (i) Let $x \in X - P$. Then by ($BCI6$) and ($BCI7$), we conclude that $x \cdot (0 \cdot (0 \cdot x)) \in B - \{0\}$. Hence by Lemma 3.16, there is a prime ideal $Q$ of $X$ such that $x \cdot (0 \cdot (0 \cdot x)) \notin Q$. Therefore, $x \notin Q$. Since if $x \in Q$, then by ($BCI4$), ($BCI5$) and ($BCI6$), we get $(x \cdot (0 \cdot (0 \cdot x))) \cdot x = 0 \cdot x \in Q$ (since $Q$ is closed) and so $x \cdot (0 \cdot (0 \cdot x)) \in Q$, which is impossible.

(ii) It is straight consequent of (i). \hfill \qed

References