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Counting certain sublattices in the subgroup lattice of a finite abelian group

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Abstract. The main goal of the current paper is to determine the total number of convex sublattices of length 2 in the subgroup lattice of a finite abelian group. This counting problem is reduced to finite p-groups. Explicit formulas are obtained in some particular cases.

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1. Introduction

The relation between the structure of a group and the structure of its lattice of subgroups constitutes an important domain of research in group theory. The topic has enjoyed a rapid development starting with the first half of the '20 century. Many classes of groups determined by different properties of partially ordered subsets of their subgroups (especially lattices of subgroups) have been identified. We refer to Suzuki's book [9], Schmidt's book [5] or the more recent book [10] by the first author for more information about this theory.

An important concept of subgroup lattice theory has been introduced by Schmidt [6] (see also [7]): given a lattice L, a group G is said to be *L-free* if the subgroup lattice $L(G)$ has no sublattice isomorphic to L. Interesting results about L-free groups have been obtained for several particular lattices L, as the diamond lattice M_5 or the pentagon lattice N_5. We recall here that a group is M_5-free if and only if it is locally cyclic and, in particular, a finite group is M_5-free if and only if it is cyclic. Notice also that the class of L-free groups can be extended to the class of groups whose subgroup lattices contain a certain number of sublattices isomorphic to L (see e.g. [13]).

Clearly, for a finite group G the above concept leads to the natural problem of counting the sublattices of $L(G)$ that are isomorphic to a given lattice L. In the general case this problem is very difficult. So, in the current paper we will treat it only for finite abelian groups G and for L of type $L_p = L(\mathbb{Z}_p^2)$ (i.e. the lattice of subspaces of a vector space of dimension 2 over $\mathbb{Z}_p$), $p \in \pi(G)$. This choice of L is very natural, since a finite abelian p-group is L_p-free if and only if it is cyclic. Moreover, we will restrict our counting only on the convex sublattices of $L(G)$. In other words, the purpose of this paper is to determine the number $cs_2(G)$ of convex sublattices of length 2 in the subgroup lattice of a finite abelian group.

The paper is organized as follows. In Section 2 we show that the study is reduced to p-groups and we develop a general method to find the number of the above sublattices.

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Section 3 and 4 deal with the particular cases of elementary abelian p-groups and of abelian p-groups of rank 2. In the final section some conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here. Basic definitions and results on lattices and groups can be found in [2, 3] and [4, 8], respectively.

2. Preliminaries

Let G be a finite abelian group and $L(G)$ be the subgroup lattice of G. It is well-known that G can be written as a direct product of p-groups

$$G=\prod_{i=1}^{k} G_i,$$

where $|G_i| = p_i^{\alpha_i}$, for all $i = 1, 2, ..., k$. Since the subgroups of a direct product of groups having coprime orders are also direct products (see Corollary of (4.19), [8], I), it follows that

$$L(G)\cong\prod_{i=1}^{k} L(G_i).$$

The above lattice direct decomposition is often used in order to reduce some combinatorial problems on $L(G)$ to the subgroup lattices of finite p-groups (see e.g. [1, 11, 12]). This technique can be also applied to our problem.

It is easy to see that a convex sublattice of length 2 of $L(G)$ can be obtained in a unique way, namely as a direct product of a convex sublattice of length 2 of some $L(G_i)$ by a subgroup contained in $\prod_{j\neq i} G_j$. This leads to the following result.

Theorem 2.1. *The number of convex sublattices of length* 2 *in the subgroup lattice of the finite abelian group* $G=\prod_{i=1}^{k} G_i$ *satisfies*

$$cs_2(G)=\sum_{i=1}^{k} cs_2(G_i)\prod_{j\neq i}\,|\,L(G_j)\,|\,. \tag{1}$$

Clearly, Theorem 2.1 shows that the computation of $cs_2(G)$ is reduced to the computation of $cs_2(G_i)$, $i = 1, 2, ..., k$.

Example 2.1. For the abelian group $G=\mathbb{Z}_2^2\times\mathbb{Z}_3^2$ we have

$$cs_2(G)=cs_2(\mathbb{Z}_2^2)\,|\,L(\mathbb{Z}_3^2)\,|+cs_2(\mathbb{Z}_3^2)\,|\,L(\mathbb{Z}_2^2)\,|=1\cdot 6+1\cdot 5=11\,.$$

Assume next that G is a finite abelian p-group, say

$$G=\prod_{i=1}^{r}\mathbb{Z}_{p^{\alpha_i}}\,,$$

where $1\leq\alpha_1\leq\alpha_2\leq...\leq\alpha_r$. For every subgroup H of G, let us denote by N_H the number of convex sublattices of length 2 of $L(G)$ with the top H. Obviously, we have

$$cs_2(G)=\sum_{H\in L(G)} N_H\,. \tag{2}$$

Let s be the rank of $H \in L(G)$. We easily infer that every convex sublattice of length 2 of $L(G)$ with the top H is perfectly determined by a quotient of H isomorphic to $\mathbb{Z}_p^2$. This implies that

$$N_H = |\{K \in L(H) \mid H/K \cong \mathbb{Z}_p^2\}| = \\ = |\{K \in L(\mathbb{Z}_p^s) \mid |K| = p^{s-2}\}| = \\ = \frac{p^{2s-1} - p^s - p^{s-1} + 1}{p^3 - p^2 - p + 1}\,.$$

Thus (2) allows us to compute $cs_2(G)$ if we know precisely the subgroup structure of G. Unfortunately, this is happening in very few cases. One of them is constituted by the finite elementary abelian p-groups and will be treated in Section 3.

For the other cases we are able to develop an alternative method. Observe that $cs_2(G)$ can be seen as a function in $\alpha_1, \alpha_2, ..., \alpha_r$, say

$$cs_2(G) = cs_2(\alpha_1, \alpha_2, ..., \alpha_r),$$

and put

$$n = \frac{p^r - 1}{p - 1}\,.$$

Then G has n maximal subgroups, say $M_1, M_2, ..., M_n$. A convex sublattice of length 2 of $L(G)$ either is contained in some M_i, $i = 1, 2, ..., n$, or has the top G. Consequently, by applying the Inclusion-Exclusion Principle, we get

$$cs_2(G)=\sum_{i=1}^{n} cs_2(M_i) - \sum_{1\le i_1<i_2\le n} cs_2(M_{i_1} \cap M_{i_2}) + ... + (-1)^{n-1} cs_2(\bigcap_{i=1}^{n} M_i) + N_G\,. \quad (3)$$

It is now clear that if all intersections of maximal subgroups of G are known, then (3) leads to a recurrence relation that permits us to determine explicitly $cs_2(G)$. This will be exemplified in Section 4 for $r = 2$.

3. Finite elementary abelian p-groups

Let $G = \mathbb{Z}_p^r$ be an elementary abelian p-group of rank r, that is a direct product of r copies of $\mathbb{Z}_p$. Under the above notation and by using the fact that all subgroups of a given order of G are isomorphic, one obtains

$$cs_2(G) = \sum_{H\in L(G)} N_H = \sum_{i=0}^{r} \sum_{H\in L(G),\ |H|=p^i} N_H = \sum_{i=0}^{r} a_{r,p}(i) N_{H_i}\,,$$

where $a_{r,p}(i)$ denotes the number of subgroups of order p^i of G and $|H_i| = p^i$, $i = 0, 1, ..., r$. By Corollary 2.14 of [11] (see also Proposition 3.1 of [12]), we have

$$a_{r,p}(i) = 1 \ \text{ if } \ i = 0 \text{ or } i = r$$

and

$$a_{r,p}(i) = \sum_{1\le j_1<j_2<...<j_i\le r} p^{j_1+j_2+...+j_i - \frac{i(i+1)}{2}} \ \text{ if } \ 1 \le i \le r-1.$$

On the other hand, all quantities N_{H_i} are known by Section 2 since the rank of H_i is i. Notice also that $N_{H_i} = 0$, for all $i \le 2$. Hence we have shown the following theorem.

Theorem 3.1. *The number of convex sublattices of length* 2 *in the subgroup lattice of the finite elementary abelian* p*-group* $G = \mathbb{Z}_p^r$ *is*

$$cs_2(G) = \sum_{i=2}^{r} a_{r,p}(i)\ \frac{p^{2i-1} - p^i - p^{i-1} + 1}{p^3 - p^2 - p + 1}\ ,$$

where the numbers $a_{r,p}(i)$, $i = 2, 3, ..., r$, *are indicated above.*

Example 3.1. For the elementary abelian p-group $\mathbb{Z}_p^3$ we have $a_{3,p}(2) = p^2 + p + 1$ and $a_{3,p}(3) = 1$. Thus the formula in Theorem 3.1 becomes

$$cs_2(\mathbb{Z}_p^3) = (p^2 + p + 1) \cdot \frac{p^3 - p^2 - p + 1}{p^3 - p^2 - p + 1} + 1 \cdot \frac{p^5 - p^3 - p^2 + 1}{p^3 - p^2 - p + 1} = 2(p^2 + p + 1)$$

and in particular

$$cs_2(\mathbb{Z}_2^3) = 14\,.$$

Remark 3.1. Theorem 3.1 can be also used to give a lower bound for $cs_2(G)$, when $G = \prod_{i=1}^{r} \mathbb{Z}_{p^{\alpha_i}}$ is an arbitrary finite abelian p-group. Put $\alpha = \sum_{i=1}^{r} \alpha_i$ and consider the first $\alpha + 1$ terms of the Frattini series of G:

$$\Phi_0(G) = G \text{ and } \Phi_j(G) = \Phi(\Phi_{j-1}(G)), \text{ for all } j = \overline{1, \alpha}.$$

Then we have

$$cs_2(G) \geq \sum_{j=1}^{\alpha} cs_2(\Phi_{j-1}(G)/\Phi_j(G)) = \sum_{i=0}^{r-1} \sum_{j=\alpha_0+...+\alpha_i+1}^{\alpha_0+...+\alpha_i+\alpha_{i+1}} cs_2(\Phi_{j-1}(G)/\Phi_j(G)),$$

where, by convention, $\alpha_0 = 0$. Since $\Phi_{j-1}(G)/\Phi_j(G) \cong \mathbb{Z}_p^{r-i}$ for any $j = \alpha_0 + ... + \alpha_i + 1, ..., \alpha_0 + ... + \alpha_i + \alpha_{i+1}$, Theorem 3.1 leads to

$$cs_2(G) \geq \sum_{i=0}^{r-1} \alpha_{i+1} \sum_{k=2}^{r-i} a_{r-i,p}(k)\ \frac{p^{2k-1} - p^k - p^{k-1} + 1}{p^3 - p^2 - p + 1}\ .$$

We remark that in the case $i = r - 1$ we have $\Phi_{j-1}(G)/\Phi_j(G) \cong \mathbb{Z}_p$ for all j, and consequently $cs_2(\Phi_{j-1}(G)/\Phi_j(G)) = 0$. In this way, the above inequality can be rewritten as

$$cs_2(G) \geq \sum_{i=0}^{r-2} \alpha_{i+1} \sum_{k=2}^{r-i} a_{r-i,p}(k)\ \frac{p^{2k-1} - p^k - p^{k-1} + 1}{p^3 - p^2 - p + 1}\ . \tag{4}$$

Finally, we exemplify (4) for abelian p-groups of rank 3:

$$cs_2(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \mathbb{Z}_{p^{\alpha_3}}) \geq \sum_{i=0}^{1} \alpha_{i+1} \sum_{k=2}^{3-i} a_{3-i,p}(k)\ \frac{p^{2k-1} - p^k - p^{k-1} + 1}{p^3 - p^2 - p + 1} =$$

$$= \alpha_1 \left[a_{3,p}(2) + a_{3,p}(3)(p^2 + p + 1)\right] + \alpha_2 a_{2,p}(2) =$$

$$= 2\alpha_1(p^2 + p + 1) + \alpha_2\,.$$

4. Finite rank 2 abelian p-groups

It is well-known that for a finite rank 2 abelian p-group $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$, $1 \le \alpha_1 \le \alpha_2$, the number of maximal subgroups is $n = p+1$ and these are of type

$$M_1, M_2, ..., M_p \cong \mathbb{Z}_{p^{\alpha_1-1}} \times \mathbb{Z}_{p^{\alpha_2}} \text{ and } M_{p+1} \cong \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2-1}}.$$

Then (3) becomes

$$cs_2(\alpha_1,\alpha_2) = p \cdot cs_2(\alpha_1-1,\alpha_2) + cs_2(\alpha_1,\alpha_2-1) - p \cdot cs_2(\alpha_1-1,\alpha_2-1) + 1\,,$$

for all $1 \le \alpha_1 \le \alpha_2$. The solution of this recurrence relation can be easily found by induction on α_1 or by using the method of generating functions, namely

$$cs_2(\alpha_1,\alpha_2) = \sum_{i=0}^{\alpha_1-1} (\alpha_1+\alpha_2-2i-1)\,p^i =$$

$$= \frac{(\alpha_2-\alpha_1+1)p^{\alpha_1+1}-(\alpha_2-\alpha_1-1)p^{\alpha_1}-(\alpha_1+\alpha_2+1)p+(\alpha_1+\alpha_2-1)}{(p-1)^2}\,.$$

Hence the following theorem holds.

Theorem 4.1. *The number of convex sublattices of length* 2 *in the subgroup lattice of the finite rank* 2 *abelian p-group* $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$, $1 \le \alpha_1 \le \alpha_2$, *is*

$$cs_2(G) = \frac{(\alpha_2-\alpha_1+1)p^{\alpha_1+1}-(\alpha_2-\alpha_1-1)p^{\alpha_1}-(\alpha_1+\alpha_2+1)p+(\alpha_1+\alpha_2-1)}{(p-1)^2}\,.$$

Example 4.1. By the above formula, we have

$$cs_2(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^3}) = \frac{2p^3-6p+4}{(p-1)^2} = 2p+4$$

and in particular

$$cs_2(\mathbb{Z}_4 \times \mathbb{Z}_8) = 8\,.$$

Remarks.

1. It is easy to see that for $\alpha_1+\alpha_2 = k =$ constant (i.e. $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ is of a fixed order, namely p^k) the function $cs_2 = cs_2(\alpha_1)$ is increasing, and therefore its maximum is obtained for $\alpha_1 = \left[\frac{k}{2}\right]$.
2. The value of $cs_2(\alpha_1,\alpha_2)$ coincides with the total number of subgroups of $\mathbb{Z}_{p^{\alpha_1-1}} \times \mathbb{Z}_{p^{\alpha_2-1}}$, which has been calculated in Proposition 2.9 of [11] (see also Theorem 3.3 of [12]).
3. By Theorem 3.1, we infer that
$$cs_2(\alpha_1,\alpha_2) \equiv \alpha_1+\alpha_2-1 \pmod p\,.$$
A standard induction on r easily shows that
$$cs_2(\alpha_1,\alpha_2,...,\alpha_r) \equiv \alpha_1+\alpha_2+...+\alpha_r-1 \pmod p\,.$$
4. A lower bound for $cs_2(\alpha_1,\alpha_2,...,\alpha_r)$ that involves the quantity $cs_2(\alpha_1,\alpha_2)$ computed above is the following
$$cs_2(\alpha_1,\alpha_2,...,\alpha_r) \ge cs_2(\alpha_1,\alpha_2)\,|L(\prod_{i=3}^{r} \mathbb{Z}_{p^{\alpha_i}})| \ge$$
$$\ge cs_2(\alpha_1,\alpha_2)\prod_{i=3}^{r}|L(\mathbb{Z}_{p^{\alpha_i}})| = cs_2(\alpha_1,\alpha_2)\prod_{i=3}^{r}(\alpha_i+1).$$

Observe that this depends on all α_i, $i = 1, 2, ..., r$, instead of the lower bound given by Remark 3.1.

5. Conclusions and further research

All our previous results show that the problem of counting the number of (convex) sublattices in the subgroup lattice of a group G that are isomorphic to a given lattice L is an interesting computational aspect of subgroup lattice theory. Clearly, the study started in this paper can be extended for other lattices L and groups G. This will surely be the subject of some further research.

Finally, we indicate three open problems concerning the above topic.

Problem 5.1. Improve Theorems 3.1 and 4.1, by obtaining explicit formulas for $cs_2(G)$ when G is of an arbitrary rank.

Problem 5.2. In the class of finite groups G of a fixed order, find the minimum/maximum of $cs_2(G)$. Is it true that the function cs_2 is strictly decreasing on the set of abelian p-groups of order p^n, totally ordered by the lexicographic order?

Problem 5.3. Determine the number of (convex) sublattices in other remarkable posets of subgroups (e.g. normal subgroup lattices) of a finite group that are isomorphic to a given lattice.

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