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On the stability of weak solutions of sediment transport models

Jean De Dieu Zabsonré

Abstract. In this paper we are concerned with the stability of weak solutions for a family of two-dimensional bed-load transport models which combines a viscous shallow water system with a transport equation that describes the bottom evolution. Our analysis is performed in a periodic domain where models with critical shear stress are used for the solid discharge.

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1. Introduction.

In this paper, we study the stability of weak solutions of a viscous sedimentation model. We consider here the system proposed by Saint-Venant-Exner to model the transport of sediment caused by a flow. It consists of a coupling between the hydrodynamic Saint-Venant system to a morphodynamic bed-load transport sediment equation. It is well known that the equation that describes sediment transport is a continuity equation. The expression of the conservation sediment volume equation is given by

$$\partial_t z_b + \beta \mathrm{div}(q_b(h,u)) = 0, \tag{1}$$

where q_b denotes the solid transport discharge, z_b is the movable bed thickness, $\beta = \frac{1}{1-\zeta}$ with ζ the porosity of the sediment layer (see [6],[7],[11]).

There are several formulae for q_b available in the literature which are obtained using empirical models. They depend on the height h of the fluid and the water discharge $q = hu$, where u is the velocity. The most important used for rivers are :

- Grass model:
$$q_b = A_g \left|\frac{q}{h}\right|^{m_g-1}\frac{q}{h}, \quad 1 \le m_g \le 4,$$
where the constant A_g (s^2/m), which is usually obtained from experimental data, takes into account the grain diameter and the kinematic viscosity, see ([6],[7],[12],[14]).
- the model proposed by Meyer-Peter & Müller [16], Van Rijn's [20], Einstein [9], Nielsen [17], Fernández-Luque & Van Beek [10] or Kalinske [13]. Such formulae can be written under a general form
$$q_b = \alpha(\tau)(|\tau| - \tau_*)_+^m, \tag{2}$$
where m is a positive real number and $\alpha = \alpha(\tau)$ depends also on the grain diameter of the sediment. These formulae imply that the movement of the sediment only begins when the modulus of the shear stress is bigger than the critical one denoted τ_*. Usually, one uses Manning's law to define the shear stress: $\tau = ghS_f$

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where $S_f = \dfrac{gM^2|u|}{R_h^{4/3}}u$, being R_h is the hydraulic ratio and M the Manning's friction coefficient. Notice that Meyer-Peter & Müller's formula [16], is the most important formula used in Fluid Mechanics.

In this paper we consider a modified formula of q_b given by (2) by choosing

$$\alpha(\tau) = \beta hu, \quad \tau = c_1|u|\,u \quad \text{ and } \tau_* = c_2$$

where β, c_1 and c_2 are real numbers. Thus, in our analysis we consider the following discharge for the solid transport:

$$q_b = \beta hu(c_1|u|^2 + c_2)_+^m \tag{3}$$

Next as it was presented in [8], in order to have a diffusion process, we add a diffusive term in the sediment transport equation. Finally, we consider the following system:

$$\partial_t h + \mathrm{div}(hu) = 0, \tag{4}$$

$$\partial_t(hu) + \mathrm{div}(hu \otimes u) + gh\nabla(h + z_b) - \nu \mathrm{div}(hD(u)) = 0, \tag{5}$$

$$\partial_t z_b + \mathrm{div}(q_b(h,u)) - \frac{\nu}{2}\Delta z_b = 0, \tag{6}$$

in a two-dimensional periodic domain with periodic boundary conditions. Here q_b is given by (3), $D(u)$ is the symmetric part of the gradient, that is, $D(u) = (\nabla u + {}^t\nabla u)/2$, and $g > 0$ denotes the gravity number. We assume that

$$c_2 < 0, \quad 0 < c_1 < \frac{1}{4m\beta}, \quad 0 < m \leq \frac{1}{2}.$$

The initial data are taken in such a way that

$$h|_{t=0} = h_0 \geq 0, \qquad z_b|_{t=0} = z_{b_0}, \qquad hu|_{t=0} = q_0, \tag{7}$$

where

$$\begin{aligned} h_0 \in L^2(\Omega), &\qquad z_{b_0} \in L^2(\Omega), \\ \frac{|q_0|^2}{h_0} \in L^1(\Omega), &\qquad \nabla\sqrt{h_0} \in (L^2(\Omega))^2. \end{aligned} \tag{8}$$

For $c_2 = 0$, we obtain the Grass model studied in [21].

Let us next recall some results on viscous sedimentation models. In [19], the authors obtained the existence of weak solutions of a viscous sedimentation model. In that work, the viscous Saint-Venant system studied in [18] is coupled with a Grass model of the type $q_b = hu$. Notice that in [18], the authors chose a viscous term of the form $\nu\Delta u$. Assuming that the initial data are small enough as in [18] and using Brower fixed point theorem, they obtained an existence result.

In [21], the authors obtained the stability of weak solutions of system (4)–(6) with Grass model ($1 \leq m_g \leq 3/2$). The key point in their analysis is the use of the nice mathematical entropy inequality namely BD entropy, developed in ([1],[2]) for shallow water equation and in ([2],[3],[4], [5]) for viscous compressible Navier-Stokes equations. In [21], the stability is obtained by using one of the multipliers used in [15]. The authors proved that the bed-load transport system considered is energetically consistent without any restriction on the data.

However, in the literature, most of the works related to sediment transport model are done by using formula with critical shear stress (2) for the solid discharge.

In this work, we extend the stability result of weak solutions considered in [21], to the more general system (4)–(6). The difficulty in this paper comes from the form of the discharge q_b which makes difficult the energy estimates and the passage to the limit in the sequence $(q_{b_n})_n = (h_n u_n (c_1|u_n|^2 + c_2)_+^m)_n$.

The rest of the paper is organized as follows: in Section 2, we state the main result. In Section 3 we state the energy inequality which is the main ingredient in proving our main result. Sections 4 and 5 contain the proof of the main result and the energy estimate respectively.

2. The main result

We start this section with the definition of the weak solution to (4)–(6).

Definition 2.1. *We shall say $(h, q = hu, z_b)$ is a weak solution of (4)–(6) on $(0,T)\times\Omega$ with initial conditions (7) if*

- *System (4)–(6) holds in $(\mathcal{D}'((0,T)\times\Omega))^4$,*
- *Equation (7) (on initial conditions) holds in $\mathcal{D}'(\Omega)$ with $h \geq 0$ a.e.,*
- *the energy inequality (11) is satisfied for a.e. non-negative t and the following regularity properties are satisfied:*

$$\sqrt{h}u \in L^\infty(0,T;(L^2(\Omega))^2), \qquad \sqrt{h}\nabla u \in L^2(0,T;(L^2(\Omega))^4),$$

$$\sqrt{h}(c_1|u|^2+c_2)_+^{(m+1)/2} \in L^\infty(0,T;(L^2(\Omega))^2), \qquad h + z_b \in L^\infty(0,T;L^2(\Omega)),$$

$$\nabla h + \nabla z_b \in L^2(0,T;(L^2(\Omega))^2), \qquad \nabla\sqrt{h} \in L^\infty(0,T;(L^2(\Omega))^2),$$

$$\sqrt{h}D(u)(c_1|u|^2+c_2)_+^{m/2} \in L^2(0,T;(L^2(\Omega))^2),$$

- *h and z_b are in $\mathcal{C}^0(0,T;H^{-s}(\Omega))$ and hu is in $\mathcal{C}^0\left(0,T;(H^{-s}(\Omega))^2\right)$ for s large enough.*

We are now in a position to state our main result.

Theorem 2.1. *Let $(h_n, q_n = h_n u_n, z_{b_n})$ denote a sequence of weak solutions of (4)–(6) which satisfy the entropy inequality (11), with initial data*

$$h_{n|_{t=0}} = h_0^n(x),\ h_n u_{n|_{t=0}} = q_0^n(x) \text{ and } z_{b_n|_{t=0}} = z_{b_0}^n(x),$$

where h_0^n, $z_{b_0}^n$ and u_0^n verify

$$h_0^n \geq 0,\ \ h_0^n \to h_0 \text{ in } L^1(\Omega),\ \ z_{b_0}^n \to z_{b_0} \text{ in } L^1(\Omega),\ \ q_0^n \to q_0 \text{ in } L^1(\Omega), \tag{9}$$

and satisfy the following bounds:

$$\int_\Omega h_0^n\frac{|u_0^n|^2}{2} + \frac{|h_0^n + z_{b_0}^n|^2}{2} + h_0^n\left(c_1|u_0^n|^2+c_2\right)_+^{m+1} < C,$$

$$\int_\Omega \left|\nabla\sqrt{h_0^n}\right|^2 < C \text{ and } \int_\Omega |h_0^n| < C. \tag{10}$$

Then, up to a subsequence, $(h_n)_n$, $(q_n)_n$ and $(z_{b_n})_n$ converge strongly in $\mathcal{C}^0(0,T;L^{2p/(2+p)}(\Omega))$, $\mathcal{C}^0(0,T;W^{-1,2p/(2+p)}(\Omega))$ and $\mathcal{C}^0(0,T;L^{2p/(2+p)}(\Omega))$ respectively to a weak solution of (4)–(6) satisfying entropy inequalities (11).

3. Energy inequality

Proposition 3.1. *For (h, q, z_b) smooth solution of the model (4)–(6), we show the following relation:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h|u+\nu\nabla\log h|^2 + \frac{\beta}{c_1(m+1)}\frac{d}{dt}\int_\Omega h\left(c_1|u|^2+c_2\right)_+^{m+1} + \frac{1}{2}\frac{d}{dt}\int_\Omega h|u|^2 + g\frac{d}{dt}\int_\Omega |z_b+h|^2 + g\nu\int_\Omega |\nabla(h+z_b)|^2 + \frac{\nu}{4}\int_\Omega h\left|\nabla u + {}^t\nabla u\right|^2 + \frac{\nu}{4}\int_\Omega h\left|\nabla u - {}^t\nabla u\right|^2 + 2(1-4m\beta c_1)\nu\int_\Omega h|D(u)|^2\left(c_1|u|^2+c_2\right)_+^m = 0. \quad (11)$$

The proof of this Proposition is postponed to Section 5.

Remark 3.1. *The energy inequality provides the following uniform estimates for a smooth solution:*

$$\|\sqrt{h}u\|_{L^\infty(0,T;(L^2(\Omega))^2)} \le c \in \mathbb{R}_+, \qquad \|\nabla\sqrt{h}\|_{L^\infty(0,T;(L^2(\Omega))^2)} \le c,$$

$$\|z_b+h\|_{L^\infty(0,T;L^2(\Omega))} \le c, \qquad \|\sqrt{h}\left(c_1|u|^2+c_2\right)_+^{(m+1)/2}\|_{L^\infty(0,T;(L^2(\Omega))^2)} \le c,$$

$$\|\nabla(h+z_b)\|_{L^2(0,T;(L^2(\Omega))^2)} \le c, \qquad \|\sqrt{h}D(u)\|_{L^2(0,T;(L^2(\Omega))^2)} \le c,$$

$$\|\sqrt{h}D(u)\left(c_1|u|^2+c_2\right)_+^{m/2}\|_{L^2(0,T;(L^2(\Omega))^2)} \le c.$$

4. Proof of Theorem 2.1

In this section we give the proof of Theorem 2.1. With the previous a priori bounds, we are able to prove the compactness of the sequence (h_n, u_n, z_{b_n}) of approximate solution of system (4)–(6) and pass to the limit in the different terms that compose the three equations. Most of the convergence relies on the approach given in [21]. For the sake of completeness, we present here a complete proof. Our argument will be divided into four steps.

4.1. First step: Convergence of the sequences $\left(\sqrt{h_n}\right)_{n\ge1}$, $(h_n)_{n\ge1}$ and $(z_{b_n})_{n\ge1}$.

Integrating the mass equation, we directly get $(\sqrt{h_n})_n$ in $L^\infty(0,T;L^2(\Omega))$. In addition, Remark 3.1 gives us $\|\nabla\sqrt{h}\|_{L^\infty(0,T;(L^2(\Omega))^2)} \le c$, so we obtain:

$$(\sqrt{h_n})_n \text{ is bounded in } L^\infty(0,T;H^1(\Omega)). \quad (12)$$

Using again the continuity equation on h_n, we have the following equality:

$$\partial_t\sqrt{h_n} = \frac{1}{2}\sqrt{h_n}\,\mathrm{div}\,u_n - \mathrm{div}\left(\sqrt{h_n}u_n\right),$$

which allows us to conclude that $(\partial_t\sqrt{h_n})_n$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.
Thanks to Aubin-Simon lemma, we can extract a subsequence, still denoted $(h_n)_{n\ge1}$, such that $\sqrt{h_n}$ strongly converges to $\sqrt{h}$ in $\mathcal{C}^0(0,T;L^2(\Omega))$.

We study now the subsequence $(h_n)_n$. According to the property (12) and Sobolev embeddings, we know that, for all finite p, $(\sqrt{h_n})_n$ is bounded in $L^\infty(0,T;L^p(\Omega))$. In the following, we will assume $p \ge 6$ in order to simplify our expressions and ensure that $(h_n)_n$ is in $L^\infty(0,T;L^2(\Omega))$.

The mass equation on h_n reads: $\partial_t h_n = -\text{div}(h_n u_n)$. Since $h_n u_n = \sqrt{h_n}\sqrt{h_n} u_n$, we deduce that $(h_n u_n)_n$ is bounded in $L^\infty(0,T;(L^{2p/(2+p)}(\Omega))^2)$. Then, the sequence $(\partial_t h_n)_n$ is bounded in $L^\infty(0,T;W^{-1,2p/(2+p)}(\Omega))$.

The equality $\nabla h_n = 2\sqrt{h_n}\,\nabla\sqrt{h_n}$ implies that the sequence $(\nabla h_n)_n$ is bounded in $L^\infty(0,T;(L^{2p/(2+p)}(\Omega))^2)$. Thus, the sequence $(h_n)_n$ is bounded in $L^\infty(0,T;W^{1,2p/(2+p)}(\Omega))$.

Thanks to Aubin-Simon lemma again, we find:

$$h_n \to h \text{ in } \mathcal{C}^0(0,T;L^{2p/(2+p)}(\Omega)).$$

Last, we perform the convergence of the bottom term $(z_{b_n})_n$: combining the bound on $(\sqrt{h_n})_n$ in $L^\infty(0,T;L^p(\Omega))$ and the bound on $(\nabla z_{b_n})_n$ in $L^2(0,T;(L^{2p/(2+p)}(\Omega))^2)$, we deduce:

$$(z_{b_n})_n \text{ is bounded in } L^\infty(0,T;W^{1,2p/(2+p)}(\Omega)).$$

Moreover, we have just shown that $(\Delta z_{b_n})_n$ belongs to $L^\infty(0,T;W^{-1,2p/(2+p)}(\Omega))$. Let us now write the discharge $(q_{b_n})_n$ under the following form:

$$h_n u_n (c_1|u_n|^2 + c_2)_+^m = \sqrt{h_n} u_n \left(\sqrt{h_n}(c_1|u_n|^2 + c_2)_+^{1/2}\right)^{2m} h_n^{\frac{1}{2}-m}. \tag{13}$$

We have $(c_1|u_n|^2 + c_2)_+ \leq c_1|u_n|^2$, so that $\left(\sqrt{h_n}(c_1|u_n|^2 + c_2)_+^{1/2}\right)$ belongs to $L^\infty(0,T;L^2(\Omega))$. The term :

- $\sqrt{h_n} u_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$,
- $\left(\sqrt{h_n}(c_1|u_n|^2 + c_2)_+^{1/2}\right)^{2m}$ is bounded in $L^\infty(0,T;L^{1/m}(\Omega))$,
- $h_n^{\frac{1}{2}-m}$ is bounded in $L^\infty(0,T;L^{p/2(\frac{1}{2}-m)}(\Omega))$.

We then have the sequence $(q_{b_n})_n$ bounded in $L^\infty(0,T;L^{p/(p+1)}(\Omega))$ and $\text{div}(q_{b_n}) \in L^\infty(0,T;W^{-1,p/(p+1)}(\Omega))$.

$$(\partial_t z_{b_n})_n \text{ is bounded in } L^\infty(0,T;W^{-1,p/(p+1)}(\Omega)).$$

For p large enough, we have the relations $W^{1,2p/(2+p)}(\Omega) \subset\subset L^{2p/(2+p)}(\Omega) \subset W^{-1,p/(p+1)}(\Omega)$. Next, by Aubin-Simon lemma we are able to assert that $(z_{b_n})_n$ converges strongly to z_b in $\mathcal{C}^0(0,T;L^{2p/(2+p)}(\Omega))$.

4.2. Second step: Convergence of the water discharge $(q_n)_{n\geq 1} = (h_n u_n)_{n\geq 1}$. We proved that the sequence $(h_n u_n)_n$ is bounded in $L^\infty(0,T;(L^{2p/(2+p)}(\Omega))^2)$ where p is an integer greater than six in the first step. Writing the gradient as follows:

$$\nabla(h_n u_n) = 2\sqrt{h_n} u_n \nabla\sqrt{h_n} + \sqrt{h_n}\sqrt{h_n}\nabla u_n,$$

since the first term is in $L^\infty(0,T;L^1(\Omega))$ and the second one belongs to $L^2(0,T;L^{2p/(2+p)}(\Omega))$, we have:

$$(h_n u_n)_n \text{ bounded in } L^2(0,T;W^{1,1}(\Omega)).$$

Moreover, the momentum equation (5) gives:

$$\partial_t(h_n u_n) = -\text{div}(h_n u_n \otimes u_n) - \frac{1}{Fr^2} h_n \nabla(h_n + z_{b_n}) + \nu\, \text{div}\left(h_n D(u_n)\right).$$

Let us study each term separately:

- $\text{div}(h_n u_n \otimes u_n) = \text{div}\big(\sqrt{h_n} u_n \otimes \sqrt{h_n} u_n\big)$ is in $L^\infty(0,T;W^{-1,1}(\Omega))$,
- as h_n is in $L^\infty(0,T;W^{1,2p/(2+p)}(\Omega))$, it is also in $L^\infty(0,T;L^p(\Omega))$ and then: $h_n\nabla(h_n + z_{b_n})$ is in $L^2(0,T;L^{2p/(2+p)}(\Omega)) \subset L^2(0,T;W^{-1,2p/(2+p)}(\Omega))$,

- remark that

$$\begin{aligned} h_n \nabla u_n &= \nabla(h_n u_n) - u_n \otimes \nabla h_n \\ &= \nabla\left(\sqrt{h_n}\sqrt{h_n}u_n\right) - 2\sqrt{h_n}u_n \nabla\sqrt{h_n}\,; \end{aligned} \tag{14}$$

  we know that the first term is in $L^\infty(0,T;W^{-1,2p/(2+p)}(\Omega))$ and the second one in $L^\infty(0,T;L^1(\Omega))$. So we have the sequence $(h_n D(u_n))_n$ bounded in $L^2(0,T;W^{-1,2p/(2+p)}(\Omega))$.

Finally, note that these three terms are included in $L^2(0,T;W^{-2,2p/(2+p)}(\Omega))$, which means that $\partial_t(h_n u_n)$ is also in this space for all $n \geq 1$.

Then, applying Aubin-Simon lemma, we obtain:

$$(h_n u_n)_n \text{ converges strongly to } q \text{ in } \mathcal{C}^0(0,T;W^{-1,2p/(2+p)}(\Omega)).$$

4.3. Third step: Convergence of $\left(\sqrt{h_n}u_n\right)_{n\geq 1}$. The product $\sqrt{h_n}u_n$ is equal to $q_n/\sqrt{h_n}$.We will prove a strong convergence for this term. We know that $(q_n/\sqrt{h_n})_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$; so Fatou lemma yields:

$$\sup_{t\in[0,T]}\int_\Omega \liminf \frac{q_n^2}{h_n} \leq \sup_{t\in[0,T]} \liminf \int_\Omega \frac{q_n^2}{h_n} < +\infty.$$

In particular, $q(t,x)$ is equal to zero for almost every x where $h(t,x)$ vanishes. Then, let us define the limit velocity by taking $u(t,x) = q(t,x)/h(t,x)$ if $h(t,x) \neq 0$ or else $u(t,x) = 0$. So we can write $q(t,x) = h(t,x)u(t,x)$ and:

$$\sup_{t\in[0,T]}\int_\Omega \frac{q^2}{h} = \sup_{t\in[0,T]}\int_\Omega h|u|^2 < +\infty.$$

Moreover, we can use Fatou lemma again to write

$$\begin{aligned} \int_\Omega h(c_1|u|^2+c_2)_+^{m+1} &\leq \int_\Omega \liminf h_n(c_1|u_n|^2+c_2)_+^{m+1} \\ &\leq \liminf \int_\Omega h_n(c_1|u_n|^2+c_2)_+^{m+1}, \end{aligned}$$

which gives $\sqrt{h}(c_1|u|^2+c_2)_+^{(m+1)/2}$ in $L^\infty(0,T;L^2(\Omega))$.

As $(q_n)_n$ and $(h_n)_n$ converge almost everywhere, the sequence $(\sqrt{h_n}u_n)_n = (q_n/\sqrt{h_n})_n$ converges almost everywhere to $\sqrt{h}u = q/\sqrt{h}$ when h does not vanish. Moreover, for all M positive, $(\sqrt{h_n}u_n\mathbf{1}_{|u_n|\leq M})_n$ converges almost everywhere to $\sqrt{h}u\mathbf{1}_{|u\leq M}$ (still assuming that h does not vanish). If h vanishes, we can write $\sqrt{h_n}u_n\mathbf{1}_{|u_n|\leq M} \leq M\sqrt{h_n}$ and then have convergence towards zero. Then, almost everywhere, we obtain the convergence of $(\sqrt{h_n}u_n\mathbf{1}_{|u_n|\leq M})_n$.

Finally, we have:

$$\begin{aligned}
&\int_0^T\int_\Omega\left|\sqrt{h_n}u_n-\sqrt{h}u\right|^2\\
&\le\int_0^T\int_\Omega\Big(\left|\sqrt{h_n}u_k\mathbf{1}_{|u_n|\le M}-\sqrt{h}u\mathbf{1}_{|u|\le M}\right|+\left|\sqrt{h_n}u_n\mathbf{1}_{|u_n|>M}\right|\\
&\qquad+\left|\sqrt{h}u\mathbf{1}_{|u|>M}\right|\Big)^2\\
&\le 3\int_0^T\int_\Omega\left|\sqrt{h_n}u_k\mathbf{1}_{|u_n|\le M}-\sqrt{h}u\mathbf{1}_{|u|\le M}\right|^2+3\int_0^T\int_\Omega\left|\sqrt{h_n}u_n\mathbf{1}_{|u_n|>M}\right|^2\\
&\qquad+3\int_0^T\int_\Omega\left|\sqrt{h}u\mathbf{1}_{|u|>M}\right|^2.
\end{aligned}$$

Since $(\sqrt{h_n})_n$ is in $L^\infty(0,T;L^p(\Omega))$, $(\sqrt{h_n}u_n\mathbf{1}_{|u_n|\le M})_n$ is bounded in this space. So, as we have seen previously, the first integral tends to zero. Let us study the other two terms:
for M large enough, we have $h_n(c_1|u_n|^2+c_2)_+\mathbf{1}_{|u_n|>M}=h_n(c_1|u_n|^2+c_2)\mathbf{1}_{|u_n|>M}$. Thus

$$\begin{aligned}
&\int_0^T\int_\Omega\left|\sqrt{h_n}u_n\mathbf{1}_{|u_n|>M}\right|^2\\
&=\frac{1}{c_1}\int_0^T\int_\Omega h_n(c_1|u_n|^2+c_2)\mathbf{1}_{|u_n|>M}-\frac{1}{c_1}\int_0^T\int_\Omega c_2h_n\mathbf{1}_{|u_n|>M}\\
&\le\frac{1}{c_1}\int_0^T\int_\Omega h_n(c_1|u_n|^2+c_2)\mathbf{1}_{|u_n|>M}+\frac{1}{c_1}\int_0^T\int_\Omega c_2h_n\mathbf{1}_{|u_n|>M}\\
&\le\frac{1}{(c_1M^2+c_2)_+^m}\int_0^T\int_\Omega h_n(c_1|u_n|^2+c_2)_+^{m+1}+\frac{c_2}{c_1M^2}\int_0^T\int_\Omega h_n|u_n|^2\mathbf{1}_{|u_n|>M}\\
&\le\frac{k}{M^2}+\frac{k}{(c_1M^2+c_2)_+^m}
\end{aligned}$$

for some fixed $k>0$ and for all M large enough. We also have

$$\int_0^T\int_\Omega\left|\sqrt{h}u\mathbf{1}_{|u|>M}\right|^2\le\frac{k}{M^2}+\frac{k}{(c_1M^2+c_2)_+^m}.$$

To conclude this part, we let M tend to infinity which finally gives

$$(\sqrt{h_n}u_n)_n \text{ strongly converges to } \sqrt{h}u \text{ in } L^2(0,T;L^2(\Omega)).$$

4.4. Fourth step: Convergence of the diffusion terms, the pressure and the solid transport flux. The sequence $(\nabla(h_nu_n))_n$ converges to $\nabla(hu)$ in the sense of the distributions, in $(\mathcal{D}'((0,T)\times\Omega))^4$. The weak convergence of the sequence $(\nabla\sqrt{h_n})_n$ in $L^2(0,T;(L^2(\Omega))^2)$ and the strong convergence of $(\sqrt{h_n}u_n)_n$ in this space give the weak convergence in $L^1(0,T;(L^1(\Omega))^4)$ of $(u_n\otimes\nabla h_n)_n$. Then, relation (14) implies that $(h_n\nabla u_n)_n$ converges to $h\nabla u$ in $(\mathcal{D}'((0,T)\times\Omega))^4$. This gives the convergence of the complete diffusion term.

Due to Remark 3.1, the sequence $(\nabla(h_n+z_{b_n}))_n$ converges weakly to $\nabla(h+z_b)$ in $L^2(0,T;(L^2(\Omega))^2)$. Moreover, the sequence $(h_n)_n$ converges strongly in $\mathcal{C}^0(0,T;L^{2p/(2+p)}(\Omega))$; so the product converges weakly to $h\nabla(h+z_b)$ in $L^2(0,T;(L^{p/(1+p)}(\Omega))^2)$.

The last term is the term of solid transport flux: first, we remark that $h_n(c_1|u_n|^2 + c_2)_+$ converges strongly to $h(c_1|u|^2 + c_2)_+$ in $L^1(0,T;L^1(\Omega))$ and then $\left(\sqrt{h_n}(c_1|u_n|^2 + c_2)_+^{1/2}\right)^{2m}$ converges strongly to $\left(\sqrt{h}(c_1|u|^2 + c_2)_+^{1/2}\right)^{2m}$ in $L^{1/m}(0,T;L^{1/m}(\Omega))$. In addition, the sequence $(h_n^{1/2-m})_n$ converges converges to $h^{1/2-m}$ in $\mathcal{C}^0(0,T;L^{2/(1-2m)}(\Omega))$ and $(\sqrt{h_n}u_n)_n$ converges strongly to $\sqrt{h}u$ in $L^2(0,T;(L^2(\Omega))^2)$. By using Equation (13), we obtain that the sequence $(h_nu_n(c_1|u_n|^2 + c_2)_+^m)_n$ converges strongly to $hu(c_1|u|^2+c_2)_+^m$ in the space $L^{2/(2m+1)}(0,T;(L^1(\Omega))^2)$.

This ends the proof of Theorem 2.1.

5. Proof of the energy inequality

Lemma 5.1. *Let (h,q,z_b) be a smooth solution of (4)–(6). Then the following energy inequality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h|u|^2 + \frac{g}{2}\frac{d}{dt}\int_\Omega |z_b+h|^2 + \frac{\beta}{2c_1(m+1)}\frac{d}{dt}\int_\Omega h\left(c_1|u|^2+c_2\right)_+^{m+1} + \frac{g\nu}{2}\int_\Omega \nabla h\cdot\nabla z_b + \frac{g\nu}{2}\int_\Omega|\nabla z_b|^2 + \frac{\nu}{4}\int_\Omega h\left|\nabla u + {}^t\nabla u\right|^2 + (1-4m\beta c_1)\nu\int_\Omega h|D(u)|^2\left(c_1|u|^2+c_2\right)_+^m \le 0. \quad (15)$$

Proof. We multiply Equation (5) by u, and integrate over Ω. Using (4) we deduce:

$$\int_\Omega h\partial_t u\cdot u + \int_\Omega (hu\cdot\nabla)u\cdot u + \frac{1}{Fr^2}\int_\Omega h\nabla(h+z_b)\cdot u - \nu\int_\Omega \operatorname{div}(hD(u))\cdot u = 0.$$

Now let us simplify each term:

- $\displaystyle\int_\Omega h\partial_t u\cdot u + \int_\Omega (hu\cdot\nabla)u\cdot u = \frac{1}{2}\frac{d}{dt}\int_\Omega h|u|^2,$
- $\displaystyle\int_\Omega h\nabla(h+z_b)\cdot u = \int_\Omega (h+z_b)\,\partial_t h = \frac{1}{2}\frac{d}{dt}\int_\Omega h^2 + \int_\Omega z_b\,\partial_t h\ ,$
- $\displaystyle\int_\Omega \operatorname{div}(hD(u))\cdot u = -\int_\Omega hD(u):\nabla u = -\frac{1}{4}\int_\Omega h\left|\nabla u + {}^t\nabla u\right|^2.$

Substituting all these terms, we find:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h|u|^2 + \frac{g}{2}\frac{d}{dt}\int_\Omega h^2 + g\int_\Omega z_b\,\partial_t h + \frac{\nu}{4}\int_\Omega h\left|\nabla u + {}^t\nabla u\right|^2 = 0. \quad (16)$$

Next, we multiply Equation (5) by $\frac{q_b(h,u)}{h}$ and we integrate on Ω:

$$\int_\Omega h\partial_t u\cdot\frac{q_b(h,u)}{h} + \int_\Omega (hu\cdot\nabla)u\cdot\frac{q_b(h,u)}{h} + \frac{1}{Fr^2}\int_\Omega h\nabla(h+z_b)\cdot\frac{q_b(h,u)}{h} - \nu\int_\Omega \operatorname{div}(hD(u))\cdot\frac{q_b(h,u)}{h} = 0.$$

Here again, we study separately each term:

- $\displaystyle\int_\Omega h\partial_t u\cdot\frac{q_b(h,u)}{h} + \int_\Omega (hu\cdot\nabla)u\cdot\frac{q_b(h,u)}{h} = \frac{\beta}{2c_1(m+1)}\frac{d}{dt}\int_\Omega hu\left(c_1|u|^2+c_2\right)_+^{m+1},$

- $g \int_\Omega h\nabla(h+z_b)\cdot \frac{q_b(h,u)}{h} = -g\beta \int_\Omega (h+z_b)\text{div}\left(hu\left(c_1|u|^2+c_2\right)_+^m\right).$

  Next we use Equation (6) to write:

$$
\begin{aligned}
g \int_\Omega h\nabla(h+z_b)\cdot \frac{q_b(h,u)}{h}& \\
&= -\frac{g\nu}{2}\int_\Omega (h+z_b)\Delta z_b + g\int_\Omega (h+z_b)\partial_t z_b \\
&= \frac{g\nu}{2}\int_\Omega \nabla h\cdot\nabla z_b + \frac{g\nu}{2}\int_\Omega |\nabla z_b|^2 + g\int_\Omega h\partial_t z_b \\
&\quad +\frac{g}{2}\frac{d}{dt}\int_\Omega {z_b}^2,
\end{aligned}
$$

-

$$
\begin{aligned}
\int_\Omega \text{div}\,(hD(u))\cdot\frac{q_b(h,u)}{h}& \\
&= -\beta c_1 \int_\Omega h\left(D(u):\nabla u\right)\cdot\left(c_1|u|^2+c_2\right)_+^m \\
&\quad -2m\beta c_1\int_\Omega h\left(D(u)u\nabla\right)u\cdot u\left(c_1|u|^2+c_2\right)_+^{m-1}
\end{aligned}
$$

  Since $c_1<1$, $c_1|u|^2+c_2\le |u|^2$ and so

$$
\begin{aligned}
\left|2mc_1\int_\Omega h\left(D(u)u\nabla\right)u\cdot u\left(c_1|u|^2+c_2\right)_+^{m-1}\right|& \\
&\le 4mc_1\int_\Omega h|D(u)|^2|u|^2\left(c_1|u|^2+c_2\right)_+^{m-1} \\
&\le 4mc_1\int_\Omega h|D(u)|^2\left(c_1|u|^2+c_2\right)_+^{m}.
\end{aligned}
$$

Gathering all these results, we are led to:

$$
\begin{aligned}
\frac{\beta}{2c_1(m+1)}\frac{d}{dt}\int_\Omega h\left(c_1|u|^2+c_2\right)_+^{m+1} + \frac{g\nu}{2}\int_\Omega \nabla h\cdot\nabla z_b + \frac{g\nu}{2}\int_\Omega|\nabla z_b|^2 \\
+g\int_\Omega h\partial_t z_b + \frac{g}{2}\frac{d}{dt}\int_\Omega {z_b}^2 + (1-4m\beta c_1)\nu\int_\Omega h|D(u)|^2\left(c_1|u|^2+c_2\right)_+^m = 0. \quad (17)
\end{aligned}
$$

Now we add Equation (17) to Equation (16): we find the proclaimed inequality. □

Next we introduce the BD entropy in order to have more information on the integral of $\nabla h\cdot\nabla z_b$.

Lemma 5.2. *If (h,q,z_b) is a smooth solution of (4)–(6), the following energy inequality holds:*

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega h|u+\nu\nabla\log h|^2 + \frac{\beta}{2c_1(m+1)}\frac{d}{dt}\int_\Omega h\left(c_1|u|^2+c_2\right)_+^{m+1} \\
+\frac{g}{2}\frac{d}{dt}\int_\Omega |h+z_b|^2 + g\nu\int_\Omega|\nabla h|^2 + \frac{2g\nu}{3}\int_\Omega\nabla h\cdot\nabla z_b + \frac{g\nu}{2}\int_\Omega|\nabla z_b|^2 \\
+\frac{\nu}{4}\int_\Omega h|\nabla u - {}^t\nabla u|^2 + (1-4m\beta c_1)\nu\int_\Omega h|D(u)|^2\left(c_1|u|^2+c_2\right)_+^m = 0 \quad (18)
\end{aligned}
$$

Proof. We use the transport equation to find:

$$\partial_t \nabla h + \operatorname{div}(h\nabla^t u) + \operatorname{div}(u \otimes \nabla h) = 0$$

Replacing ∇h by $h\nabla \log h$ and introducing the viscosity ν, it becomes

$$\partial_t(\nu h\nabla \log h) + \nu \operatorname{div}(h\nabla^t u) + \operatorname{div}(hu \otimes \nu\nabla \log h) = 0$$

Next, we add the momentum equation to obtain:

$$\partial_t(hu + \nu\nabla \log h) + \operatorname{div}(hu \otimes (u + \nu\nabla \log h)) \\ - \nu \operatorname{div}(h(D(u) - \nabla^t u)) + gh\nabla(h + z_b) = 0 \quad (19)$$

We multiply that equation by $(u + \nu\nabla \log h)$ and integrate over Ω. We study each term which has not appeared in the classical energy:

- $$\int_\Omega \partial_t(hu + \nu\nabla \log h)(u + \nu\nabla \log h) \\ + \int_\Omega \operatorname{div}(hu \otimes (u + \nu\nabla \log h))(u + \nu\nabla \log h) = \frac{1}{2}\frac{d}{dt}\int_\Omega h|u + \nu\nabla \log h|^2.$$
- Using the definition of stress tensor, we deduce
$$\int_\Omega \operatorname{div}(h(D(u) - \nabla^t u))(u + \nu\nabla \log h) = \frac{\nu}{4}\int_\Omega h|\nabla u - {}^t\nabla u|^2$$
- $$\int_\Omega h\nabla(h + z_b)\cdot \nu\nabla \log h = \nu\int_\Omega |\nabla h|^2 + \nu\int_\Omega \nabla h \cdot \nabla z_b$$

Finally, we obtain the following equality :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h|u + \nu\nabla \log h|^2 + \frac{\nu}{4}\int_\Omega h|\nabla u - {}^t\nabla u|^2 + g\nu\int_\Omega |\nabla h|^2 \\ + g\nu\int_\Omega \nabla h\cdot\nabla z_b + \frac{g}{2}\frac{d}{dt}\int_\Omega h^2 + g\int_\Omega z_b\, \partial_t h = 0 \quad (20)$$

We add this equality to (17) to deduce the proclaimed result. □

The proof of Proposition 3.1 is now achieved by adding the estimate (15) with the equality (18).

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(Jean De Dieu Zabsonré) Institut des Sciences Exactes et Appliquées (ISEA), Université Polytechnique de Bobo-Dioulasso (UPB), 01 BP 1091 Bobo-Dioulasso 01 Burkina Faso.
E-mail address: jzabsonre@gmail.com