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Approximate analytical solutions to the Bagley-Torvik equation by the Fractional Iteration Method

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Abstract. In this paper we solve the Bagley-Torvik equation, which is an ordinary fractional differential equation, where the solution procedure is easier, more effective and straightforward. The validity and the accuracy of this method is shown by the obtained results.

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1. Introduction

Owing to their frequent appearances in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering, fractional differential equations have been the focus of numerous studies. Therefore, considerable attention has been given to how solving fractional ordinary differential equations, integral equations and fractional partial differential equations. However, there exists no method that yields exact solutions to differential equations of fractional order, consequently some analytical techniques to handle such equations have been proposed, for example, Adomian decomposition method [4], [12], the homotopy analysis method [3], the homotopy perturbation method [1], the Taylor series method [10], variational iteration method (VIM) [8], [15], etc. Recently, Ghorbani [7] introduced a new alternative technique for solving nonlinear differential equations of fractional order which requires no Lagrange multiplier or variational theory. This method can be very effective and reliable for solving nonlinear fractional differential equations.

The paper is organized as follows: section 2 is devoted to describe some necessary definitions on fractional calculus which will be used throughout the paper, in section 3 we describe briefly the basic concept of the fractional iteration method (FIM). In section 4 we apply successfully the FIM to solve the fractional Bagley-Torvik equation. Some concluding remarks are also given in section 5.

2. Preliminaries definitions

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695 where half-order derivative was mentioned. The commonly used definitions for the general fractional are Riemann-Liouville and Caputo. In this paper, the derivatives are considered in the Caputo sense, which has the advantage of defining integer order initial conditions for fractional order differential equations.

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Definition 2.1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exits a real number $\lambda > \mu$ such that $f(x) = x^\lambda g(x)$, where $g(x) \in C[0, \infty)$ and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$ for $m \in \mathbb{N}$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator of order α of a real function $f(x) \in C_\mu,\ \mu \geq -1$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0 \quad and \quad J^0 f(x) = f(x). \quad (1)$$

The the operators J^α have the following proprieties, for $\alpha, \beta \geq 0$, $\gamma, \mu \geq -1$:

- $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
- $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
- $J^\alpha x^\xi = \frac{\Gamma(\xi+1)}{\Gamma(\alpha+\xi+1)} x^{\alpha+\xi}$.

Next we define the Caputo fractional derivatives D^α of a function $f(x)$ of any real number α such that $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, for $x > 0$ and $f \in C_{-1}^m$ in the terms of J^α as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt \quad (2)$$

and has the following proprieties for $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, $\mu \geq -1$ and $f \in C_\mu^m$

- $D^\alpha J^\alpha f(x) = f(x)$,
- $J^\alpha D^\alpha = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}$, for $x > 0$.

3. The Variational Iteration Method

In this section we recall briefly the basic concept of VIM [8]. We begin with considering a differential equation in the general form,

$$L(y(t)) + N(y(t)) = F(t, y(t)), \quad (3)$$

where L is a linear operator, N a nonlinear operator and F is the source term. According to the variational iteration method [8], one can construct a correction functional as follow

$$y_{n+1}(t) = y_n(t) + J^\alpha \left(\lambda(\tau)[L(y_n(\tau)) + N(\tilde{y}_n(\tau)) - F(\tau, y(\tau))]\right) \quad (4)$$

where λ is a general Lagrangian multiplier, the subscript n denotes the n^{th} order approximation, $y_0(t)$ is an initial approximation which can be known according to the initial conditions or the boundary conditions, and the function $\tilde{y}_n$ is a restricted variation which means $\delta\tilde{y}_n = 0$. It is clear that the successive approximation $y_n,\ n \geq 1$, can be established by determining a general Lagrangian multiplier λ, which can be identified optimally via the variational theory. The successive approximations $y_{n+1},\ \ n \geq 0$ of the solution $y(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $y_0(t)$. When λ is known, then other several approximations $y_n,\ n \geq 1$, follow immediately. Consequently, the exact solution can be obtained by using

$$y(t) = \lim_{n\to\infty} y_n(t). \quad (5)$$

4. The Fractional Iteration Method

Based on [7], we will explain briefly the basic concept of FIM, by considering a nonlinear fractional differential equation in the following form:

$$D_t^\alpha y(t) - F(t, y(t)) = 0, \tag{6}$$

with the condition

$$y^{(k)}(0) = a_k, \qquad k = 0, 1, ..., m-1, \tag{7}$$

where $m - 1 \le \alpha \le m$, $m \in \mathbb{N}$, F is a given nonlinear function of y and y is the unknown function to be determined. The main task is to fond a solution y to problem (6), (7) under the form

$$y(t) = \lim_{n\to\infty} y_n(t). \tag{8}$$

Let $H(t) \neq 0$ denote the so-called auxiliary function. Multiplying (6) by $H(t)$ and applying the Riemann-Liouville integral operator J^α on both sides we get

$$J^\alpha(H(t)[D_t^\alpha y(t) - F(t, y(t))]) = 0. \tag{9}$$

Let h be the so-called auxiliary parameter. Multiplying (9) by h and adding y on both sides we get

$$y(t) = y(t) + hJ^\alpha(H(t)[D_t^\alpha y(t) - F(t, y(t))]). \tag{10}$$

Consequently, equation (9) can be solved iteratively as follows

$$y_{n+1}(t) = y_n(t) + hJ^\alpha(H(t)[D_t^\alpha y_n(t) - F(t, y_n(t))]). \tag{11}$$

Where the subscript n denotes the iteration order. The convergence of (11) is ensured by the Banach's fixed point theorem, provided that the right hand of (11) is a contractive mapping [7]. On the other hand the convergence region and the rate of the convergence of the obtained series solutions can be controlled by the convergence parameter h as it is described in the framework of the homotopy analysis method (HAM) [9]. Actually, by plotting the solution (or one of its derivatives) at a particular point with respect to the auxiliary parameter h which is the so-called $h-$curve, one can get a proper value of h that ensures the convergence of the obtained solution series. For more details about the mathematical properties of the $h-$curve we refer the readers to a recent paper by Abbasdandy et al. [2].

5. Numerical implementation and discussion

In this section we will solve the Bagley Torvik equation by the so-called fractional iteration method. The obtained solutions will be compared to the exact ones and to those obtained via the variational iteration method.

The Bagley-Torvik equation is originally formulated in the studies on behavior of real material by use of fractional calculus [5]-[13]. It has raised its importance since than in many engineering and applied sciences applications. In particular, the equation with $\frac{1}{2}$-order derivative or $\frac{3}{2}$-order derivative can model the frequency-dependent damping materials quite satisfactorily. It can also describe motion of real physical systems, the modeling of the motion of a rigid plate immersed in a Newtonian fluid and a gas in a fluid, respectively [14]. Approximate solutions have recently been proposed in the book and papers of Podlubny in which the solution obtained with approximate methods is compared to the exact solution.

Let us consider the following Bagley-Torvik equation with fractional order

$$y'(t) + D_t^{\frac{3}{2}} y(t) + y(t) = F(t), \tag{12}$$

with the initial conditions

$$y(0) = 1, \qquad y'(0) = 1. \tag{13}$$

where

$$F(t) = 7\,t + 8\,\frac{t^{3/2}}{\sqrt{\pi}} + t^3 + 1. \tag{14}$$

and the exact solution is

$$y(t) = t^3 + t + 1. \tag{15}$$

First, we begin by solving the Bagley-Torvik equation by VIM as follows.
The Lagrange multiplier can be identified as $\lambda = -1$, and from (4), we have the following iteration formula for solving equation (12):

$$y_{n+1}(t) = y_n(t) + J^{\alpha}\left(\lambda(\tau)[\tilde{y_n'}(t) + D_t^{\frac{3}{2}}\tilde{y_n}(t) + \tilde{y_n}(t) - F(t)]\right), \tag{16}$$

In view of (13), we choose the initial guess in the form of $y_0(t) = 1 + t$, we have then the following recursive relation:

$$\begin{aligned}
&y_0(t) = 1 + t,\\
&y_1(t) = 1 + t + \frac{16}{5}\,\frac{t^{5/2}}{\sqrt{\pi}} + t^3 + \frac{64}{315}\,\frac{t^{9/2}}{\sqrt{\pi}}\\
&y_2(t) = 1 + t + t^3 - 3\,t^2 - 1/2\,t^4 - \frac{1}{120}\,t^6\\
&y_3(t) = 1 + t + t^3 - 6\,t^2 - t^4 - \frac{1}{60}\,t^6 + 8\,\frac{t^{3/2}}{\sqrt{\pi}} - \frac{96}{35}\,\frac{t^{7/2}}{\sqrt{\pi}} - \frac{128}{1155}\,\frac{t^{11/2}}{\sqrt{\pi}} - \frac{512}{675675}\,\frac{t^{15/2}}{\sqrt{\pi}}\\
&\quad .\\
&\quad .\\
&\quad .
\end{aligned} \tag{17}$$

and so on. On the other hand, using FIM (11) and starting with the same initial approximation $y_0(t) = 1 + t$ and the auxiliary function $H(t) = 1$, we obtain the following

$$\begin{aligned}
&y_0(t) = 1 + t,\\
&y_1(t) = 1 + t - \frac{16}{5}\,\frac{ht^{5/2}}{\sqrt{\pi}} - ht^3 - \frac{64}{315}\,\frac{ht^{9/2}}{\sqrt{\pi}}\\
&y_2(t) = 1 + t - \frac{32}{5}\,\frac{ht^{5/2}}{\sqrt{\pi}} - 2\,ht^3 - \frac{128}{315}\,\frac{ht^{9/2}}{\sqrt{\pi}} - 3\,h^2t^2 - \frac{32}{5}\,\frac{h^2t^{5/2}}{\sqrt{\pi}}\\
&-1/2\,h^2t^4 - h^2t^3 - \frac{128}{315}\,\frac{h^2t^{9/2}}{\sqrt{\pi}} - \frac{1}{120}\,h^2t^6\\
&.\\
&.\\
&.
\end{aligned} \tag{18}$$

and so on. In general, by means of the h-curve, it is straightforward to choose a proper value of h which ensures that the solution series is convergent. This proper value of h corresponds to the curve segment nearly parallel to the horizontal axis. For this study we take $h = -0.4$ (see Figure 1).

TABLE 1. Comparison of the FIM solutions of (12) with the exact and the VIM solution

t	y_{VIM}	y_{exact}	y_{FIM}	$\lvert y_{VIM} - y_{exact}\rvert$	$\lvert y_{FIM} - y_{exact}\rvert$
0.10	1.183140356	1.101000	1.103763584	0.082140356	0.00276358
0.25	1.43878394	1.265625	1.269040456	0.173158944	0.00341545
0.50	1.51984451	1.625000	1.623997167	0.105155480	0.00100283
0.75	0.83083557	2.171875	2.166900262	1.341039427	0.00497473
1.00	−1.11359385	3.000000	2.994988879	4.113593854	0.00501112

In order to verify numerically whether the proposed methodology leads to high accuracy, we evaluate the numerical solutions using third-order approximation and compare it with both of the exact analytical solution and the VIM solution. Table 1 shows the absolute errors between exact solution and the FIM and VIM solutions. Table 1 and Figures 2(a) and (b) show that the FIM numerical approximate solution has a high degree of accuracy, compared with the VIM solution. As we know, the more terms added to the approximate solution, the more accurate it will be.

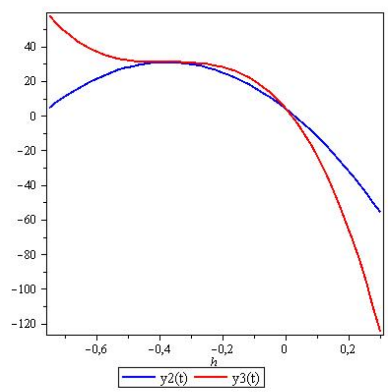
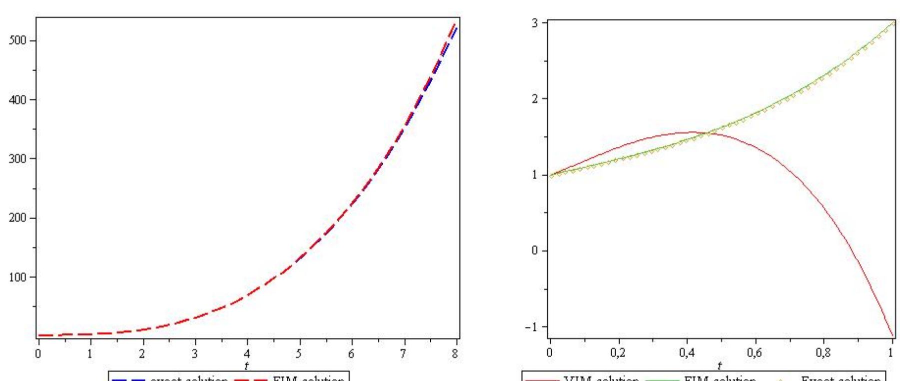


FIGURE 1. $h-$curve of $y(t)$ for the second and third order approximation by FIM.



(a) (b)

FIGURE 2. Comparison of the exact solution of (12) with: (a) the FIM solution, (b) both of the FIM and VIM solutions.

6. Conclusion

The fractional iteration method has proven as an efficient technique to solve nonlinear fractional differential equations. Comparison with the variational method has been shown, the simplicity of the method and the obtained exact results show that it is a powerful mathematical tool for solving nonlinear fractional differential equations. The method was used in a direct way without need for the Lagrange multiplier, correction functional, stationary conditions, linearization or discretization. It also provides more realistic series solutions that converge very rapidly in real physical problems.

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