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Distributive residuated lattices

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Abstract. The aim of this paper is to put in evidence some sufficient conditions for the distributivity in the residuated lattices.

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1. Introduction

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([12]), Dilworth ([5]), Ward and Dilworth ([18]), Ward ([17]), Balbes and Dwinger ([2]) and Pavelka ([14]). In [10], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: *BCK-latices* in [9], *full BCK-algebras* in [12], *FL_{ew}-algebras* in [13], and *integral, residuated, commutative l-monoids* in [3].

The paper is organized as follows. In Section 2 we recall the basic definitions and we put in evidence many rules of calculus in a residuated lattice which we need in the rest of the paper.

In Section 3 we give examples of residuated lattices which are not distributive and this is the reason why we start the study when a residuated lattice becomes distributive.

2. Preliminaries

Definition 2.1. ([16]) An algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ will be called *residuated lattice* if

Lr_1 : $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;

Lr_2 : $(L, \odot, 1)$ is a commutative monoid;

Lr_3 : For every $x, y, z \in L$, $x \leq y \rightarrow z \Leftrightarrow x \odot y \leq z$.

For examples of residuated lattices see [11], [15]-[18].

In what follow by L we denote the univers of a residuated lattice. For $x \in L$ and $n \geq 1$ we define $x^* = x \rightarrow 0$, $x^{**} = (x^*)^*$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geqslant 1$.

Theorem 2.1. *([15]-[18]) Let L be a residuated lattice. Then for every $x, y, z \in L$, we have:*

$$1 \rightarrow x = x; \tag{1}$$

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$$x \to x = 1; \tag{2}$$

$$x \odot y \leq x, y, \ \textit{so}\ x \odot y \leq x \wedge y; \tag{3}$$

$$y \leq x \to y; \tag{4}$$

$$x \leq y \Leftrightarrow x \to y = 1; \tag{5}$$

$$x \to y = y \to x = 1 \Rightarrow x = y; \tag{6}$$

$$x \to 1 = 1; \tag{7}$$

$$0 \to x = 1; \tag{8}$$

$$x \odot (x \to y) \leq y, \ \textit{so}\ x \leq (x \to y) \to y; \tag{9}$$

$$x \to y \leq (x \odot z) \to (y \odot z); \tag{10}$$

$$\textit{If}\ x \leq y, \ \textit{then}\ x \odot z \leq y \odot z; \tag{11}$$

$$x \to y \leq (y \to z) \to (x \to z) \ \textit{and}\ x \to y \leq (z \to x) \to (z \to y); \tag{12}$$

$$x \leq y \Rightarrow z \to x \leq z \to y \ \textit{and}\ x \leq y \Rightarrow y \to z \leq x \to z; \tag{13}$$

$$x \to (y \to z) = (x \odot y) \to z, \ \textit{so}\ x \to (y \to z) = y \to (x \to z); \tag{14}$$

$$x \odot (y \vee z) = (x \odot y) \vee (x \odot z) \ \textit{and}\ x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z); \tag{15}$$

$$x \odot x^* = 0, \ 1^* = 0, \ 0^* = 1, \ x \leq x^{**}; \tag{16}$$

$$x \to y \leq y^* \to x^*; \tag{17}$$

$$x^{***} = x^*. \tag{18}$$

Consider the following identities:

$$x \wedge y = x \odot (x \to y) \qquad (\textit{divisibility}), \tag{19}$$

$$(x \to y) \vee (y \to x) = 1 \qquad (\textit{pre} - \textit{linearity}). \tag{20}$$

Definition 2.2. A *residuated lattice* L is called:
(i) *Divisible* if L verify (19);
(ii) *MTL-algebra* if L verify (20);
(iii) *BL-algebra* if L verify (19) and (20);
(iv) *G-algebra* if $x^2 = x$ for every $x \in L$.

We denote by $\mathcal{RL}_d$, ($\mathcal{MTL}$, $\mathcal{BL}$) the classes of *divisible residuated lattices (MTL-algebras,* respectively *BL-algebras*).

Proposition 2.1. *([7]) For a residuated lattice L, the following assertions are equivalent:*
(*i*) $L \in \mathcal{MTL}$;
(*ii*) *L is subdirect product of linearly ordered residuated lattices;*
(*iii*) $x \to (y \vee z) = (x \to y) \vee (x \to z)$ *for every* $x, y, z \in L$;
(*iv*) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$ *for every* $x, y, z \in L$.

Corollary 2.1. *([7],[8]) If $L \in \mathcal{MTL}$, then for every $x, y, z \in L$ we have:*
(*i*) $(x \wedge y)^* = x^* \vee y^*$;
(*ii*) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;
(*iii*) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(*iv*) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$.

Proposition 2.2. *([17],[18]) For a residuated lattice L, the following assertions are equivalent:*
(*i*) $L \in \mathcal{RL}_d$;
(*ii*) *For any $x, y \in L$ such $x \leq y$, there is $z \in L$ such that $x = y \odot z$;*
(*iii*) $x \to (y \wedge z) = (x \to y) \odot [(x \wedge y) \to z]$ *for every* $x, y, z \in L$.

Remark 2.1. *([4]) In a $G-algebra$ L, $x \odot y = x \wedge y$ for every $x, y \in L$.*

Proposition 2.3. *([4]) In a residuated lattice L the following assertions are equivalent:*
(i) L is a $G-algebra$,
(ii) $x \odot (x \to y) = x \odot y = x \wedge y$ for any $x, y \in L$.

Remark 2.2. In [6] MTL-algebras are known under the name of *normal residuated lattices.*

Definition 2.3. ([4]) An algebra $(L, \to, *, 1)$ of type $(2, 1, 0)$ will be called *Wajsberg algebra* if for every $x, y, z \in L$ are verified the axioms (1), (12) from the Theorem 2.1 and the axioms:

$$(x \to y) \to y = (y \to x) \to x, \tag{21}$$

$$(x^* \to y^*) \to (y \to x) = 1. \tag{22}$$

Remark 2.3. *In a residuated lattice L the axiom* (22) *is equivalent with* $x^{**} = x$, *for every* $x \in L$.

Proof. $x^* \to y^* = (x \to 0) \to (y \to 0) = y \to ((x \to 0) \to 0) = y \to x^{**} = y \to x$, so $(x^* \to y^*) \to (y \to x) = 1$, for every $x, y \in L$. □

Remark 2.4. *A residuated lattice with* (21) *becomes a Wajsberg algebra.*

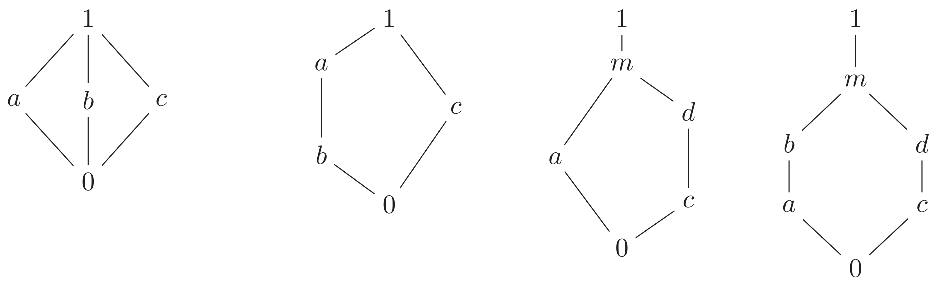
Proof. $(x \to 0) \to 0 = (0 \to x) \to x$, so $x^{**} = 1 \to x = x$ for any $x \in L$. □

Theorem 2.2. *([16], Theorem 10, pag. 46) A residuated lattice $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a Wajsberg algebra $(L, \to, *, 1)$ if, and only if it satisfy an additional condition*

$$(x \to y) \to y = (y \to x) \to x, for\ any\ x, y \in L,$$

with an abbreviation $x^* = x \to 0$.

FIGURE 1. (M_5) (N_5) *Example* 2.1 *Example* 2.2



Theorem 2.3. *(*[1],[2]*) For a lattice* $(L, \vee, \wedge)$ *the following assertions are equivalent:*
(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*, for all* $x, y, z \in L$;
(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*, for all* $x, y, z \in L$;
(iii) *If* $x, y, z \in L$ *and* $x \wedge z = y \wedge z$, $x \vee z = y \vee z$*, then* $x = y$;
(iv) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$*, for every* $x, y, z \in L$;
(v) *Neither of the lattices* (M_5) *and* (N_5) *are sublattices of* L;
(vi) *For all* $a, b \in L$ *with* $a \leq b$*, there exists a join-endomorphism* f *of* L *such that* $f(b) = a$ *and* $f(x) \leq x$*, for all* $x \in L$;
(vii) *For all* $a, b \in L$ *whit* $a \leq b$*, there exists a meet-endomorphism* f *of* L *such that* $f(a) = b$ *and* $x \leq f(x)$*, for all* $x \in L$.

Definition 2.4. The lattice $(L, \vee, \wedge)$ is called *distributive* if L verify one of the equivalent conditions $(i) - (vii)$ from the Theorem 2.3.

For the next result see Corollary 5.13.7 from ([4]).

Remark 2.5. *Every Wajsberg algebra is a MTL-algebra, so it is distributive.*

Remark 2.6. *There are residuated lattices which are not distributive.*

For this we will offer the following two examples:

Example 2.1. *Let* $L = \{0, a, c, d, m, 1\}$ *with* $0 < a < m < 1$, $0 < c < d < m < 1$*, but* a *incomparable with* c *and* d*. Then (*[11]*, pag.* 233*)* L *becomes a residuated lattice relative to the following operations:*

$\rightarrow$	0	a	c	d	m	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
c	a	a	1	1	1	1
d	a	a	m	1	1	1
m	0	a	d	d	1	1
1	0	a	c	d	m	1

$\odot$	0	a	c	d	m	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
c	0	0	c	c	c	c
d	0	0	c	c	c	d
m	0	a	c	c	m	m
1	0	a	c	d	m	1

L is not distributive because $c \vee (a \wedge d) = c \vee 0 = c$, $(c \vee a) \wedge (c \vee d) = m \wedge d = d$ and $c \neq d$.

Example 2.2. *Let* $L = \{0, a, b, c, d, m, 1\}$ *with* $0 < a < b < m < 1$, $0 < c < d < m < 1$ *and elements* $\{a, c\}$ *and* $\{b, d\}$ *are pairwise incomparable. Then (*[11]*, pag.* 234*)* L

becomes a residuated lattice relative to the following operations:

$\to$	0	a	b	c	d	m	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	m	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	m	1	1	1
m	0	b	b	d	d	1	1
1	0	a	b	c	d	m	1

$\odot$	0	a	b	c	d	m	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
m	0	a	a	c	c	m	m
1	0	a	b	c	d	m	1

L is not distributive because $c \vee a \wedge d) = c \vee 0 = c,\ (c \vee a) \wedge (c \vee d) = m \wedge d = d$ and $c \neq d$.

3. Sufficient conditions for distributivity

In this section we put in evidence some sufficient conditions for the distributivity of a residuated lattice.

Theorem 3.1. *For a residuated lattice* $(L, \vee, \wedge, \odot, \to, 0, 1)$ *we consider the following assertions:*

(i) $(L, \vee, \wedge)$ *is a distributive residuated lattice;*
(ii) $x \odot y = x \wedge y$ *for all* $x, y \in L$;
(iii) *L is divisible;*
(iv) *L is a MTL-algebra;*
(v) *If* $x, y, z \in L$, *such that* $z \to x = z \to y$ *and* $z \geq x, y$, *then* $x = y$;
(vi) $x \vee y = (x \to y) \to y$ *for all* $x, y \in L$;
(vii) *If* $a, x, y \in L$, $a^* \odot x = a^* \odot y$ *and* $a \to x = a \to y$, *then* $x = y$;
$(viii)$ $x \to (y \to z) = (x \to y) \to (x \to z)$ *for all* $x, y, z \in L$.
Then,
a) $(ii), (iii), (iv), (v), (vi), (vii), (viii) \Rightarrow (i)$;
b) $(i) \not\Rightarrow (ii), (iii), (iv), (v), (vi), (vii), (viii)$.

Proof. $(ii) \Rightarrow (i)$ Since for every $x, y, z \in L$, $x \odot (y \vee z) \overset{(15)}{=} (x \odot y) \vee (x \odot z)$, by (ii) we deduce that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, that is, L is *distributive.*

$(i) \not\Rightarrow (ii)$ Consider the following counterexample:

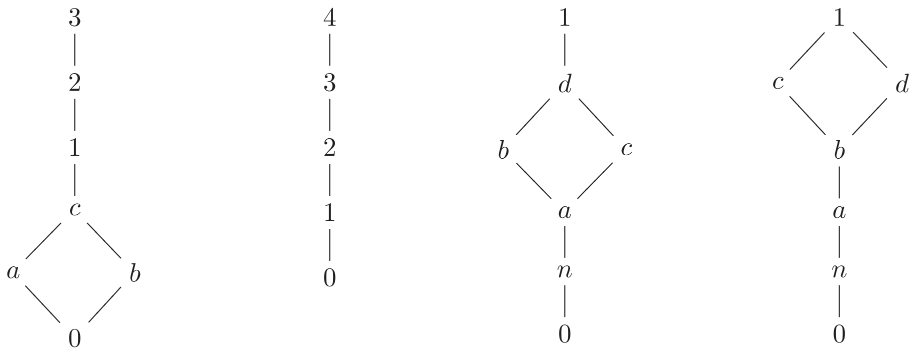
Example 3.1. *Let* $L = \{0, a, b, c, 1, 2, 3\}$, *with* $0 < a < c < 1 < 2 < 3,\ 0 < b < c < 1 < 2 < 3$, *and the elements* a, b *are incomparable. Then (*[11], *pag.* 232*) L becomes a distributive residuated lattice relative to the operations:*

$\to$	0	a	b	c	1	2	3
0	3	3	3	3	3	3	3
a	b	3	b	3	3	3	3
b	a	a	3	3	3	3	3
c	0	a	b	3	3	3	3
1	0	a	b	2	3	3	3
2	0	a	b	1	1	3	3
3	0	a	b	c	1	2	3

$\odot$	0	a	b	c	1	2	3
0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a
b	0	0	b	b	b	b	b
c	0	a	b	c	c	c	c
1	0	a	b	c	c	c	1
2	0	a	b	c	c	2	2
3	0	a	b	c	1	2	3

Since for $1, 2 \in L$ we have $1 \odot 2 = c,\ 1 \wedge 2 = 1$ and $c \neq 1$, then it is not a G-algebra.

FIGURE 2. *Example* 3.1 *Example* 3.2 *Example* 3.3 *Example* 3.4



$(iii) \Rightarrow (i)$ Let $x, y, z \in L$. Clearly, $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$. We have $x \wedge (y \vee z) = (y \vee z) \odot ((y \vee z) \to x) \overset{(15)}{=} \{y \odot [(y \vee z) \to x]\} \vee \{z \odot [(y \vee z) \to x]\} \leq [y \odot (y \to x)] \vee [z \odot (z \to x)] \overset{(19)}{=} (y \wedge x) \vee (z \wedge x) = (x \wedge y) \vee (x \wedge y)$, hence $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y)$.

$(i) \not\Rightarrow (iii)$ Consider the following counterexample:

Example 3.2. *(*[11]*, pag.* 218*) Consider the chain* $L_5 = \{0, 1, 2, 3, 4\}$*, organized as a distributive residuated lattice by natural ordering, with the operations* $\to$ *and* $\odot$ *as in the following tables:*

$\to$	0	1	2	3	4
0	4	4	4	4	4
1	0	4	4	4	4
2	0	3	4	4	4
3	0	3	3	4	4
4	0	1	2	3	4

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	1	1	2
3	0	1	1	1	3
4	0	1	2	3	4

Since for $2, 3 \in L_5$ we have $3 \wedge 2 = 2$, $3 \odot (3 \to 2) = 1$ and $2 \neq 1$, then L_5 is not divisible.

$(iv) \Rightarrow (i)$ See Corollary 2.1, (iii).

$(i) \not\Rightarrow (iv)$ Consider the following counterexample:

Example 3.3. *Let* $L = \{0, n, a, b, c, d, 1\}$ *with* $0 < n < a < b, c < d < 1$*, but* b *and* c *are incomparable. Then (*[11]*, pag.* 231*)* L *becomes a distributive residuated lattice relative to the operations:*

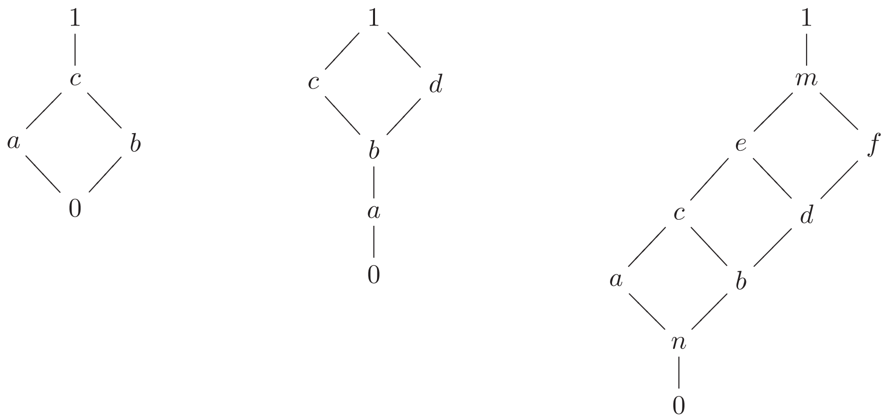
$\to$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	c	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	0	n	c	1	c	1	1
c	n	n	b	b	1	1	1
d	0	n	a	b	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	n	0	n	n
a	0	0	a	a	a	a	a
b	0	n	a	b	a	b	b
c	0	0	a	a	c	c	c
d	0	n	a	b	c	d	d
1	0	n	a	b	c	d	1

Since for $b, c \in L, (b \to c) \vee (c \to b) = c \vee b = d \neq 1$, then L is not a MTL-algebra.

$(v) \Rightarrow (i)$ Consider $x, y \in L$. Since $x \odot (x \to y) \leq y \Rightarrow x \to [x \odot (x \to y)] \leq x \to y$. Since $x \to y \leq x \to [x \odot (x \to y)]$ we deduce that $x \to [x \odot (x \to y)] = x \to y$.

FIGURE 3. *Example* 3.5 *Example* 3.6 *Example* 3.7



From $x \to (x \wedge y) = (x \to x) \wedge (x \to y) = x \to y$ and $x \to [x \odot (x \to y)] = x \to y$ we deduce that

$$x \to (x \wedge y) = x \to [x \odot (x \to y)]. \tag{23}$$

But $x \wedge y,\ x \odot (x \to y) \leq x$, so $x \wedge y = x \odot (x \to y)$, that is, L is *divisible*, hence *distributive.*

$(i) \not\Rightarrow (v)$ Consider the following counterexample:

Example 3.4. *Let* $L = \{0, n, a, b, c, d, 1\}$ *with* $0 < n < a < b < c, d < 1$*, but* c *and* d *are incomparable. Then (*[11]*, pag.* 229*)* L *becomes a distributive residuated lattice relative to the following operations:*

$\to$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	d	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	n	n	a	1	1	1	1
c	0	n	a	d	1	d	1
d	n	n	a	c	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	0	n	0	n
a	0	0	a	a	a	a	a
b	0	0	a	b	b	b	b
c	0	n	a	b	c	b	c
d	0	0	a	b	b	d	d
1	0	n	a	b	c	d	1

Since for $a, n, 0 \in L,\ a \to n = a \to 0 = n,\ a \geq n, 0$ and $n \neq 0$, we have that L not verify (v).

$(vi) \Rightarrow (i)$ Since for any $x, y \in L$,

$$x \vee y = (x \to y) \to y = (y \to x) \to x,$$

then by Theorem 2.2, L becomes a *Wajsberg algebra*, so it is a *distributive MTL-algebra* (see [16]).

$(i) \not\Rightarrow (vi)$ Consider the following counterexample:

Example 3.5. *Let* $L = \{0, a, b, c, 1\}$ *with* $0 < a, b < c < 1$*, but* a *and* b *are incomparable. Then (*[4]*, pag.* 239*)* L *becomes a distributive residuated lattice relative to the following operations:*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Since for $a, b \in L \Rightarrow a \vee b = c,\ (a \to b) \to b = b \to b = 1$ and $c \neq 1$, then L not verify (vi).

$(vii) \Rightarrow (i)$ it can be proved the fact that $(vii) \Rightarrow (iii)$: We have by (23), $x \to [x \odot (x \to y)] = x \to (x \wedge y)$. We shall prove that

$$x^* \odot (x \wedge y) = x^* \odot [x \odot (x \to y)]. \tag{24}$$

From $x \odot (x \to y) \leq x, y \Rightarrow x \odot (x \to y) \leq x \wedge y$, hence, by (11) and the operation $\odot$ is commutative,

$$x^* \odot [x \odot (x \to y)] \leq x^* \odot (x \wedge y). \tag{25}$$

Conversely, by (15), $x^* \odot (x \wedge y) \leq (x^* \odot x) \wedge (x^* \odot y) \overset{(16)}{=} 0 \wedge (x^* \odot y) = 0$. Hence $x^* \odot (x \wedge y) = 0$. It follows,

$$x^* \odot (x \wedge y) \leq x^* \odot [x \odot (x \to y)]. \tag{26}$$

Therefore, by (25) and (26), (24) holds. It follows, by (vii), that $x \wedge y = x \odot (x \to y)$. That is, L is *divisible*, hence *distributive.*

$(i) \not\Rightarrow (vii)$ Consider the following counterexample:

Example 3.6. *Let $L = \{0, a, b, c, d, 1\}$ with $0 < a < b < c, d < 1$, but c and d are incomparable. Then ([11], pag. 228) L becomes a distributive residuated lattice relative to the following operations:*

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Since for $a, b, c \in L,\ a^* = a \to 0 = d,\ a^* \odot b = d \odot b = b = d \odot c = a^* \odot c,\ a \to b = 1 = a \to c$ and $b \neq c$.

$(viii) \Rightarrow (i)$ Since $x \to (y \to z) = (x \odot y) \to z$ and $x \to (y \to z) = (x \to y) \to (x \to z)$ from

$$(x \to y) \to (x \to z) = ((x \odot (x \to y)) \to z,$$

we deduce that $x \odot y = x \odot (x \to y)$, so L is divisible, hence distributive.

$(i) \not\Rightarrow (viii)$ for this we offer the example from the proof of $(i) \not\Rightarrow (v)$, because for $x = y = n$ and $z = 0$ we have $n \to (n \to 0) = n \to d = 1,\ (n \to n) \to (n \to 0) = 1 \to d = d$ and $1 \neq d$.

□

Remark 3.1. *There are distributive residuated lattices which don't verify the conditions $(ii) - (viii)$ from the Theorem 3.1, like in the example from below.*

Example 3.7. *Let* $L=\{0,n,a,b,c,d,e,f,m,1\}$ *with* $0<n<a<c<e<m<1$, $0<n<b<d<f<m<1$ *and the elements* $\{a,b\}$, $\{c,d\}$, $\{e,f\}$ *are pairwise incomparable. Then* L *becomes a distributive residuated lattice relative to the following operations:*

$\rightarrow$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

$\odot$	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

Then
(ii) $n\odot n=0\neq n=n\wedge n$;
(iii) $a\wedge n=n\neq 0=a\odot f=a\odot(a\rightarrow n)$;
(iv) $(b\rightarrow a)\vee(a\rightarrow b)=e\vee f=m\neq 1$;
(v) $d\geq b,c$ and $d\rightarrow b=d\rightarrow c=e$, but $b\neq c$;
(vi) $m\vee d=m\neq 1=(m\rightarrow d)\rightarrow d$;
(vii) $c^{*}=c\rightarrow 0=d$ and $d\odot m=d\odot 1=d,\ c\rightarrow m=c\rightarrow 1=1$, but $m\neq 1$;
$(viii)$ $n\rightarrow(n\rightarrow 0)=n\rightarrow m=1\neq m=1\rightarrow m=(n\rightarrow n)\rightarrow(n\rightarrow 0)$.

Remark 3.2. Unfortunately, we did not find any references in the literature to the necessary and sufficient conditions for the distributivity of the residuated lattices.

In consequence we propose the following
Open problem : Find necessary conditions for a residuated lattice to become distributive.

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