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Localization of MTL - algebras

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Abstract. The aim of the present paper is to define the localization MTL - algebra of a $MTL-$ algebra A with respect to a topology $\mathcal{F}$ on A. In the last part of the paper is proved that the maximal MTL - algebra of quotients (defined in [15]) and the MTL - algebra of fractions relative to an $\wedge-$ closed system (defined in [3]) are MTL - algebras of localization.

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Basic Fuzzy logic (BL from now on) is the many-valued residuated logic introduced by Hájek in [10] to cope with the logic of continuous t-norms and their residua. Monoidal logic (ML from now on), is a logic whose algebraic counterpart is the class of residuated; MTL-algebras (see [5]) are algebraic structures for the Esteva-Godo monoidal t-norm based logic (MTL), a many-valued propositional calculus that formalizes the structure of the real unit interval $[0, 1]$, induced by a left–continuous t-norm. MTL algebras were independently introduced in [6] under the name *weak-BL algebras*.

A remarkable construction in ring theory is the *localization ring* $A_{\mathcal{F}}$ associated with a Gabriel topology $\mathcal{F}$ on a ring A.

Using the model of localization ring, in [9], G. Georgescu defined for a bounded distributive lattice L *the localization lattice* $L_{\mathcal{F}}$ *of* L *with respect to a topology* $\mathcal{F}$ *on* L and prove that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of regular ideals); analogous results we have for lattices of fractions of bounded distributive lattices relative to $\wedge-$ closed systems.

The main aim of this paper is to develop a theory of localization for MTL - algebras. Since $BL-$ algebras are particular classes of $MTL-$ algebras, the results of this paper generalize a part of the results from [2] for $BL-$ algebras. The main difference is that the axiom $x \odot (x \to y) = x \wedge y$ is not valid for MTL-algebras.

1. Definitions and preliminaries

Definition 1.1. A *residuated lattice* ([1], [18]) is an algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:
(a_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice relative to the order $\leq$;
(a_2) $(A, \odot, 1)$ is a commutative ordered monoid;
(a_3) $(\odot, \to)$ is an adjoint pair, i.e. $z \leq x \to y$ iff $x \odot z \leq y$ for every $x, y, z \in A$.

The class $\mathcal{RL}$ of residuated lattices is equational (see [11]).
For examples of residuated lattices see [3] and [18].

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In what follows by A we denote the universe of a residuated lattice. For $x \in A$, we denote $x^* = x \to 0$ and $(x^*)^* = x^{**}$.

We review some rules of calculus for residuated lattices A used in this paper:

Theorem 1.1. *([1], [18]) Let $x, y, z \in A$. Then we have the following:*

(c_1) $1 \to x = x, x \to x = 1, y \leq x \to y, x \odot (x \to y) \leq y, x \to 1 = 1, 0 \to x = 1, x \odot 0 = 0$;

(c_2) $x \leq y$ *iff* $x \to y = 1$;

(c_3) $x \leq y$ *implies* $x \odot z \leq y \odot z, z \to x \leq z \to y$ *and* $y \to z \leq x \to z$;

(c_4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$, *so* $(x \odot y)^* = x \to y^* = y \to x^*$;

(c_5) $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \leq y^*$;

If A is a complete residuated lattice and $(y_i)_{i \in I}$ is a family of elements of A, then:

(c_6) $x \odot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \odot y_i)$;

(c_7) $x \to (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \to y_i)$.

By $B(A)$ we denote the set of all complemented elements in the lattice $L(A) = (A, \wedge, \vee, 0, 1)$. Complements are generally not unique, unless the lattice is distributive; in the case of residuated lattices, however, although the underlying lattices need not be distributive, the complements are unique ([8]). Also, if b is the complement of a, then a is the complement of b, $b = a^*, a^2 = a$ and $a^{**} = a$ ([1], [3]). So, $B(A)$ is a Boolean subalgebra of A, called the *Boolean center* of A.

Theorem 1.2. *([3]) For $e \in A$ the following assertions are equivalent:*

(i) $e \in B(A)$;

(ii) $e \vee e^* = 1$.

Theorem 1.3. *([3]) If $e, f \in B(A)$ and $x, y \in A$, then:*

(c_8) $e \odot x = e \wedge x$;

(c_9) $x \odot (x \to e) = e \wedge x, e \odot (e \to x) = e \wedge x$;

(c_{10}) $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)]$;

(c_{11}) $x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)]$.

Definition 1.2. *([5], [6], [7])* A $MTL-$ algebra is a residuated lattice satisfying the *preliniarity equation*:

(c_{12}) $(x \to y) \vee (y \to x) = 1$.

The variety of $MTL-$ algebras will be denoted by $\mathcal{MTL}$.

Proposition 1.1. *([5]) For a residuated lattice, the following conditions are equivalent:*

(i) $A \in \mathcal{MTL}$;

(ii) *A is a subdirect product of linearly ordered residuated lattices;*

(iii) *For every $x, y, z \in A$ we have:*

(c_{13}) $x \to (y \vee z) = (x \to y) \vee (x \to z)$;

(iv) *For every $x, y, z \in A$ we have:*

(c_{14}) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

Corollary 1.1. *([5]) Let $A \in \mathcal{MTL}$. Then for every $x, y, z \in A$ we have:*

(c_{15}) $(x \wedge y)^* = x^* \vee y^*$;

(c_{16}) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;

(c_{17}) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

(c_{18}) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$.

Remark 1.1. *From* (c_{18}) *we deduce that a* $MTL-$ *algebra is a semi-Boolean lattice (see* [13]*).*

Remark 1.2. *Every linearly ordered residuated lattice is a* $MTL-$ *algebra. A* $MTL-$ *algebra* A *is a* $BL-$ *algebra iff in* A *is verified the divisibility condition:* $x \odot (x \to y) = x \wedge y$. *So,* $BL-$ *algebras are examples of* $MTL-$ *algebras; for an example of* $MTL-$ *algebra which is not* $BL-$ *algebra consider the residuated lattice defined on the unit interval* $A = [0, 1]$, *for all* $x, y \in A$, *such that*

$$x \odot y = 0 \text{ if } x + y \leq \frac{1}{2} \text{ and } x \wedge y \text{ elsewhere,}$$

$$x \to y = 1 \text{ if } x \leq y \text{ and } \max\left\{\frac{1}{2} - x, y\right\} \text{ elsewhere (see [18], p.16).}$$

Let $0 < y < x$, $x + y < \frac{1}{2}$. *Then* $y < \frac{1}{2} - x$ *and* $0 \neq y = x \wedge y$, *but* $x \odot (x \to y) = x \odot (\frac{1}{2} - x) = 0$. *This residuated lattice is a chain, so is a* $MTL-$*algebra, but the divisibility condition not hold.*

Definition 1.3. Let $(P, \leq)$ an ordered set. A nonempty subset I of P is called *order ideal* if, whenever $x \in I, y \in P$ and $y \leq x$, we have $y \in I$; we denote by $I(P)$ the set of all order ideals of P.

For a MTL-algebra A we denote by $Id(A)$ the set of all ideals of the lattice $L(A)$.

Remark 1.3. *Clearly,* $Id(A) \subseteq I(A)$ *and if* $I_1, I_2 \in I(A)$, *then* $I_1 \cap I_2 \in I(A)$. *Also, if* $I \in I(A)$, *then* $0 \in I$.

2. Topologies on a MTL-algebra

Definition 2.1. A non-empty set $\mathcal{F}$ of elements $I \in I(A)$ will be called a *topology* on A if the following axioms hold:

(a_4) If $I_1 \in \mathcal{F}, I_2 \in I(A)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$ (hence $A \in \mathcal{F}$);

(a_5) If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$.

Remark 2.1.
1. $\mathcal{F}$ *is a topology on* A *iff* $\mathcal{F}$ *is a filter of the lattice of power set of* A*; for this reason a topology on* $I(A)$ *is usually called a Gabriel filter on* $I(A)$.
2. *Clearly, if* $\mathcal{F}$ *is a topology on* A, *then* $(A, \mathcal{F} \cup \{\emptyset\})$ *is a topological space.*

Any intersection of topologies on A is a topology; so, the set $T(A)$ of all topologies of A is a complete lattice with respect to inclusion.

Example 2.1. *If* $I \in I(A)$, *then the set* $\mathcal{F}(I) = \{I' \in I(A) : I \subseteq I'\}$ *is a topology on* A.

Remark 2.2. *If in particular* $A = [0, 1]$ *is the* MTL *- algebra from Remark 1.2, then* $I(A) = \{[0, x] : x \in A\}$. *For* $x = 0$, $\mathcal{F}(\{0\}) = I(A)$; *for* $x \in (0, 1), \mathcal{F}([0, x]) = \{[0, y] : x \leq y, y \in A\}$.

Definition 2.2. *(*[15]*)* A non-empty set $I \subseteq A$ will be called *regular* if for every $x, y \in A$ such that $e \wedge x = e \wedge y$ for every $e \in I \cap B(A)$, then $x = y$.

Example 2.2. *If we denote* $R(A) = \{I \subseteq A : I$ *is a regular subset of* $A\}$, *then* $I(A) \cap R(A)$ *is a topology on* A.

Remark 2.3. *Clearly, if* $A = [0, 1]$ *is the* MTL *-algebra from Remark 1.2, since* $B(A) = \{0, 1\} = L_2$ *then only* $I = A$ *is a regular subset of* A *(*$I = [0, x]$ *with* $x \neq 1$ *are non regular because contain* 0 *and for example we have* $0 \wedge a = 0 \wedge b$ *for every* $a, b \in A$ *and* $a \neq b$*). So, in this case* $\mathcal{F} = I(A) \cap R(A) = \{A\}$.

Example 2.3. *A nonempty set $I \subseteq A$ will be called* dense *(see* [9]*) if for $x \in A$ such that $e \wedge x = 0$ for every $e \in I \cap B(A)$, then $x = 0$. If we denote by $D(A)$ the set of all dense subsets of A, then $R(A) \subseteq D(A)$ and $\mathcal{F} = I(A) \cap D(A)$ is a topology on A.*

Remark 2.4. *As above, for $MTL-$ algebra $A = [0, 1]$ from Remark 1.2, $D(A) = \{A\}$ (because $I \in D(A)$ if $1 \in I$).*

Definition 2.3. *(*[3]*)* A subset $S \subseteq A$ is called $\wedge-$ *closed* if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

Example 2.4. *For any $\wedge-$ closed subset S of A, the set $\mathcal{F}_S = \{I \in I(A) : I \cap S \cap B(A) \neq \emptyset\}$ is a topology on A .*

Remark 2.5. *In the case of MTL- algebra $A = [0, 1]$ from Remark 1.2, $S \subseteq [0, 1]$ is a $\wedge-$ closed subset if $1 \in S$. Since $B(A) = \{0, 1\} = L_2$ then for $S \subseteq A$ a $\wedge-$ closed system, $\mathcal{F}_S = \{I \in I(A) : I \cap S \cap \{0, 1\} \neq \emptyset\}$.*

1. *If S is a $\wedge-$closed systems of A such that $0 \in S$ we have $I \cap S \cap B(A) \neq \emptyset$ for every $I \in I(A)$, so $\mathcal{F}_S = I(A)$.*
2. *If $0 \notin S$ then $\mathcal{F}_S = \{A\}$ (because, if $I \in I(A)$ and $1 \in I$ implies $I = A$).*

3. $\mathcal{F}$-multipliers and localization MTL-algebras

Let $\mathcal{F}$ be a topology on a $MTL-$algebra A and we consider the relation $\theta_{\mathcal{F}}$ of A defined in the following way: $(x, y) \in \theta_{\mathcal{F}} \Leftrightarrow$ there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for any $e \in I \cap B(A)$.

Lemma 3.1. *$\theta_{\mathcal{F}}$ is a congruence on A.*

Proof. See [2] for the case of $BL-$ algebras. □

We shall denote by $a/\theta_{\mathcal{F}}$ the congruence class of an element $a \in A$ and by $p_{\mathcal{F}} : A \to A/\theta_{\mathcal{F}}$ the canonical morphism of MTL-algebras.

Proposition 3.1. *For $a \in A$, $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$ iff there exists $I \in \mathcal{F}$ such that $a \vee a^* \geq e$ for every $e \in I \cap B(A)$. So, if $a \in B(A)$, then $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

Proof. Using Theorem 1.2, for $a \in A$, we have $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}) \Leftrightarrow a/\theta_{\mathcal{F}} \vee (a/\theta_{\mathcal{F}})^* = 1/\theta_{\mathcal{F}} \Leftrightarrow (a \vee a^*)/\theta_{\mathcal{F}} = 1/\theta_{\mathcal{F}} \Leftrightarrow$ there exist $I \in \mathcal{F}$ such that $(a \vee a^*) \wedge e = 1 \wedge e = e$, for every $e \in I \cap B(A) \Leftrightarrow a \vee a^* \geq e$, for every $e \in I \cap B(A)$. If $a \in B(A)$, then for every $I \in \mathcal{F}$, $1 = a \vee a^* \geq e$, for every $e \in I \cap B(A)$, hence $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$. □

Corollary 3.1. *If $\mathcal{F} = I(A) \cap R(A)$, then for $a \in A$, $a \in B(A)$ iff $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

Definition 3.1. Let $\mathcal{F}$ be a topology on A. A $\mathcal{F}-$ *multiplier* is a mapping $f : I \to A/\theta_{\mathcal{F}}$ where $I \in \mathcal{F}$ and for every $x \in I$ and $e \in B(A)$ the following axioms are fulfilled:

(a_6) $f(e \odot x) = e/\theta_{\mathcal{F}} \wedge f(x) = e/\theta_{\mathcal{F}} \odot f(x)$;
(a_7) $f(x) \leq x/\theta_{\mathcal{F}}$;
(a_8) $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f(x)) = f(x)$.

Remark 3.1. *If A is a BL algebra, then the axiom (a_8) is a consequence of (a_7) (because in this case $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f(x)) = x/\theta_{\mathcal{F}} \wedge f(x) \overset{a_7}{=} f(x)$, for every $x \in I$).*

By $dom(f) \in \mathcal{F}$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify language, we will use $\mathcal{F}-$ *multiplier* instead *partial $\mathcal{F}-$ multiplier*, using *total* to indicate that the domain of a certain $\mathcal{F}-$ multiplier is A.

If $\mathcal{F} = \{A\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of A so a $\mathcal{F}-$ multiplier is a total multiplier in sense of [15], Definition 3, which verify the conditions M_1, M_2 and M_3.

The maps $\mathbf{0}, \mathbf{1} : A \to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in A$ are $\mathcal{F}-$ multipliers in the sense of Definition 3.1.

Also, for $a \in B(A)$, $f_a : A \to A/\theta_{\mathcal{F}}$ defined by $f_a(x) = a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$ for every $x \in A$, is a $\mathcal{F}-$ multiplier. If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

We shall denote by $M(I, A/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}$ and $M(A/\theta_{\mathcal{F}}) = \bigcup_{I \in \mathcal{F}} M(I, A/\theta_{\mathcal{F}})$. If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$ we have a canonical mapping $\varphi_{I_1,I_2} : M(I_2, A/\theta_{\mathcal{F}}) \to M(I_1, A/\theta_{\mathcal{F}})$ defined by $\varphi_{I_1,I_2}(f) = f_{|I_1}$ for $f \in M(I_2, A/\theta_{\mathcal{F}})$. Let us consider the directed system of sets

$\langle \{M(I, A/\theta_{\mathcal{F}})\}_{I \in \mathcal{F}},\ \{\varphi_{I_1,I_2}\}_{I_1,I_2 \in \mathcal{F}, I_1 \subseteq I_2} \rangle$ and denote by $A_{\mathcal{F}}$ the inductive limit (in the category of sets) $A_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, A/\theta_{\mathcal{F}})$. For any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$ we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $A_{\mathcal{F}}$.

Remark 3.2. *If $f_i : I_i \to A/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}-$ multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $A_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.*

Proposition 3.2. *If $I_1, I_2 \in \mathcal{F}$ and $f_i \in M(I_i, A/\theta_{\mathcal{F}}), i = 1, 2$, then*

(c_{19}) $f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] = f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)]$, *for every $x \in I_1 \cap I_2$.*

Proof. For $x \in I_1 \cap I_2$ we have $f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{a_8}{=} x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \odot (x/\theta_{\mathcal{F}} \to f_2(x)) = [x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_2(x))] \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \overset{a_8}{=} f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)]$. □

Let $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), $\mathcal{F}-$multipliers. Let us consider the mappings $f_1 \wedge f_2, f_1 \vee f_2, f_1 \odot f_2, f_1 \to f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$ defined by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x), (f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$

$$(f_1 \odot f_2)(x) = f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{c_{19}}{=} f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)],$$

$$(f_1 \to f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)],$$

for any $x \in I_1 \cap I_2$, and let

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2), \widehat{(I_1, f_1)} \curlyvee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2),$$

$$\widehat{(I_1, f_1)} \otimes \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \odot f_2), \widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \to f_2).$$

Clearly, the definitions of the operations $\curlywedge, \curlyvee, \otimes$ and $\longmapsto$ on $A_{\mathcal{F}}$ are correct.

Lemma 3.2. $f_1 \wedge f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. It is suffice to verify only a_8 (for a_6 and a_7, see [2]).

For every $x \in I_1 \cap I_2$ we have $x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (f_1 \wedge f_2)(x)] = x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (f_1(x) \wedge f_2(x))] \overset{c_7}{=} x/\theta_{\mathcal{F}} \odot [(x/\theta_{\mathcal{F}} \to f_1(x)) \wedge (x/\theta_{\mathcal{F}} \to f_2(x))] \overset{c_{16}}{=} [x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x))] \wedge [x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_2(x))] \overset{a_8}{=} f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x)$, that is, $f_1 \wedge f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. □

Lemma 3.3. $f_1 \vee f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. The axioms a_6 and a_7 are verified as in the case of $BL-$algebras (see [2]). To verify a_8, let $x \in I_1 \cap I_2$. Then $x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (f_1 \vee f_2)(x)] = x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (f_1(x) \vee f_2(x))] \overset{c_{13}}{=} x/\theta_{\mathcal{F}} \odot [(x/\theta_{\mathcal{F}} \to f_1(x)) \vee (x/\theta_{\mathcal{F}} \to f_2(x))] \overset{c_6}{=} [x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x))] \vee [x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_2(x))] \overset{a_8}{=} f_1(x) \vee f_2(x) = (f_1 \vee f_2)(x)$, that is, $f_1 \vee f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. □

Lemma 3.4. $f_1 \odot f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. By using c_{10}, a_6 and a_7 are verified as in the case of $BL-$algebras (see [2]). For a_8 let $x \in I_1 \cap I_2$ and denote $f = f_1 \odot f_2$.

To prove the equality $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f(x)) = f(x)$ it is suffice (using c_1) to prove that $f(x) \leq x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f(x))$. We have $f(x) = f_1(x) \odot (x/\theta_{\mathcal{F}} \to f_2(x)) = x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \odot (x/\theta_{\mathcal{F}} \to f_2(x))$ and $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f(x)) = x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (f_1(x) \odot (x/\theta_{\mathcal{F}} \to f_2(x)))] = x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \odot (x/\theta_{\mathcal{F}} \to f_2(x)))]$. So, to prove that $f(x) \leq x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f(x))$ it is suffice to prove that $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \odot (x/\theta_{\mathcal{F}} \to f_2(x)) \leq x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \odot (x/\theta_{\mathcal{F}} \to f_2(x)))]$, that is, $\alpha \leq x/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot \alpha)$ (with $\alpha \overset{not}{=} (x/\theta_{\mathcal{F}} \to f_1(x)) \odot (x/\theta_{\mathcal{F}} \to f_2(x))$), which is clearly, since $\alpha \to [x/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot \alpha)] \overset{c_4}{=} (\alpha \odot x/\theta_{\mathcal{F}}) \to (x/\theta_{\mathcal{F}} \odot \alpha) = 1$, that is, $f_1 \odot f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. □

Lemma 3.5. $f_1 \to f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. By using c_{10}, a_6 and a_7 are verified as in the case of $BL-$algebras (see [2]). For a_8, let $x \in I_1 \cap I_2$ and denote $f = f_1 \to f_2$; then $f(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)]$. We have $f_1(x) \to f_2(x) \leq x/\theta_{\mathcal{F}} \to [x/\theta_{\mathcal{F}} \odot (f_1(x) \to f_2(x))]$, hence $x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)] \leq x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot (f_1(x) \to f_2(x)))] \Leftrightarrow f(x) \leq x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to f(x)] \overset{c_1}{\Leftrightarrow} f(x) = x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to f(x)]$, that is, $f_1 \to f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. □

Proposition 3.3. $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \otimes, \longmapsto, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ *is a MTL-algebra.*

Proof. We verify the axioms of MTL-algebras.

(a_1). Obviously $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ is a bounded lattice.

(a_2). As in the case of $BL-$ algebras (see [2]), by using c_{19} and a_8.

(a_3). $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1, 2, 3$.

Since $f_1 \leq f_2 \to f_3$ for $x \in I_1 \cap I_2 \cap I_3$ we have $f_1(x) \leq (f_2 \to f_3)(x) \Leftrightarrow f_1(x) \leq x/\theta_{\mathcal{F}} \odot [f_2(x) \to f_3(x)]$. So, by c_3, $f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \leq x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \odot [f_2(x) \to f_3(x)] \overset{a_8}{\Leftrightarrow} f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \leq f_2(x) \odot [f_2(x) \to f_3(x)] \leq f_3(x) \Leftrightarrow (f_1 \odot f_2)(x) \leq f_3(x)$, for every $x \in I_1 \cap I_2 \cap I_3$, that is, $f_1 \odot f_2 \leq f_3$. Conversely, if $(f_1 \odot f_2)(x) \leq f_3(x)$ we have $f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)] \leq f_3(x)$, for every $x \in I_1 \cap I_2 \cap I_3$. Obviously, $x/\theta_{\mathcal{F}} \to f_1(x) \leq f_2(x) \to f_3(x) \overset{c_3}{\Leftrightarrow} x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \leq x/\theta_{\mathcal{F}} \odot (f_2(x) \to f_3(x)) \Leftrightarrow f_1(x) \leq (f_2 \to f_3)(x)$. So $f_1 \leq f_2 \to f_3$ iff $f_1 \odot f_2 \leq f_3$ for all $f_1, f_2, f_3 \in M(A/\theta_{\mathcal{F}})$ and so $\widehat{(I_1, f_1)} \leq \widehat{(I_2, f_2)} \longmapsto \widehat{(I_3, f_3)}$ iff $\widehat{(I_2, f_2)} \otimes \widehat{(I_1, f_1)} \leq \widehat{(I_3, f_3)}$. Since the preliniarity equation c_{12} is proved as in the case of $BL-$ algebras (see [2]) we deduce that $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \otimes, \longmapsto, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ is a MTL-algebra. □

Remark 3.3. $(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \odot, \to, \mathbf{0}, \mathbf{1})$ *is a MTL-algebra.*

Definition 3.2. The MTL-algebra $A_{\mathcal{F}}$ will be called the *localization MTL-algebra* of A with respect to the topology $\mathcal{F}$.

Definition 3.3. ([5], [7]) A MTL-algebra A is called

(i) An *$IMTL$-algebra (involutive algebra)* if it satisfies the equation
(I) $x^{**} = x$;

(ii) a *$SMTL$-algebra* if it satisfies the equation
(S) $x \wedge x^* = 0$;

(iii) a *$WNM-$algebra (weak nilpotent minimum)* if it satisfies the equation
(W) $(x \odot y)^* \vee [(x \wedge y) \to (x \odot y)] = 1$;

(iv) a $\Pi SMTL-$ *algebra* if it is a $SMTL-$algebra satisfying the equation
(Π) $[z^{**} \odot ((x \odot z) \to (y \odot z))] \to (x \to y) = 1$.

Theorem 3.1. *If MTL-algebra A is a $BL-$algebra (resp. an $IMTL$-algebra, a $SMTL$-algebra, a WNM-algebra, a $\Pi SMTL$-algebra), then $A_{\mathcal{F}}$ is also a $BL-$algebra (resp. an $IMTL$-algebra, a $SMTL$-algebra, a WNM-algebra, a $\Pi SMTL$-algebra).*

Proof. Suppose that A is a $BL-$algebra (see Remark 1.2). Since for $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in A_{\mathcal{F}}$, where $I_i \in \mathcal{F}$, $i = 1, 2$, $\widehat{(I_1, f_1)} \otimes (\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)}) = \widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} \Leftrightarrow (I_1 \cap I_2, \widehat{f_1 \odot (f_1 \to f_2)}) = (I_1 \cap I_2, \widehat{f_1 \wedge f_2})$, to prove that $A_{\mathcal{F}}$ is a $BL-$algebra, it is suffice to prove that for every $x \in I_1 \cap I_2$, $(f_1 \odot (f_1 \to f_2))(x) = (f_1 \wedge f_2)(x) \Leftrightarrow (f_1 \to f_2)(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)] = f_1(x) \wedge f_2(x) \Leftrightarrow x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)] \odot [x/\theta_{\mathcal{F}} \to f_1(x)] = f_1(x) \wedge f_2(x) \Leftrightarrow (x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to f_1(x)]) \odot [f_1(x) \to f_2(x)] = f_1(x) \wedge f_2(x) \overset{a_8}{\Leftrightarrow} f_1(x) \odot [f_1(x) \to f_2(x)] = f_1(x) \wedge f_2(x)$, which is true because A is supposed a $BL-$algebra, so $A/\theta_{\mathcal{F}}$ is also a $BL-$algebra.

Suppose that A is an $IMTL-$algebra; obviously, $A/\theta_{\mathcal{F}}$ is also an $IMTL-$algebra. For $\alpha = \widehat{(I, f)} \in A_{\mathcal{F}}$, where $I \in \mathcal{F}$, we have $f^{**} = (f \to \mathbf{0}) \to \mathbf{0}$ so $f^{**}(x) = x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f(x))^*]^* \overset{c_4}{=} x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (f(x))^{**}] \overset{(I)}{=} x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to f(x)] \overset{a_8}{=} f(x)$, for $x \in I$, hence $\alpha^{**} = \alpha$, that is, $A_{\mathcal{F}}$ is an $IMTL$-algebra.

Suppose that A is a $SMTL-$algebra; obviously, $A/\theta_{\mathcal{F}}$ is also a $SMTL-$algebra. If $\alpha = \widehat{(I, f)} \in A_{\mathcal{F}}$, then the equation $\alpha \curlywedge \alpha^* = \mathbf{0}$ is equivalent with $f \wedge (f \to \mathbf{0}) = \mathbf{0} \Leftrightarrow f(x) \wedge [x/\theta_{\mathcal{F}} \odot (f(x))^*] = 0$, for every $x \in I$, which is clearly (since $f(x) \wedge [x/\theta_{\mathcal{F}} \odot (f(x))^*] \leq f(x) \wedge (f(x))^* = 0$), hence $\alpha \curlywedge \alpha^* = \mathbf{0}$, that is, $A_{\mathcal{F}}$ is a $SMTL$-algebra.

Suppose that A is a $WNM-$algebra. Let $\alpha = \widehat{(I, f)}, \beta = \widehat{(J, g)}$ and denote $a = f(x), b = g(x)$ for $x \in I \cap J$. We have $(\alpha \otimes \beta)^* \curlyvee ((\alpha \curlywedge \beta) \longmapsto (\alpha \otimes \beta)) = [I \cap J, \widehat{(f \odot g)^* \vee ((f \wedge g) \to (f \odot g))}]$ and $((f \odot g)^* \vee ((f \wedge g) \to (f \odot g)))(x) = ((f \odot g)^*(x)) \vee (x/\theta_{\mathcal{F}} \odot ((f \wedge g)(x) \to (f \odot g)(x))) = (x/\theta_{\mathcal{F}} \odot (a \odot (x/\theta_{\mathcal{F}} \to b))^*) \vee (x/\theta_{\mathcal{F}} \odot ((a \wedge b) \to (a \odot (x/\theta_{\mathcal{F}} \to b)))) \overset{c_6}{=} x/\theta_{\mathcal{F}} \odot ((a \odot (x/\theta_{\mathcal{F}} \to b))^* \vee ((a \wedge b) \to (a \odot (x/\theta_{\mathcal{F}} \to b))))$.

Since $b \leq x/\theta_{\mathcal{F}} \to b$ we deduce that $a \wedge b \leq a \wedge (x/\theta_{\mathcal{F}} \to b)$, hence, using c_3, $(a \wedge (x/\theta_{\mathcal{F}} \to b)) \to (a \odot (x/\theta_{\mathcal{F}} \to b)) \leq (a \wedge b) \to (a \odot (x/\theta_{\mathcal{F}} \to b))$.

Since A is supposed a $WNM-$algebra we deduce that $A/\theta_{\mathcal{F}}$ is also a $WNM-$algebra, so we obtain $1/\theta_{\mathcal{F}} = (a \odot (x/\theta_{\mathcal{F}} \to b))^* \vee ((a \wedge (x/\theta_{\mathcal{F}} \to b)) \to (a \odot (x/\theta_{\mathcal{F}} \to b))) \leq (a \odot (x/\theta_{\mathcal{F}} \to b))^* \vee ((a \wedge b) \to (a \odot (x/\theta_{\mathcal{F}} \to b)))$, hence $(a \odot (x/\theta_{\mathcal{F}} \to b))^* \vee ((a \wedge b) \to (a \odot (x/\theta_{\mathcal{F}} \to b))) = 1/\theta_{\mathcal{F}}$. Then $((f \odot g)^* \vee ((f \wedge g) \to (f \odot g)))(x) = x/\theta_{\mathcal{F}} \odot 1/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} = \mathbf{1}(x) \Leftrightarrow (\alpha \otimes \beta)^* \curlyvee ((\alpha \curlywedge \beta) \longmapsto (\alpha \otimes \beta)) = \mathbf{1}$, that is $A_{\mathcal{F}}$ is a $WNM-$algebra.

Suppose now A is a $\Pi SMTL-$algebra, so $A/\theta_{\mathcal{F}}$ is also a $\Pi SMTL-$algebra. From the condition $x \wedge x^* = 0$ $(x \in A)$, we deduce that $x^* \vee x^{**} \overset{c_{15}}{=} (x \wedge x^*)^* = 0^* = 1$, that is, $x^* \in B(A)$. Let $\alpha = \widehat{(I, f)}, \beta = \widehat{(J, g)}, \gamma = \widehat{(K, h)} \in A_{\mathcal{F}}$. Consider $x \in I \cap J \cap K$ and denote $a = f(x), b = g(x)$ and $c = h(x)$. Then $h^{**}(x) = x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to c^{**}) \overset{c_9}{=} x/\theta_{\mathcal{F}} \wedge c^{**} \overset{c_8}{=} x/\theta_{\mathcal{F}} \odot c^{**}$, $[h^{**} \odot ((f \odot h) \to (g \odot h))](x) = [x/\theta_{\mathcal{F}} \to h^{**}(x)] \odot [x/\theta_{\mathcal{F}} \odot [(f \odot h)(x) \to (g \odot h)(x)]] = [x/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot c^{**})] \odot [x/\theta_{\mathcal{F}} \odot [((x/\theta_{\mathcal{F}} \to a) \odot c) \to ((x/\theta_{\mathcal{F}} \to b) \odot c)]] = [x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot c^{**}))] \odot [((x/\theta_{\mathcal{F}} \to a) \odot c) \to ((x/\theta_{\mathcal{F}} \to b) \odot c)] \overset{c_1}{\leq} (x/\theta_{\mathcal{F}} \odot c^{**}) \odot [((x/\theta_{\mathcal{F}} \to a) \odot c) \to ((x/\theta_{\mathcal{F}} \to b) \odot c)] = x/\theta_{\mathcal{F}} \odot [c^{**} \odot [((x/\theta_{\mathcal{F}} \to a) \odot c) \to ((x/\theta_{\mathcal{F}} \to b) \odot c)]] \overset{(\Pi)}{\leq} x/\theta_{\mathcal{F}} \odot [(x/\theta_{\mathcal{F}} \to a) \to (x/\theta_{\mathcal{F}} \to$

$b)] \overset{c_4}{=} x/\theta_{\mathcal{F}} \odot [(x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to a)) \to b] \overset{a_8}{=} x/\theta_{\mathcal{F}} \odot (a \to b) = (f \to g)(x)$, hence $[\gamma^{**} \otimes ((\alpha \otimes \gamma) \longmapsto (\beta \otimes \gamma))] \longmapsto (\alpha \longmapsto \beta) = \mathbf{1}$, so $A_{\mathcal{F}}$ is a $\Pi SMTL-$algebra. □

Remark 3.4. *If $MTL-$ algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a $BL-$ algebra (resp. an $IMTL$-algebra, a $SMTL$-algebra, a WNM-algebra, a $\Pi SMTL$-algebra), then $MTL-$ algebra $(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \odot, \to, \mathbf{0}, \mathbf{1})$ is a $BL-$ algebra (resp. an $IMTL$-algebra, a $SMTL$-algebra, a WNM-algebra, a $\Pi SMTL$-algebra).*

Remark 3.5. *If $MTL-$ algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a $BL-$ algebra in* [2] *will be called $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \otimes, \longmapsto, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ the localization BL-algebra of A with respect to the topology $\mathcal{F}$.*

Lemma 3.6. *Let the map $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ defined by $v_{\mathcal{F}}(a) = \widehat{(A, \overline{f_a})}$ for every $a \in B(A)$. Then:*

(i) $v_{\mathcal{F}}$ is a morphism of MTL-algebras;

(ii) For $a \in B(A)$, $\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}})$;

(iii) $v_{\mathcal{F}}(B(A)) \in R(A_{\mathcal{F}})$.

Proof. $(i), (iii)$. As in the case of $BL-$ algebras (see [2]).

(ii). For $a \in B(A)$ we have $a \vee a^* = 1$, hence $(a \wedge x) \vee [x \odot (a \wedge x)^*] \overset{c_{15}}{=} (a \wedge x) \vee [x \odot (a^* \vee x^*)] \overset{c_6}{=} (a \wedge x) \vee [(x \odot a^*) \vee (x \odot x^*)] \overset{c_5}{=} (a \wedge x) \vee [(x \odot a^*) \vee 0) \overset{c_8}{=} (a \wedge x) \vee (x \wedge a^*) \overset{c_{17}}{=} x \wedge (a \vee a^*) = x \wedge 1 = x$, for every $x \in A$. Since $A \in \mathcal{F}$ we deduce that $(a \wedge x)/\theta_{\mathcal{F}} \vee [x/\theta_{\mathcal{F}} \odot ((a \wedge x)/\theta_{\mathcal{F}})^*] = x/\theta_{\mathcal{F}}$ hence $\overline{f_a} \vee (\overline{f_a})^* = \mathbf{1}$, that is, $\widehat{(A, \overline{f_a})} \curlyvee \widehat{(A, \overline{f_a})}^* = \widehat{(A, \mathbf{1})}$, so $\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}})$. □

4. Applications

In the following we describe the localization MTL-algebra $A_{\mathcal{F}}$ in some special instances.

1. If $I \in I(A)$, and $\mathcal{F}$ is the topology $\mathcal{F}(I) = \{I' \in I(A) : I \subseteq I'\}$ (see Example 2.1), then $A_{\mathcal{F}}$ is isomorphic with $M(I, A/\theta_{\mathcal{F}})$ and $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ is defined by $v_{\mathcal{F}}(a) = \overline{f_{a|I}}$ for every $a \in B(A)$.

If I is a regular subset of A, then $\theta_{\mathcal{F}}$ is the identity, hence $A_{\mathcal{F}}$ is isomorphic with $M(I, A)$ (see [15], Definition 3, conditions M_1, M_2 and M_3), which in generally is not a Boolean algebra. For example, if $I = A = [0, 1]$ is the *Lukasiewicz structure* (see [18]) then $A_{\mathcal{F}}$ is not a Boolean algebra (see [2]).

Remark 4.1. *If consider $MTL-$algebra $A = [0, 1]$ from Remark 1.2, then*

1. *If $I = \{0\}$, then $\mathcal{F}(\{0\}) = I(A)$ (see Remark 2.2), so $A_{\mathcal{F}} \approx M(I, A/\theta_{\mathcal{F}}) = M(\{0\}, A/\theta_{\mathcal{F}}) = \mathbf{0}$.*
2. *If $I = A$, then $\mathcal{F}(A) = \{A\}$ and $\theta_{\mathcal{F}}$ is the identity, so $A_{\mathcal{F}} \approx M(A, A)$. Since $B(A) = L_2 = \{0, 1\}$, then $f \in M(A, A)$ iff $f(x) \leq x$ and $x \odot (x \to f(x)) = f(x)$, for every $x \in A$. So, $f(0) = 0$. For $x \geq \frac{1}{2}$ if we denote $f(x) = y$, then $y \leq x$ and we deduce that $x \odot (x \to f(x)) = x \odot (x \to y) = x \odot \max(\frac{1}{2} - x, y) = x \odot y = x \wedge y = y = f(x)$, so for $x \geq \frac{1}{2}, f \in M(A, A)$ iff $f(x) \leq x$. If consider $f \in A_{\mathcal{F}} = M(A, A)$ such that $f(\frac{3}{4}) = \frac{1}{2}$, then $(f \vee f^*)(\frac{3}{4}) = f(\frac{3}{4}) \vee f^*(\frac{3}{4}) = f(\frac{3}{4}) \vee [\frac{3}{4} \odot (f(\frac{3}{4}))^*] = \frac{1}{2} \vee [\frac{3}{4} \odot (\frac{1}{2})^*] = \frac{1}{2} \vee (\frac{3}{4} \odot 0) = \frac{1}{2} \vee 0 = \frac{1}{2} \neq \mathbf{1}(\frac{3}{4}) = \frac{3}{4}$, hence f is not a boolean element in $A_{\mathcal{F}}$ (hence in this case $A_{\mathcal{F}}$ is not a Boolean algebra). Also, f is not a principal multiplier (because $B(A) = \{0, 1\}$ hence the only principal multipliers are $f_0 = \mathbf{0}$ and $f_1 = \mathbf{1}$).*

3. *If* $I=[0,x]$ *with* $x\neq 0,1$, $\mathcal{F}(I)=\{[0,a]:x\leq a, a\in(0,1]\}$. *Since* $0\in[0,a], a\neq 1$ *and* $0\wedge x=0\wedge y$, *then* $(x,y)\in\theta_{\mathcal{F}}$ *for every* $x,y\in A$, *hence in this case* $A_{\mathcal{F}}\approx M(I,\mathbf{0})=0$.

2. **Main remark.** To obtain the maximal MTL -algebra of quotients $Q(A)$ as a localization relative to a topology $\mathcal{F}$ we have to develope another theory of multipliers (meaning we add new axioms for $\mathcal{F}$-multipliers).

Definition 4.1. Let $\mathcal{F}$ be a topology on A. A *strong - $\mathcal{F}-$ multiplier* is a mapping $f:I\to A/\theta_{\mathcal{F}}$ (where $I\in\mathcal{F}$) which verifies the axioms a_6, a_7 and a_8 (see Definition 3.1) and

(a_9) If $e\in I\cap B(A)$, then $f(e)\in B(A/\theta_{\mathcal{F}})$;

(a_{10}) $(x/\theta_{\mathcal{F}})\wedge f(e)=(e/\theta_{\mathcal{F}})\wedge f(x)$, for every $e\in I\cap B(A)$ and $x\in I$.

Remark 4.2. *If* $(A,\wedge,\vee,\odot,\to,0,1)$ *is a* $MTL-$ *algebra, the maps* $\mathbf{0},\mathbf{1}:A\to A/\theta_{\mathcal{F}}$ *defined by* $\mathbf{0}(x)=0/\theta_{\mathcal{F}}$ *and* $\mathbf{1}(x)=x/\theta_{\mathcal{F}}$ *for every* $x\in A$ *are strong - $\mathcal{F}-$ multipliers. We recall that if* $f_i:I_i\to A/\theta_{\mathcal{F}}$ *, (with* $I_i\in\mathcal{F}$, $i=1,2$*) are $\mathcal{F}-$multipliers* $f_1\wedge f_2, f_1\vee f_2, f_1\odot f_2, f_1\to f_2 : I_1\cap I_2\to A/\theta_{\mathcal{F}}$ *defined by* $(f_1\wedge f_2)(x)=f_1(x)\wedge f_2(x), (f_1\vee f_2)(x)=f_1(x)\vee f_2(x), (f_1\odot f_2)(x)=f_1(x)\odot[x/\theta_{\mathcal{F}}\to f_2(x)]\overset{c_{19}}{=}f_2(x)\odot[x/\theta_{\mathcal{F}}\to f_1(x)], (f_1\to f_2)(x)=x/\theta_{\mathcal{F}}\odot[f_1(x)\to f_2(x)]$, *for any* $x\in I_1\cap I_2$ *are $\mathcal{F}-$multipliers. If* f_1,f_2 *are strong - $\mathcal{F}-$ multipliers then the multipliers* $f_1\wedge f_2, f_1\vee f_2, f_1\odot f_2, f_1\to f_2$ *are also strong - $\mathcal{F}-$ multipliers (the proof is as in the case of $BL-$algebras, see* [2]*).*

Remark 4.3. *Analogous as in the case of $\mathcal{F}-$ multipliers if we work with strong-$\mathcal{F}-$ multipliers we obtain a $MTL-$ subalgebra of $A_{\mathcal{F}}$ denoted by $s-A_{\mathcal{F}}$ which will be called the strong-localization $MTL-$ algebra of A with respect to the topology $\mathcal{F}$.*

So, if $\mathcal{F}=I(A)\cap R(A)$ is the topology of regular ideals, then $\theta_{\mathcal{F}}$ is the identity congruence of A and we obtain the definition for multipliers on A, so

$$s-A_{\mathcal{F}}=\varinjlim_{I\in\mathcal{F}}(s-M(I,A)),$$

where $s-M(I,A)$ is the set of strong multipliers of A having the domain I (see [15], Definition 3, conditions M_1-M_5).

In this situation we obtain:

Proposition 4.1. *In the case $\mathcal{F}=I(A)\cap R(A)$, $A_{\mathcal{F}}$ is exactly the maximal MTL-algebra $Q(A)$ of quotients of A (introduced in* [15]*) which is a Boolean algebra (for the proof, see* [14] *Proposition 6.12, p.194, for the case of $BL-$ algebras). If $MTL-$ algebra A is a $BL-$ algebra, $A_{\mathcal{F}}$ is exactly the maximal BL-algebra $Q(A)$ of quotients of A.*

Remark 4.4. *If consider in particular MTL- algebra $A=[0,1]$ from Remark 1.2, then $\mathcal{F}=\{A\}$, hence $A_{\mathcal{F}}\approx s-M(A,A)$. Consider $f\in s-M(A,A)$. Clearly, $f(0)=0$ and $f(1)\in\{0,1\}$. If $f(1)=0$, then for every $x\in A$, $x\wedge f(1)=1\wedge f(x)\Leftrightarrow x\wedge 0=f(x)\Leftrightarrow f(x)=0\Leftrightarrow f=\mathbf{0}$. If $f(1)=1$ then from a_{10}, $f(x)=x=\mathbf{1}(x)$, hence $f=\mathbf{1}$. So, in this case $s-A_{\mathcal{F}}\approx s-M(A,A)=L_2$.*

3. Denoting by $\mathcal{D}$ the topology of dense ordered ideals of A, then (since $R(A)\subseteq D(A)$) there exists a morphism of MTL -algebras $\alpha:Q(A)\to s-A_{\mathcal{D}}$ such that the diagrame

$$\begin{array}{ccc} B(A) & \xrightarrow{\overline{v_A}} & Q(A) \\ v_D\searrow & & \alpha\swarrow \\ & s-A_{\mathcal{D}} & \end{array}$$

is commutative (i.e. $\alpha \circ \overline{v_A} = v_{\mathcal{D}}$). Indeed, if $[f, I] \in Q(A)$ (with $I \in I(A) \cap R(A)$ and $f : I \to A$ a strong multiplier in the sense of [15]) we denote by $f_{\mathcal{D}}$ the strong - $\mathcal{D}-$multiplier $f_{\mathcal{D}} : I \to A/\theta_{\mathcal{D}}$ defined by $f_{\mathcal{D}}(x) = f(x)/\theta_{\mathcal{D}}$ for every $x \in I$. Thus, α is defined by $\alpha([f, I]) = [f_{\mathcal{D}}, I]$.

4. Let $S \subseteq A$ a $\wedge-$closed system of MTL- algebra A. Consider the following congruence on $A : (x, y) \in \theta_S \Leftrightarrow$ there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ (see [3]). $A[S] = A/\theta_S$ is called in [3] the *MTL-algebra of fractions of A relative to the $\wedge-$closed system S*.

As in the case of $BL-$algebras we obtain the following result:

Proposition 4.2. *If $\mathcal{F}_S$ is the topology associated with a $\wedge-$closed system $S \subseteq A$, then the MTL-algebra $s - A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.*

Remark 4.5. *In the proof of Proposition 4.2 the axiom a_{10} is not necessarily.*

Remark 4.6. *If A is $MTL-$ algebra $A = [0, 1]$, from Remark 1.2, since $B(A) = \{0, 1\} = L_2$ then for $S \subseteq A$ a $\wedge-$ closed system, $\mathcal{F}_S = \{I \in I(A) : I \cap S \cap \{0, 1\} \neq \oslash\}$ and $s - A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$:*

1. *If S is a $\wedge-$closed systems of A such that $0 \in S$, then $\mathcal{F}_S = I(A)$ (see Remark 2.5) and $s - A_{\mathcal{F}_S} = s - A_{I(A)} \approx B(A[S]) = B(\mathbf{0}) = \mathbf{0}$.*
2. *If $0 \notin S$, $\mathcal{F}_S = A$ (see Remark 2.5) and $s - A_{\mathcal{F}_S} = s - A_A \approx B(A[S]) = B(A) = \{0, 1\} = L_2$.*

Concluding remarks

Since in particular a $MTL-$ algebra is a $BL-$ algebra we obtain a part of the results about localization of $BL-$ algebras (see [2]), so we deduce that the main results of this paper are generalization of the analogous result relative to $BL-$ algebras from [2].

We use in the construction of localization $MTL-$ algebra $A_{\mathcal{F}}$ the Boolean center $B(A)$ of $MTL-$ algebra A; as a consequence of this fact, $s - A_{\mathcal{F}}$ is a Boolean algebra in some particular cases.

A very interesting subject for future research would be a treatment of the localization for MTL algebras or residuated lattices without use the Boolean center.

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