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On a result by Niculescu and Spiridon

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Abstract. In the present paper we are concerned with the Jensen type inequality based on the recent results for a class of functions which are not totally convex on their domain of definition.

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Recently, Niculescu and Spiridon [3] have proved the following result, which extends Jensen's inequality to the framework of almost convex functions:

Theorem 1. *Suppose that $f : [-b, b] \to \mathbb{R}$ is an odd function, whose restriction to $[0, b]$ is convex and $p : [-b, b] \to [0, \infty)$ is a nondecreasing function that does not vanish on $(-b/3, b]$. Then for every $a \in [-b/3, b)$,*

$$f\left(\frac{1}{\int_a^b p(x)dx}\int_a^b xp(x)dx\right) \leq \frac{1}{\int_a^b p(x)dx}\int_a^b f(x)p(x)dx. \tag{1}$$

The discrete version of this theorem (see Corollary 3 in [3]) allows easily to extend the validity of some classical concrete inequalities outside the realm of convexity, for example,

$$\tan\left(\frac{x+y+z}{3}\right) \leq \frac{\tan x + \tan y + \tan z}{3}, \tag{2}$$

for all $x, y, z \in (-\pi/6, \pi/2)$, with $x + y + z + \min\{x, y, z\} \geq 0$.

The aim of the present note is to prove that actually the inequality (2) works for all $x, y, z \in (-\pi/2, \pi/2)$, with $x + y + z + \min\{x, y, z\} \geq 0$. This follows easily from the following general result:

Theorem 2. *Suppose that $f : (-a, a) \to \mathbb{R}$ verifies the following three properties:*
(a) f is convex on $[0, a)$;
(b) f is an odd function;
(c) $f(x) + f(y) \leq f(x + y)$, for all $x, y \in [0, a)$, with $x + y < a$.
Then we have the inequality:

$$f\left(\frac{x+y+z}{3}\right) \leq \frac{f(x) + f(y) + f(z)}{3}, \tag{3}$$

for all $x, y, z \in (-a, a)$, such that

$$x + y + z + \min\{x, y, z\} \geq 0. \tag{4}$$

Proof. Without loss of generality we can assume that $x \leq y \leq z$. The case $x \geq 0$ is clear. Assume $-a < x < 0 \leq y \leq z < a$ and put $t = -x \in (0, a)$ and $v = (y+z)/2 - t$.

By (4), we obtain $v \in [0, a)$. Under the property (b), the inequality (3) becomes:

$$3f\left(\frac{1}{3}t + \frac{2}{3}v\right) \le -f(t) + f(y) + f(z).$$

Hence we obtain:

$$\begin{aligned} 3f\left(\frac{x+y+z}{3}\right) &= 3f\left(\frac{1}{3}t + \frac{2}{3}v\right) \overset{(a)}{\le} 3\left(\frac{1}{3}f(t) + \frac{2}{3}f(v)\right) \\ &= f(t) + 2f(v) = -f(t) + 2(f(t) + f(v)) \\ &\overset{(c)}{\le} -f(t) + 2f(t+v) = -f(t) + 2f((y+z)/2) \overset{(a)}{\le} -f(t) + f(y) + f(z) \\ &= f(x) + f(y) + f(z). \end{aligned}$$

Finally, let us discuss the case where $x \le y < 0 \le z$. We put $t = -x \in (0, a),\ s = -y$ and $v = z - s - 2t$. From (4) it follows that $v \in [0, a)$. Then,

$$\begin{aligned} 3f\left(\frac{x+y+z}{3}\right) &= 3f\left(\frac{v+t}{3}\right) \\ &\overset{(a)}{\le} f(v) + 2f(t/2) = -f(t) - f(s) + [f(t) + f(s) + 2f(t/2) + f(v)]. \end{aligned}$$

Using the mathematical induction we infer from (c) that

$$\sum_{k=1}^{n} f(x_k) \le f\left(\sum_{k=1}^{n} x_k\right),\ \forall\ x_k \in [0, a),\ \ \sum_{k=1}^{n} x_k < a.$$

Therefore

$$\begin{aligned} 3f\left(\frac{x+y+z}{3}\right) &\le -f(t) - f(s) + [f(t) + f(s) + 2f(t/2) + f(v)] \\ &\le -f(t) - f(s) + f(v + s + 2t) = \\ &= f(x) + f(y) + f(z) \end{aligned}$$

and the proof is complete. □

Corollary 1. *Suppose that $f : (-a, a) \to \mathbb{R}$ verifies the following three properties*:
(a') f is concave on $[0, a)$;
(b') f is an odd function;
(c') $f(x) + f(y) \ge f(x + y)$, whenever $x, y \in [0, a)$, with $x + y < a$.
Then:

$$f\left(\frac{x+y+z}{3}\right) \ge \frac{f(x) + f(y) + f(z)}{3}, \tag{5}$$

for all $x, y, z \in (-a, a)$, such that

$$x + y + z + \min\{x, y, z\} \ge 0. \tag{6}$$

Proof. Indeed, if the function f satisfies the properties (a')-(c'), then $-f$ satisfies (a)-(c) of Theorem 2. □

An illustration of this corollary is offered by the following inequality:

$$\sin\left(\frac{x+y+z}{3}\right) \ge \frac{\sin x + \sin y + \sin z}{3},$$

for all $x, y, z \in (-\pi, \pi)$, such that $x + y + z + \min\{x, y, z\} \ge 0$.

Many others generalizations of Jensen's inequality maybe found in [1], [2], [4] and [3].

References

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