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Inequalities involving Mellin transform, integral mean, exponential and logarithmic mean

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Abstract. In this paper the Mellin transform in complex domain is considered for functions f which vanish beyond a finite domain $[a,b] \subset [0,\infty\rangle$ and such that $f' \in L_p[a,b]$. New inequalities involving the Mellin transform of f, integral mean of f, exponential mean and logarithmic mean of the endpoints of the domain of f are presented.

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1. Introduction

The Mellin transform $\mathcal{M}(f)$ of a Lebesgue integrable mapping $f:[a,b]\to\mathbb{R}$, where $[a,b]\subset[0,\infty\rangle$, is defined by

$$\mathcal{M}(f)(z) = \int_a^b f(t)\,t^{z-1}dt, \tag{1.1}$$

for every $z\in\mathbb{C}$ for which the integral on the right hand side of (1.1) exists, i.e. $\left|\int_a^b f(t)\,t^{z-1}dt\right| < \infty$ (see for instance [5]).

The exponential mean $E(z,w)$ of two complex numbers $z,w\in\mathbb{C}$ is defined by

$$E(z,w) = \begin{cases} \frac{e^z-e^w}{z-w}, & \text{if } z\neq w, \\ e^w, & \text{if } z=w. \end{cases} \tag{1.2}$$

In recent paper [2] bounds of the difference between the Laplace transform

$$\mathcal{L}(f)(z) = \int_a^b f(t)\,e^{-zt}dt$$

and the product of the exponential mean $E(-za,-zb)$ and the integral mean of f were obtained.

Theorem 1.1. [2] *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous such that $f'\in L_p[a,b]$. Then for $z\neq 0$, $1<p\leq\infty$, the following inequalities hold*

$$\left|\mathcal{L}(f)(z) - E(-za,-zb)\int_a^b f(s)\,ds\right| \leq \frac{2e^{-a\operatorname{Re}z}(b-a)^{\frac{1}{q}}}{|z|}\left\|f'\right\|_p \quad \text{if } \operatorname{Re}z\geq 0,$$

and

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$$\left|\mathcal{L}(f)(z) - E(-za,-zb)\int_a^b f(s)\,ds\right| \le \frac{2e^{-b\operatorname{Re} z}(b-a)^{\frac{1}{q}}}{|z|}\left\|f'\right\|_p \quad \text{if } \operatorname{Re} z < 0,$$

while for $p = 1$

$$\left|\mathcal{L}(f)(z) - E(-za,-zb)\int_a^b f(s)\,ds\right| \le \frac{2e^{-a\operatorname{Re} z}}{|z|}\left\|f'\right\|_1 \quad \text{if } \operatorname{Re} z \ge 0,$$

and

$$\left|\mathcal{L}(f)(z) - E(-za,-zb)\int_a^b f(s)\,ds\right| \le \frac{2e^{-b\operatorname{Re} z}}{|z|}\left\|f'\right\|_1 \quad \text{if } \operatorname{Re} z < 0.$$

Inequalities of the similar type involving the Fourier transform of functions in L_p spaces and also of functions of bounded variation were obtained in [1] and [4] respectively.

The aim of this paper is to obtain analogue inequalities for the Mellin transform $\mathcal{M}(f)(z)$ in the complex domain for functions $f : [a,b] \to \mathbb{R}$, $[a,b] \subset [0,\infty\rangle$, and $f' \in L_p[a,b]$. Beside integral and exponential means these inequalities involve also the logarithmic mean $L(a,b)$ of $a, b \in \mathbb{R}$, defined by

$$L(a,b) = \begin{cases} \frac{a-b}{\ln a - \ln b}, & \text{if } a \ne b, \\ a, & \text{if } a = b. \end{cases} \tag{1.3}$$

In Section 2 estimate of difference between Mellin transform $\mathcal{M}(f)(z)$ and

$$E(z\ln a, z\ln b)(L(a,b))^{-1}\int_a^b f(s)\,ds$$

is given. In Section 3 two further generalizations of the inequality from Section 2 are obtained by means of the difference between two weighted integral means.

2. Estimates of difference between Mellin transform and product of integral, exponential and logarithmic mean

Next theorem is the analogue of Theorem 1.1 for the Mellin transform $\mathcal{M}(f)(z)$ in the complex domain.

Theorem 2.1. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous such that $[a,b] \subset \langle 0,\infty\rangle$ and $f' \in L_p[a,b]$. Then for $z \ne 0$, $1 < p \le \infty$, the following inequalities hold*

$$\left|\mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)}\int_a^b f(s)\,ds\right| \le \frac{2b^{\operatorname{Re} z}(b-a)^{\frac{1}{q}}}{|z|}\left\|f'\right\|_p \quad \text{if } \operatorname{Re} z \ge 0,$$

and

$$\left|\mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)}\int_a^b f(s)\,ds\right| \le \frac{2a^{\operatorname{Re} z}(b-a)^{\frac{1}{q}}}{|z|}\left\|f'\right\|_p \quad \text{if } \operatorname{Re} z < 0,$$

while for $p = 1$

$$\left|\mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)}\int_a^b f(s)\,ds\right| \le \frac{2b^{\operatorname{Re} z}}{|z|}\left\|f'\right\|_1 \quad \text{if } \operatorname{Re} z \ge 0,$$

and

$$\left|\mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)}\int_a^b f(s)\,ds\right| \le \frac{2a^{\operatorname{Re} z}}{|z|}\left\|f'\right\|_1 \quad \text{if } \operatorname{Re} z < 0.$$

Here $E(z,w)$ is exponential mean given by (1.2) and $L(a,b)$ is logarithmic mean given by (1.3).

Proof. **Montgomery identity** states (see [6]):

$$f(t)=\frac{1}{b-a}\int_a^b f(s)\,ds+\int_a^b P(t,s)\,f'(s)\,ds,$$

where $P(t,s)$ is the Peano kernel, defined by

$$P(t,s)=\begin{cases}\frac{s-a}{b-a}, & a\le s\le t,\\ \\ \frac{s-b}{b-a} & t<s\le b.\end{cases}$$

Multiplying the Montgomery identity by t^{z-1} and then integrating from a to b with respect to t we have

$$\begin{aligned}\mathcal{M}(f)(z)&=\int_a^b f(t)\,t^{z-1}dt\\ &=\frac{1}{b-a}\int_a^b\left[\int_a^b f(s)\,ds+\int_a^t (s-a)\,f'(s)\,ds+\int_t^b (s-b)\,f'(s)\,ds\right]t^{z-1}dt.\end{aligned}$$

Since $\frac{d}{dt}t^z=zt^{z-1}$ for $z\in\mathbb{C}$ and thus $\int_a^b t^{z-1}dt=\frac{b^z-a^z}{z}$, by an interchange of the order of integration we get

$$\begin{aligned}&\int_a^b\left(\int_a^b f(s)\,ds\right)t^{z-1}dt=\int_a^b\left(\int_a^b t^{z-1}dt\right)f(s)\,ds=\int_a^b\left(\frac{b^z-a^z}{z}\right)f(s)\,ds\\ &=\frac{e^{z\ln b}-e^{z\ln a}}{z}\int_a^b f(s)\,ds=E(z\ln b,z\ln a)\left(\ln\frac{b}{a}\right)\int_a^b f(s)\,ds,\end{aligned}$$

$$\begin{aligned}\int_a^b\left(\int_a^t (s-a)\,f'(s)\,ds\right)t^{z-1}dt&=\int_a^b\left(\int_s^b t^{z-1}dt\right)(s-a)\,f'(s)\,ds\\ &=\int_a^b\left(\frac{b^z-s^z}{z}\right)(s-a)\,f'(s)\,ds,\end{aligned}$$

$$\begin{aligned}\int_a^b\left(\int_t^b (s-b)\,f'(s)\,ds\right)t^{z-1}dt&=\int_a^b\left(\int_a^s t^{z-1}dt\right)(s-b)\,f'(s)\,ds\\ &=\int_a^b\left(\frac{s^z-a^z}{z}\right)(s-b)\,f'(s)\,ds.\end{aligned}$$

So we have

$$\begin{aligned}\mathcal{M}(f)(z)-\frac{E(z\ln a,z\ln b)}{L(a,b)}\int_a^b f(s)\,ds&=-\int_a^b\frac{s^z}{z}f'(s)\,ds\\ &+\left[\int_a^b\frac{b^z}{z}\left(\frac{s-a}{b-a}\right)f'(s)\,ds+\int_a^b\frac{a^z}{z}\left(\frac{b-s}{b-a}\right)f'(s)\,ds\right].\end{aligned}$$

For $1 < p \leq \infty$, by applying Hölder inequality we obtain

$$\begin{aligned}
&\left|\mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)}\int_a^b f(s)\,ds\right| \\
&\quad = \left|\int_a^b \left[-\frac{s^z}{z} + \left(\frac{s-a}{b-a}\right)\frac{b^z}{z} + \left(\frac{b-s}{b-a}\right)\frac{a^z}{z}\right] f'(s)\,ds\right| \\
&\quad \leq \left\|-\frac{s^z}{z} + \left(\frac{s-a}{b-a}\right)\frac{b^z}{z} + \left(\frac{b-s}{b-a}\right)\frac{a^z}{z}\right\|_q \|f'\|_p .
\end{aligned}$$

Now, if $\operatorname{Re} z \geq 0$, by applying the triangle inequality we have

$$\begin{aligned}
&\left\|-\frac{s^z}{z} + \left(\frac{s-a}{b-a}\right)\frac{b^z}{z} + \left(\frac{b-s}{b-a}\right)\frac{a^z}{z}\right\|_q \\
&\quad \leq \left\|\frac{s^z}{z}\right\|_q + \left\|\left(\frac{s-a}{b-a}\right)\frac{b^z}{z} + \left(\frac{b-s}{b-a}\right)\frac{a^z}{z}\right\|_q \\
&\quad \leq \left\|\frac{e^{z\ln s}}{z}\right\|_q + \left\|\left(\frac{s-a}{b-a}\right)\frac{e^{z\ln b}}{z} + \left(\frac{b-s}{b-a}\right)\frac{e^{z\ln a}}{z}\right\|_q \\
&\quad \leq \frac{e^{\operatorname{Re} z\ln b}}{|z|}\left(\|1\|_q + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_q\right) = \frac{2e^{\operatorname{Re} z\ln b}(b-a)^{\frac{1}{q}}}{|z|},
\end{aligned}$$

and if $\operatorname{Re} z < 0$ we have

$$\begin{aligned}
&\left\|-\frac{s^z}{z} + \left(\frac{s-a}{b-a}\right)\frac{b^z}{z} + \left(\frac{b-s}{b-a}\right)\frac{a^z}{z}\right\|_q \\
&\quad \leq \frac{e^{\operatorname{Re} z\ln a}}{|z|}\left(\|1\|_q + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_q\right) = \frac{2e^{\operatorname{Re} z\ln a}(b-a)^{\frac{1}{q}}}{|z|}.
\end{aligned}$$

Similarly for $p = 1$ we have

$$\begin{aligned}
&\left|\mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)}\int_a^b f(s)\,ds\right| \\
&\quad \leq \left\|-\frac{s^z}{z} + \left(\frac{s-a}{b-a}\right)\frac{b^z}{z} + \left(\frac{b-s}{b-a}\right)\frac{a^z}{z}\right\|_\infty \|f'\|_1 .
\end{aligned}$$

If $\operatorname{Re} z \geq 0$

$$\begin{aligned}
&\left\|-\frac{s^z}{z} + \left(\frac{s-a}{b-a}\right)\frac{b^z}{z} + \left(\frac{b-s}{b-a}\right)\frac{a^z}{z}\right\|_\infty \\
&\quad \leq \frac{e^{\operatorname{Re} z\ln b}}{|z|}\left(\|1\|_\infty + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_\infty\right) = \frac{2e^{\operatorname{Re} z\ln b}}{|z|},
\end{aligned}$$

and if $\operatorname{Re} z < 0$ we have

$$\begin{aligned}
&\left\|\frac{e^{-sz}}{z} - \left(\frac{s-a}{b-a}\right)\frac{e^{-bz}}{z} - \left(\frac{b-s}{b-a}\right)\frac{e^{-az}}{z}\right\|_\infty \\
&\quad \leq \frac{e^{\operatorname{Re} z\ln a}}{|z|}\left(\|1\|_\infty + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_\infty\right) = \frac{2e^{\operatorname{Re} z\ln a}}{|z|},
\end{aligned}$$

and the proof is done. $\square$

Remark 2.1. *In case $a = 0$ and $\operatorname{Re} z \geq 0$ proceeding in the same way as in the previous proof and using the fact that $0^z = 0$ and $\frac{b^z - a^z}{z(b-a)} = \frac{b^{z-1}}{z}$ we obtain*

$$\left| \mathcal{M}(f)(z) - \frac{b^{z-1}}{z} \int_a^b f(s)\, ds \right| \leq \frac{2b^{\operatorname{Re} z + \frac{1}{q}}}{|z|} \|f'\|_p ,$$

and

$$\left| \mathcal{M}(f)(z) - \frac{b^{z-1}}{z} \int_a^b f(s)\, ds \right| \leq \frac{2b^{\operatorname{Re} z}}{|z|} \|f'\|_1 .$$

3. Further generalizations by means of the difference between two weighted integral means

Let $w : [a,b] \to \mathbb{R}$ be an integrable weight function such that $\int_a^b w(t)\, dt \neq 0$ and $W(x) = \int_a^x w(t)\, dt$, $x \in [a,b]$. Then **weighted Montgomery identity** states (given by Pečarić in [7])

$$f(x) - \frac{1}{\int_a^b w(t)\, dt} \int_a^b f(t) w(t)\, dt = \int_a^b P_w(x,t) f'(t)\, dt \tag{3.1}$$

where $P_w(t,s)$ the weighted Peano kernel, defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \leq s \leq x, \\ \frac{W(t)}{W(b)} - 1, & x < s \leq b. \end{cases} \tag{3.2}$$

By subtracting two weighted Montgomery identities, one for the interval $[a,b]$ and the other for $[c,d]$, the next result is obtained (see [1]).

Lemma 3.1. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be an absolutely continuous function on $[a,b] \cup [c,d]$, $w : [a,b] \to \mathbb{R}$ and $u : [c,d] \to \mathbb{R}$ some weight functions, such that $\int_a^b w(t)\, dt \neq 0$, $\int_c^d u(t)\, dt \neq 0$ and*

$$W(x) = \begin{cases} 0, & t < a, \\ \int_a^x w(t)\, dt, & a \leq t \leq b, \\ \int_a^b w(t)\, dt, & t > b, \end{cases} \qquad U(x) = \begin{cases} 0, & t < c, \\ \int_c^x u(t)\, dt, & c \leq t \leq d, \\ \int_c^d u(t)\, dt, & t > d, \end{cases}$$

and $[a,b] \cap [c,d] \neq \emptyset$. Then, for both cases $[c,d] \subseteq [a,b]$ and $[a,b] \cap [c,d] = [c,b]$, (and also for $[a,b] \subseteq [c,d]$ and $[a,b] \cap [c,d] = [a,d]$) the next formula is valid

$$\frac{1}{\int_a^b w(t)\, dt} \int_a^b w(t) f(t)\, dt - \frac{1}{\int_c^d u(t)\, dt} \int_c^d u(t) f(t)\, dt = \int_{\min\{a,c\}}^{\max\{b,d\}} K(t) f'(t)\, dt \tag{3.3}$$

where

$$K(t) = P_u(x,t) - P_w(x,t), \; t \in [\min\{a,c\}, \max\{b,d\}]$$

and $P_u(x,t)$, $P_w(x,t)$ are given by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \leq s \leq x, \\ \frac{W(t)}{W(b)} - 1, & x < s \leq b, \end{cases} , \quad P_u(x,t) = \begin{cases} \frac{U(t)}{U(b)}, & c \leq s \leq x, \\ \frac{U(t)}{U(b)} - 1, & x < s \leq d, \end{cases}$$

thus

$$K(t)=\begin{cases} -\frac{W(t)}{W(b)}, & t\in[a,c],\\ -\frac{W(t)}{W(b)}+\frac{U(t)}{U(d)}, & t\in\langle c,d\rangle, \qquad \text{if} \quad [c,d]\subseteq[a,b],\\ 1-\frac{W(t)}{W(b)}, & t\in[d,b],\end{cases} \tag{3.4}$$

$$K(t)=\begin{cases} -\frac{W(t)}{W(b)}, & t\in[a,c],\\ -\frac{W(t)}{W(b)}+\frac{U(t)}{U(d)}, & t\in\langle c,b\rangle, \qquad \text{if} \quad [a,b]\cap[c,d]=[c,b].\\ \frac{U(t)}{U(d)}-1, & t\in[b,d],\end{cases} \tag{3.5}$$

Remark 3.1. *It is easy to check that weighted Montgomery identity (3.1) and the previous Lemma hold also for $w:[a,b]\to\mathbb{C}$ integrable and such that $\int_a^b w(t)\,dt\neq 0$.*

In case $w(t)=t^{z-1}$, $t\in[a,b]$ we have

$$\int_a^b w(t)\,dt=\frac{b^z-a^z}{z}\neq 0$$

since for $z=x+iy$

$$b^z=a^z \;\Leftrightarrow\; e^{z\ln a}=e^{z\ln b} \;\Leftrightarrow\; e^{x\ln a}(\cos(y\ln b)+i\sin(y\ln b)) \;\Leftrightarrow\; a=b.$$

Remark 3.2. *The Lemma 3.1 for normalized weight function w, i.e. such that $\int_a^b w(t)\,dt=1$, was proved in* [3].

Next, we apply identity for the difference of the two weighted integral means (3.3) with two special weight functions: uniform weight function and kernel of the Mellin transform. In such a way new generalizations of the results from the previous section are obtained. In the special case, for $c=a$ and $d=b$, both reduce to the results of the Theorem 2.1.

Theorem 3.1. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous, $[a,b]\subset\langle 0,\infty\rangle$, $f'\in L_p[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $z\neq 0$, $\operatorname{Re} z\geq 0$ and $1<p\leq\infty$, the following inequality holds*

$$\left|\frac{d-c}{b-a}\mathcal{M}(f)(z)-\frac{E(z\ln a,z\ln b)}{L(a,b)}\int_c^d f(t)\,dt\right|\leq b^{\operatorname{Re} z}\frac{2(d-c)(b-a)^{\frac{1}{q}-1}}{|z|}\|f'\|_p,$$

while for $p=1$ it holds

$$\left|\frac{d-c}{b-a}\mathcal{M}(f)(z)-\frac{E(z\ln a,z\ln b)}{L(a,b)}\int_c^d f(t)\,dt\right|\leq b^{\operatorname{Re} z}\frac{2(d-c)}{(b-a)|z|}\|f'\|_1.$$

Here $E(z,w)$ and $L(a,b)$ are exponential and logarithmic mean given by (1.2) and (1.3) respectively.

Proof. If we apply identity (3.3) with $w(t)=t^{z-1}$, $t\in[a,b]$ and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ again we have $W(t)=\frac{E(z\ln a,z\ln t)}{L(a,t)}(t-a)$, $t\in[a,b]$; $U(t)=\frac{t-c}{d-c}$, $t\in[c,d]$ and

$$\frac{L(a,b)}{(b-a)E(z\ln a,z\ln b)}\mathcal{M}(f)(z)-\frac{1}{d-c}\int_c^d f(t)\,dt=\int_a^b K(t)f'(t)\,dt.$$

Since $[c,d] \subseteq [a,b]$ we use (3.4) so

$$K(t) = \begin{cases} -\frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{E(z\ln a, z\ln b)}\frac{L(a,b)}{L(a,t)}, & t \in [a,c], \\ -\frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{E(z\ln a, z\ln b)}\frac{L(a,b)}{L(a,t)} + \frac{t-c}{d-c}, & t \in \langle c,d\rangle, \\ 1 - \frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{E(z\ln a, z\ln b)}\frac{L(a,b)}{L(a,t)}, & t \in [d,b]. \end{cases}$$

Thus

$$\frac{d-c}{b-a}\mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)}\int_c^d f(t)\,dt = (d-c)\frac{E(z\ln a, z\ln b)}{L(a,b)}\int_a^b K(t) f'(t)\,dt$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a}\mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)}\int_c^d f(t)\,dt\right| \le (d-c)\left\|\frac{E(z\ln a, z\ln b)}{L(a,b)}K(t)\right\|_q \|f'\|_p.$$

Now, for $1 < p \le \infty$ (for $1 \le q < \infty$) we have

$$\begin{aligned}\left\|\frac{E(z\ln a, z\ln b)}{L(a,b)}K(t)\right\|_q = \Bigg(&\int_a^c \left|\frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{L(a,t)}\right|^q dt \\ &+ \int_c^d \left|\frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{L(a,t)} - \frac{t-c}{d-c}\frac{E(z\ln a, z\ln b)}{L(a,b)}\right|^q dt \\ &+ \int_d^b \left|\frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{L(a,t)} - \frac{E(z\ln a, z\ln b)}{L(a,b)}\right|^q dt\Bigg)^{\frac{1}{q}}\end{aligned}$$

and since $\left|\frac{E(z\ln a, z\ln t)}{L(a,t)}\right| = \left|\frac{e^{z\ln t} - e^{z\ln a}}{z(t-a)}\right| \le \frac{2e^{\operatorname{Re} z\ln b}}{|z||t-a|}$ for $t \in [a,b]$, we have

$$\int_a^c \left|\frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{L(a,t)}\right|^q dt \le \int_a^c \left(\frac{2e^{\operatorname{Re} z\ln b}}{(b-a)|z|}\right)^q dt = (c-a)\left(\frac{2e^{\operatorname{Re} z\ln b}}{(b-a)|z|}\right)^q,$$

$$\begin{aligned}&\int_c^d \left|\frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{L(a,t)} - \frac{t-c}{d-c}\frac{E(z\ln a, z\ln b)}{L(a,b)}\right|^q dt \\ &= \frac{1}{((b-a)|z|)^q}\int_c^d \left|\frac{d-t}{d-c}e^{z\ln a} + \frac{t-c}{d-c}e^{z\ln b} - e^{z\ln t}\right|^q dt \\ &\le \frac{1}{((b-a)|z|)^q}\left(\int_c^d \left|\frac{d-t}{d-c}e^{z\ln a} + \frac{t-c}{d-c}e^{z\ln b}\right|^q dt + \int_c^d \left|e^{z\ln t}\right|^q dt\right) \\ &\le \frac{e^{q\operatorname{Re} z\ln b}}{((b-a)|z|)^q}\left(\int_c^d \left|\frac{d-t}{d-c} + \frac{t-c}{d-c}\right|^q dt + \int_c^d |1|^q\,dt\right) \le \frac{2(d-c)e^{q\operatorname{Re} z\ln b}}{((b-a)|z|)^q},\end{aligned}$$

$$\begin{aligned}\int_d^b \left|\frac{t-a}{b-a}\frac{E(z\ln a, z\ln t)}{L(a,t)} - \frac{E(z\ln a, z\ln b)}{L(a,b)}\right|^q dt &= \frac{1}{((b-a)|z|)^q}\int_d^b \left|e^{z\ln b} - e^{z\ln t}\right|^q dt \\ &\le \frac{1}{((b-a)|z|)^q}\int_d^b \left(2e^{\operatorname{Re} z\ln b}\right)^q dt = \frac{\left(2e^{\operatorname{Re} z\ln b}\right)^q (b-d)}{((b-a)|z|)^q}.\end{aligned}$$

Thus

$$\left\| \frac{E(z\ln a, z\ln b)}{L(a,b)} K(t) \right\|_q \le e^{\operatorname{Re} z \ln b} \left(\frac{2^q (c-a) + 2(d-c) + 2^q (b-d)}{((b-a)|z|)^q} \right)^{\frac{1}{q}}$$
$$\le b^{\operatorname{Re} z} \frac{2(b-a)^{\frac{1}{q}-1}}{|z|}$$

and the first inequality is proved. For $p = 1$ we have

$$\left\| \frac{E(z\ln a, z\ln b)}{L(a,b)} K(t) \right\|_\infty = \max \left\{ \sup_{t\in[a,c]} \left| \frac{t-a}{b-a} \frac{E(z\ln a, z\ln t)}{L(a,t)} \right|, \right.$$
$$\sup_{t\in[c,d]} \left| \frac{t-a}{b-a} \frac{E(z\ln a, z\ln t)}{L(a,t)} - \frac{t-c}{d-c} \frac{E(z\ln a, z\ln b)}{L(a,b)} \right|,$$
$$\left. \sup_{t\in[d,b]} \left| \frac{t-a}{b-a} \frac{E(z\ln a, z\ln t)}{L(a,t)} - \frac{E(z\ln a, z\ln b)}{L(a,b)} \right| \right\}$$

and

$$\sup_{t\in[a,c]} \left| \frac{t-a}{b-a} \frac{E(z\ln a, z\ln t)}{L(a,t)} \right| \le \frac{2e^{\operatorname{Re} z \ln b}}{(b-a)|z|},$$

$$\sup_{t\in[c,d]} \left| \frac{t-a}{b-a} \frac{E(z\ln a, z\ln t)}{L(a,t)} - \frac{t-c}{d-c} \frac{E(z\ln a, z\ln b)}{L(a,b)} \right|$$
$$= \frac{1}{(b-a)|z|} \sup_{t\in[c,d]} \left| \frac{d-t}{d-c} e^{z\ln a} + \frac{t-c}{d-c} e^{z\ln b} - e^{z\ln t} \right|$$
$$\le \frac{e^{\operatorname{Re} z \ln b}}{(b-a)|z|} \sup_{t\in[c,d]} \left| \frac{d-t}{d-c} + \frac{t-c}{d-c} + 1 \right| = \frac{2e^{\operatorname{Re} z \ln b}}{(b-a)|z|},$$

$$\sup_{t\in[d,b]} \left| \frac{t-a}{b-a} \frac{E(z\ln a, z\ln t)}{L(a,t)} - \frac{E(z\ln a, z\ln b)}{L(a,b)} \right|$$
$$= \frac{1}{(b-a)|z|} \sup_{t\in[d,b]} \left| e^{z\ln b} - e^{z\ln t} \right| \le \frac{2e^{\operatorname{Re} z \ln b}}{(b-a)|z|}.$$

Thus

$$\left\| \frac{E(z\ln a, z\ln b)}{L(a,b)} K(t) \right\|_\infty \le \frac{2b^{\operatorname{Re} z}}{(b-a)|z|}$$

and the proof is completed. □

Remark 3.3. *The inequalities from the previous Theorem hold for* $\operatorname{Re} z \ge 0$. *Similarly it can be proved that in case* $\operatorname{Re} z < 0$ *and* $1 < p \le \infty$ *the following inequality holds*

$$\left| \frac{d-c}{b-a} \mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)} \int_c^d f(t)\,dt \right| \le a^{\operatorname{Re} z} \frac{2(d-c)(b-a)^{\frac{1}{q}-1}}{|z|} \|f'\|_p,$$

while for $\operatorname{Re} z < 0$ *and* $p = 1$ *it holds*

$$\left| \frac{d-c}{b-a} \mathcal{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)} \int_c^d f(t)\,dt \right| \le a^{\operatorname{Re} z} \frac{2(d-c)}{(b-a)|z|} \|f'\|_1.$$

Theorem 3.2. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous, $[a,b]\subset\langle 0,\infty\rangle$, $f'\in L_p[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $z\neq 0$, $\operatorname{Re} z\geq 0$ and $1<p\leq\infty$, the following inequality holds*

$$\left|\frac{d-c}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\int_a^b f(t)\,dt-\int_c^d t^{z-1}f(t)\,dt\right|\leq d^{\operatorname{Re} z}\frac{2(b-a)^{\frac{1}{q}}}{|z|}\left\|f'\right\|_p,$$

while for $p=1$ it holds

$$\left|\frac{d-c}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\int_a^b f(t)\,dt-\int_c^d t^{z-1}f(t)\,dt\right|\leq d^{\operatorname{Re} z}\frac{2}{|z|}\left\|f'\right\|_1,$$

Here $E(z,w)$ and $L(a,b)$ are exponential and logarithmic mean given by (1.2) and (1.3) respectively.

Proof. We apply identity (3.3) again with $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $u(t)=t^{z-1}$, $t\in[c,d]$, so we have $W(t)=\frac{t-a}{b-a}$, $t\in[a,b]$; $U(t)=(t-c)\frac{E(z\ln c,z\ln t)}{L(c,t)}$, $t\in[c,d]$; and

$$\frac{1}{(b-a)}\int_a^b f(t)\,dt-\frac{L(c,d)}{(d-c)E(z\ln c,z\ln d)}\int_c^d t^{z-1}f(t)\,dt=\int_a^b K(t)f'(t)\,dt.$$

Since $[c,d]\subseteq[a,b]$ we use (3.4) so

$$K(t)=\begin{cases}-\frac{t-a}{b-a}, & t\in[a,c],\\ \frac{t-c}{d-c}\frac{E(z\ln c,z\ln t)}{E(z\ln c,z\ln d)}\frac{L(c,d)}{L(c,t)}-\frac{t-a}{b-a}, & t\in\langle c,d\rangle,\\ \frac{b-t}{b-a}, & t\in[d,b].\end{cases}$$

Thus

$$\begin{aligned}&\frac{d-c}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\int_a^b f(t)\,dt-\int_c^d t^{z-1}f(t)\,dt\\ &\quad=(d-c)\frac{E(z\ln c,z\ln d)}{L(c,d)}\int_a^b K(t)f'(t)\,dt\end{aligned}$$

and by taking the modulus and applying Hölder inequality we obtain

$$\begin{aligned}&\left|\frac{d-c}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\int_a^b f(t)\,dt-\int_c^d t^{z-1}f(t)\,dt\right|\\ &\quad\leq(d-c)\left\|\frac{E(z\ln c,z\ln d)}{L(c,d)}K(t)\right\|_q\left\|f'\right\|_p.\end{aligned}$$

Now, for $1<p\leq\infty$ (for $1\leq q<\infty$) we have

$$\begin{aligned}\left\|\frac{E(z\ln c,z\ln d)}{L(c,d)}K(t)\right\|_q&=\left(\int_a^c\left|\frac{t-a}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q dt\right.\\ &+\int_c^d\left|\frac{t-c}{d-c}\frac{E(z\ln c,z\ln t)}{L(c,t)}-\frac{t-a}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q dt\\ &\left.+\int_d^b\left|\frac{b-t}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

and since $\left|\frac{E(z\ln c,z\ln t)}{L(c,t)}\right| = \left|\frac{e^{z\ln t}-e^{z\ln c}}{z(t-c)}\right| \leq \frac{2e^{\operatorname{Re} z\ln d}}{|z||t-c|}$ for $t\in[c,d]$ we have

$$\int_a^c \left|\frac{t-a}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q dt = \left|\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q \int_a^c \left(\frac{t-a}{b-a}\right)^q dt$$
$$= \left|\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q \frac{(c-a)^{q+1}}{(q+1)(b-a)^q} \leq \frac{2^q e^{q\operatorname{Re} z\ln d}(c-a)^{q+1}}{(q+1)(|z|(d-c)(b-a))^q},$$

$$\int_c^d \left|\frac{t-c}{d-c}\frac{E(z\ln c,z\ln t)}{L(c,t)} - \frac{t-a}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q dt$$
$$= \frac{1}{((d-c)|z|)^q}\int_c^d \left|\frac{b-t}{b-a}e^{z\ln c}+\frac{t-a}{b-a}e^{z\ln d}-e^{z\ln t}\right|^q dt$$
$$\leq \frac{1}{((d-c)|z|)^q}\left(\int_c^d \left|\frac{b-t}{b-a}e^{z\ln c}+\frac{t-a}{b-a}e^{z\ln d}\right|^q + \int_c^d \left|e^{z\ln t}\right|^q\right) dt$$
$$= \frac{e^{q\operatorname{Re} z\ln d}}{((d-c)|z|)^q} 2\int_c^d |1|^q\, dt \leq \frac{2e^{q\operatorname{Re} z\ln d}(d-c)}{((d-c)|z|)^q},$$

$$\int_d^b \left|\frac{b-t}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q dt = \left|\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q \int_d^b \left(\frac{b-t}{b-a}\right)^q dt$$
$$= \left|\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|^q \frac{(b-d)^{q+1}}{(q+1)(b-a)^q} \leq \frac{2^q e^{q\operatorname{Re} z\ln d}(b-d)^{q+1}}{(q+1)(|z|(d-c)(b-a))^q}.$$

Thus

$$\left\|\frac{E(z\ln c,z\ln d)}{L(c,d)}K(t)\right\|_q$$
$$\leq e^{\operatorname{Re} z\ln d}\left(\frac{\frac{2^q(c-a)^{q+1}}{q+1}+2(d-c)(b-a)^q+\frac{2^q(b-d)^{q+1}}{q+1}}{((d-c)(b-a)|z|)^q}\right)^{\frac{1}{q}}$$
$$\leq d^{\operatorname{Re} z}\frac{2(b-a)^{\frac{1}{q}}}{(d-c)|z|}$$

and the first inequality is proved. For $p=1$ we have

$$\left\|\frac{E(z\ln c,z\ln d)}{L(c,d)}K(t)\right\|_\infty = \max\left\{\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|,\right.$$
$$\sup_{t\in[c,d]}\left|\frac{t-c}{d-c}\frac{E(z\ln c,z\ln t)}{L(c,t)}-\frac{t-a}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|,$$
$$\left.\sup_{t\in[d,b]}\left|\frac{b-t}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right|\right\}$$

and

$$\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}\frac{E(z\ln c,z\ln d)}{L(c,d)}\right| \leq \frac{2e^{\operatorname{Re} z\ln d}(c-a)}{(d-c)(b-a)|z|},$$

$$\sup_{t\in[c,d]}\left|\frac{t-c}{d-c}\frac{E\left(z\ln c,z\ln t\right)}{L\left(c,t\right)}-\frac{t-a}{b-a}\frac{E\left(z\ln c,z\ln d\right)}{L\left(c,d\right)}\right|$$
$$=\frac{1}{\left(d-c\right)\left|z\right|}\sup_{t\in[c,d]}\left|\frac{b-t}{b-a}e^{z\ln c}+\frac{t-a}{b-a}e^{z\ln d}-e^{z\ln t}\right|$$
$$\leq\frac{e^{\operatorname{Re}z\ln d}}{\left(d-c\right)\left|z\right|}\sup_{t\in[c,d]}\left|\frac{d-t}{d-c}+\frac{t-c}{d-c}+1\right|=\frac{2e^{\operatorname{Re}z\ln d}}{\left(d-c\right)\left|z\right|},$$
$$\sup_{t\in[d,b]}\left|\frac{b-t}{b-a}\frac{E\left(z\ln c,z\ln d\right)}{L\left(c,d\right)}\right|\leq\frac{2e^{\operatorname{Re}z\ln d}\left(b-d\right)}{\left(d-c\right)\left(b-a\right)\left|z\right|}.$$

Thus

$$\left\|\frac{E\left(z\ln c,z\ln d\right)}{L\left(c,d\right)}K\left(t\right)\right\|_{\infty}\leq 2d^{\operatorname{Re}z}\frac{\max\left\{\left(c-a\right),\left(b-a\right),\left(b-d\right)\right\}}{\left(d-c\right)\left(b-a\right)\left|z\right|}=\frac{2d^{\operatorname{Re}z}}{\left(d-c\right)\left|z\right|}$$

and the proof is completed. □

Remark 3.4. *The inequalities from the previous Theorem hods for* $\operatorname{Re}z\geq 0$. *Similarly it can be proved that in case* $\operatorname{Re}z<0$ *and* $1<p\leq\infty$ *the following inequality holds*

$$\left|\frac{d-c}{b-a}\frac{E\left(z\ln c,z\ln d\right)}{L\left(c,d\right)}\int_a^b f\left(t\right)dt-\int_c^d t^{z-1}f\left(t\right)dt\right|\leq c^{\operatorname{Re}z}\frac{2\left(b-a\right)^{\frac{1}{q}}}{\left|z\right|}\left\|f'\right\|_p,$$

while for $\operatorname{Re}z<0$ *and* $p=1$ *it holds*

$$\left|\frac{d-c}{b-a}\frac{E\left(z\ln c,z\ln d\right)}{L\left(c,d\right)}\int_a^b f\left(t\right)dt-\int_c^d t^{z-1}f\left(t\right)dt\right|\leq c^{\operatorname{Re}z}\frac{2}{\left|z\right|}\left\|f'\right\|_1,$$

Remark 3.5. *The results of the Theorems 3.1 and 3.2 in case* $c=a$ *and* $d=b$ *reduce to the results of the Theorem 2.1.*

Remark 3.6. *In case* $a=0$ *and whenever* $\operatorname{Re}z\geq 0$ *all inequalities from the Theorem 2 and Theorem 3 hold with the term* $\frac{b^{z-1}}{z}$ *instead of* $\frac{E(z\ln a,z\ln b)}{L(a,b)}$. *In case* $a=c=0$ *and whenever* $\operatorname{Re}z\geq 0$ *all inequalities from the Theorem 4 hold with the term* $\frac{d^{z-1}}{z}$ *instead of* $\frac{E(z\ln c,z\ln d)}{L(c,d)}$ *(see Remark 1).*

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