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Algebraic view of MTL-filters

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Abstract. This paper deals with filters on MTL-algebras. We introduce some types of filters on MTL-algebras such as weak implicative and weak positive implicative and attempt to obtain some of the properties of these filters. Then we investigate some relationships between these filters and some types of filters that were already defined.

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1. Introduction

In [4], Esteva and Godo introduced a logic weaker than BL [5], with a real-valued semantics, namely MTL. Although weaker than BL, this logic is strong enough to prove the residuation property of implication $\rightarrow$ with respect to &. In [6], MTL has been shown to be complete with respect to interpretations where & and the corresponding implication $\rightarrow$ are interpreted as a left-continuous t-norm and its residuum, respectively. MTL is equivalently obtained as the extension of Höhle's Monoidal logic with the pre-linearity axiom $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$. The logic MTL is also related to Ono's family of substructural logics [11], which are different extensions of the Full Lambek calculus FL. In fact, Höhle's Monoidal logic is indeed equivalent to the extension of FL with exchange, i.e. $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$, and weakening, i.e. $(\varphi \rightarrow \psi) \rightarrow (\varphi \& \chi \rightarrow \psi)$, denoted FL_{ew}, whose algebraic semantics is the variety of (commutative, integral) residuated lattices. Hence, MTL is equivalent to the extension of FL_{ew} with pre-linearity axiom.

In [2], Borzooei et al. introduced $IMTL$, $EIMTL$, associative and strong filters on MTL-algebras and showed that $IMTL$ and strong filters are related to $IMTL$ and strong MTL-algebras. They proved that strong filters include some filters such as implicative, positive implicative and fantastic filters.

In this paper, we extend some Borzooei et al.'s results and prove that associative filters on MTL-algebras are trivial. We introduce some various types of filters on MTL-algebras such as weak implicative, weak positive implicative and prime and obtain some equivalent conditions for $EIMTL$, weak implicative and weak positive implicative filters on MTL-algebras. Then we prove that for any positive implicative filter F, the cut F_a is an $EIMTL$ filter. Also we investigate some relationships between these filters and some types of filters that were already defined.

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2. Preliminaries

Definition 2.1. [1, 4, 5] A *residuated lattice* is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice with 1 as the greatest element and 0 as the smallest element,
(2) $(L, \odot, 1)$ is a commutative monoid,
(3) $a \leq b \rightarrow c$ if and only if $a \odot b \leq c$, for all $a, b, c \in L$.

A residuated lattice L is called an MTL-algebra, if $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y \in L$.

Proposition 2.1. [1, 4, 5, 12] *The following properties hold for any residuated lattice* ((R1)-(R12)) *and MTL-algebra* ((R1)-(M2))*:*

(R1) $x \leq y \Leftrightarrow x \rightarrow y = 1$,
(R2) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$, $x \rightarrow x = 1$, $0 \rightarrow x = 1$ *and* $x \rightarrow (y \rightarrow x) = 1$,
(R3) $x \leq y \rightarrow z \Leftrightarrow y \leq x \rightarrow z$,
(R4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(R5) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$,
(R6) $z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y)$ *and* $z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x)$,
(R7) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,
(R8) $x''' = x'$ *and* $x \leq x''$, *where* $x' = x \rightarrow 0$,
(R9) $x' \wedge y' = (x \vee y)'$,
(R10) $x \vee x' = 1$ *implies* $x \wedge x' = 0$,
(R11) $x \odot y \leq x \wedge y$,
(R12) $x \leq y$ *implies* $x \odot z \leq y \odot z$,
(M1) $x' \vee y' = (x \wedge y)'$,
(M2) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,

Definition 2.2. [2, 3, 5, 8, 10, 12, 13] Let F be a non-empty subset of MTL-algebra L containing 1. Then F is called:

(i) a *filter* if F is closed with respect to $\odot$ and $x \in F$, $x \leq y$ imply $y \in F$, for all $x, y \in L$,
(ii) an $EIMTL$*-filter* if $(x \rightarrow y)'' \in F$ and $x \in F$ imply $y \in F$, for all $x, y \in L$,
(iii) a *fantastic filter* if $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$ imply $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$, for all $x, y, z \in L$,
(iv) an *associative filter* if $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $z \in F$, for all $x, y, z \in L$,
(v) an $IMTL$*-filter*, if F is a filter and $x'' \rightarrow x \in F$, for all $x \in L$,
(vi) a *strong filter*, if F is a filter and $(x'' \rightarrow x)'' \in F$, for all $x \in L$,
(vii) a *prime filter* if F is a proper filter and $x \vee y \in F$ implies $x \in F$ or $y \in F$, for all $x, y \in L$,
(viii) an *ultra filter* if F is a proper filter and $x \in F$ or $x' \in F$, for all $x \in L$,
(ix) an *implicative filter* if $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z \in F$, for all $x, y, z \in L$,
(x) a *positive implicative filter* if $x \in F$ and $x \rightarrow ((y \rightarrow z) \rightarrow y) \in F$ imply $y \in F$, for all $x, y, z \in L$,
(xi) a *Boolean filter* if F is a filter such that $x \vee x' \in F$, for all $x \in L$.

We note that, any $IMTL$-filter of MTL-algebra L is an $EIMTL$-filter and strong filter (see [2, Theorem 3.14 and Theorem 4.2]).
For any filter F of MTL-algebra L we can define a relation $\equiv_F$ on L by $x \equiv_F y \Longleftrightarrow$

$x \to y, y \to x \in F$, for all $x, y \in L$. Then $\equiv_F$ is a congruence relation on L. Let $A/F = \{[x] | \ x \in L\}$. For all $x, y \in L$, define $[x] \vee [y] = [x \vee y]$, $[x] \wedge [y] = [x \wedge y]$, $[x] \odot [y] = [x \odot y]$ and $[x] \to [y] = [x \to y]$. Then $(L/F, \vee, \wedge, \odot, \to, [0], [1])$ is an MTL-algebra. It is called the quotient MTL-algebra with respect to F (see [4]).

Lemma 2.2. [2] *Let F be a filter of MTL-algebra L. Then F is an $EIMTL$-filter if and only if $x'' \in F$ implies $x \in F$, for all $x \in L$.*

Theorem 2.3. [2] *Any fantastic (positive implicative) filter of MTL-algebra L is an $EIMTL$-filter.*

Theorem 2.4. [2] *Let F be a non-empty subset of MTL-algebra L. Then F is a positive implicative filter of L if and only if F is an implicative and $EIMTL$-filter.*

Theorem 2.5. [2, 7, 8] *Let F be a filter of MTL-algebra L. The following are equivalent:*
(i) *F is a positive implicative filter,*
(ii) *$(x \to y) \to x \in F$ implies $x \in F$, for all $x, y \in L$,*
(iii) *$(x' \to x) \to x \in F$, for all $x \in L$.*

Theorem 2.6. [2, 7, 8] *Let F be a non-empty subset of MTL-algebra L. The following are equivalent: for all $x, y, z \in F$,*
(i) *F is an implicative filter of L,*
(ii) *$1 \in F$, $z \to (y \to (y \to x)) \in F$ and $z \in F$ imply $y \to x \in F$,*
(iii) *F is a filter and $x \to x^2 \in F$,*
(iv) *F is a filter and $F_a = \{x \in L | \ a \to x \in F\}$ is a filter of L, for any $a \in L$.*

Theorem 2.7. [9] *Let F a be filter of MTL-algebra L. Then F is a Boolean filter if and if it is a positive implicative filter.*

Theorem 2.8. [2] *Any implicative, positive implicative and fantastic filter of MTL-algebra L is a strong filter.*

Theorem 2.9. [14] *Let F be a filter of residuated lattice L. Then F is fantastic filter if and only if $((x \to y) \to y) \to ((y \to x) \to x) \in F$, for any $x, y \in L$.*

3. Some algebraic results on MTL-filters

In this section, $(L, \vee, \wedge, \odot, \to, 0, 1)$ or simply L will denote an MTL-algebra, unless otherwise specified.

Definition 3.1. An MTL-algebra L is called an *$EIMTL$-algebra* if $x'' = 1 \Leftrightarrow x = 1$, for any $x \in L$.

Theorem 3.1. *Let F be a filter of L. The following conditions are equivalent on L: for all $x, y \in L$,*
(i) *F is an $EIMTL$-filter of L,*
(ii) *$x \to y \in F$ and $x'' \in F$ imply $y \in F$,*
(iii) *$(x \to y)'' \in F$ and $x'' \in F$ imply $y \in F$,*
(iv) *$x \to y'' \in F$ and $x'' \in F$ imply $y \in F$,*
(v) *$x \to y'' \in F$ and $x \in F$ imply $y \in F$,*
(vi) *L/F is an $EIMTL$-algebra.*

Proof. (i) $\Rightarrow$ (ii) Let $x \to y \in F$ and $x'' \in F$. Then by Lemma 2.2, $x \to y \in F$ and $x \in F$, so $y \in F$.

(ii) $\Rightarrow$ (iii) Let $(x \to y)'' \in F$ and $x'' \in F$. Since $(x \to y) \to (x \to y) \in F$ and $(x \to y)'' \in F$, then by (ii), $x \to y \in F$. On the other hand, $x \to x \in F$ and $x'' \in F$, so $x \in F$. Therefore, $y \in F$.

(iii) $\Rightarrow$ (iv) Let $x \to y'' \in F$ and $x'' \in F$. Then by $(x \to x)'' \in F$, $x'' \in F$ and (iii), we get $x \in F$, so $y'' \in F$. Now, by $(y \to y)'' \in F$ and $y'' \in F$ we get $y \in F$.

(iv) $\Rightarrow$ (v) Let $x \to y'' \in F$ and $x \in F$. Then $x \to y'' \in F$ and $x'' \in F$ and by (iv), we get $y \in F$.

(v) $\Rightarrow$ (vi) Let $[x]'' = [1]$, for some $x \in L$. Then $x'' \to 1$ and $1 \to x'' \in F$. Since $1 \in F$, then by (v), we get $x \in F$. Hence $[x] = [1]$. Therefore, L/F is an $EIMTL$-algebra.

(vi) $\Rightarrow$ (i) It follows from Lemma 2.2. $\square$

Now, we want to review Theorem 2.4 in details. We show that the intersection of the set of all $IMTL$-filters and the set of all implicative filters of L is exactly the set of all positive implicative filters of L.

Theorem 3.2. *F is a positive implicative filter of L if and only if F is an implicative and $IMTL$-filter.*

Proof. Let F be a positive implicative filter of L and $x \in L$. Then by Theorem 2.5, $(x' \to x) \to x \in F$. Since $0 \leq x$, then $x'' = x' \to 0 \leq x' \to x$ and so $(x' \to x) \to x \leq x'' \to x$. Therefore, $x'' \to x \in F$ and the result is obtained. The proof of the converse is straight consequent of Theorems 2.4. $\square$

By Theorem 2.7, it can be easily obtained that F is an ultra filter of L if and only if F is prime and positive implicative. In the next theorem, we want to investigate relation between positive implicative (ultra) filters and $\{x \in L|\ a \to x \in F\}$, for any $a \in L$.

Theorem 3.3. *Let F be a non-empty subset of L and $F_a = \{x \in L| \ \ a \to x \in F\}$, for all $a \in L$.*

(i) *F is a positive implicative filter of L if and only if F_a is an $EIMTL$-filter of L, for all $a \in L$,*

(ii) *Let F be a filter of L. Then F is an ultra filter if and only if $\{F_x| \ \ x \in L\} = \{F, L\}$.*

Proof. (i) Suppose that F is a positive implicative filter of L and $a \in L$. Let $(x \to y)'' \in F_a$ and $x \in F_a$, for some $x, y \in L$. Then $a \to (x \to y)'' \in F$ and $a \to x \in F$. Since F is a filter and

$$(a \to (x \to y)'') \to (a \to (x \to y)) \quad \geq \quad ((x \to y)'') \to (x \to y) \in F,$$

(by Theorem 3.2), then we conclude that $a \to (x \to y) \in F$. Since each positive implicative filter is implicative, we obtain $a \to y \in F$ and so $y \in F_a$. Therefore, F_a is an $EIMTL$-filter of L. Conversely, let F_a be an $EIMTL$-filter of L, for any $a \in L$. First, we show that F is a filter of L. Let $x \to y \in F$ and $x \in F$. Then $1 \to x \in F$ and $1 \to (x \to y) \in F$ and so $x, x \to y \in F_1$. Since F_1 is a filter, then we have $y \in F_1$ and so $y = 1 \to y \in F$. Hence F is a filter. Now, by Theorem 2.6, F is an implicative filter of L. Suppose that $x'' \in F$, for some $x \in L$. Since $x'' \in F_{x''}$, then by assumption, we get $x \in F_{x''}$ and so $x'' \to x \in F$. Hence F is an $IMTL$-filter. Therefore, by Theorem 3.2, F is a positive implicative filter of L.

(ii) Let F be an ultra filter and $x \in L$. Then by Theorem 2.7 and 2.5, F is an implicative filter and so by Theorem 3.3(i), F_x is a filter. If $x' \in F$, then $0 \in F_x$. It follows that $F_x = L$. Now, let $x' \notin F$. Then $x \in F$ and so $F_x = F$. Conversely, let $\{F_x|\ x \in L\} = \{F, L\}$ and $a \in L$. Then $F_a = L$ or $F_a = F$. If $F_a = L$, then $a' = a \to 0 \in F$. If $F_a = F$, then $a \in F_a = F$. Therefore, F is an ultra filter. □

Definition 3.2. Let F be a subset of L. The least $EIMTL$-filter of L containing F is called *$EIMTL$-filter generated by F* and is defined by $\langle F\rangle_E$.

By Lemma 2.2, we obtain $\langle F\rangle_E = \cap\{G|\ G$ is an $EIMTL$-filter of L containing F $\}$.

Theorem 3.4. *Let F be a filter of L. Then*

$\langle F\rangle_E = \{u \in L|\ (x_n \to (\cdots \to (x_2 \to (x_1 \to u)'')'' \cdots)'')'' \in F,\ \exists x_1, ..., x_n \in F,\ \exists n \in \mathbb{N}\}.$

Proof. Suppose that

$$A = \{u \in L|\ (x_n \to (\cdots \to (x_2 \to (x_1 \to u)'')'' \cdots)'')'' \in F,\ \exists x_1, ..., x_n \in F,\ \exists n \in \mathbb{N}\}.$$

(1) Since $(x \to x)'' = 1 \in F$, for any $x \in F$, $x \in A$. So, $F \subseteq A$.
(2) If $a'' \in A$, for some $a \in L$, then there exists $x_1, ..., x_n \in F$, such that

$$(x_n \to (\cdots \to (x_2 \to (x_1 \to a'')'')'' \cdots)'')'' \in F.$$

By (R2), we get

$$(x_n \to (\cdots \to (x_2 \to (x_1 \to (1 \to a)'')'')'' \cdots)'')'' \in F$$

and so $a \in A$.
(3) Let $a \to b \in A$ and $a \in A$. Then there are $x_1, ..., x_n, y_1, ..., y_m \in F$ such that

$$\alpha = (x_n \to (\cdots \to (x_2 \to (x_1 \to a)'')'' \cdots)'')'' \in F,$$

$$\beta = (y_m \to (\cdots \to (y_2 \to (y_1 \to (a \to b))'')'' \cdots)'')'' \in F.$$

In the following, we show that $b \in A$. By (R6) and (R4), we obtain

$$\beta \to [\alpha \to [(x_n \to (\cdots \to (x_2 \to (x_1 \to (y_m \to (\cdots \to (y_2 \to (y_1 \to b)'')'' \cdots)'')'')'')'' \cdots)'')'']] \geq$$

$$\beta \to [(x_n \to (\cdots \to (x_1 \to a)'' \cdots)'') \to (x_n \to (\cdots \to (x_1 \to (y_m \to (\cdots \to (y_1 \to b)'' \cdots)'')'')'' \cdots)'')] \geq$$

$$\beta \to [(x_{n-1} \to (\cdots \to (x_1 \to a)'' \cdots)'')'' \to (x_{n-1} \to (\cdots \to (x_1 \to (y_m \to (\cdots \to (y_1 \to b)'' \cdots)'')'')'' \cdots)'')''] \geq$$

$$\vdots$$

$$\beta \to [a \to (y_m \to (\cdots \to (y_1 \to b)'' \cdots)'')''] =$$

$$a \to \beta \to [(y_m \to (\cdots \to (y_1 \to b)'' \cdots)'')''] \geq$$

$$a \to [(y_m \to (\cdots \to (y_2 \to (y_1 \to (a \to b))'')'' \cdots)'') \to (y_m \to (\cdots \to (y_1 \to b)'' \cdots)'')] \geq$$

$$a \to [(y_{m-1} \to (\cdots \to (y_2 \to (y_1 \to (a \to b))'')'' \cdots)'')'' \to (y_{m-1} \to (\cdots \to (y_1 \to b)'' \cdots)'')''] \geq$$

$$\vdots$$

$$a \to [(a \to b) \to b] = 1 \in F.$$

Since $\alpha \in F$, $\beta \in F$ and F is a filter, then

$$(x_n \to (\cdots \to (x_2 \to (x_1 \to (y_m \to (\cdots \to (y_2 \to (y_1 \to b)'')'' \cdots)'')'')'')'' \cdots)'')'' \in F.$$

Hence by definition of A, $b \in A$ and so A is a filter of L. From (1), (2), (3) and Lemma 2.2, we obtain A is an $EIMTL$-filter of L containing F. Clearly $A \subseteq G$, for any $EIMTL$-filter G of L, which is containing F. Therefore, $\langle F\rangle_E = A$. □

Corollary 3.5. *$F = \{x \in L|\ x' = 0\}$ is the least $EIMTL$-filter of L and*

$$\theta = \{(x, y) \in L \times L|\ (x \to y)' = (y \to x)' = 0\}$$

is the least congruence relation on L such that L/θ is an $EIMTL$-algebra.

Proof. Since $G = \{1\}$ is a filter, then clearly, by Theorem 3.4 and $(R8)$,

$$\langle G\rangle_E = \{x \in L|\ x' = 0\}$$

and so F is a filter. Moreover, $F \subseteq E$, for any $EIMTL$-filter E of L. By $\theta = \equiv_F$, we conclude that θ is the least congruence relation on L such that L/θ is an $EIMTL$-algebra. □

We know that, in each BL-algebra, the concept of $EIMTL$-filter and fantastic filter are the same (see [2, Corollary 3.11]). In the next theorem, we want to answer this question, "under what condition these concepts are the same?"

Theorem 3.6. *Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a residuated lattice. Then any $EIMTL$-filter of L is fantastic if and only if L satisfies the condition $(x\Delta y)' = 0$, where $x\Delta y = ((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)$.*

Proof. Assume that L is a residuated lattice such that any $EIMTL$-filter of L is fantastic. Let $x, y \in L$. By Corollary 3.5, $F = \{x \in L|\ x' = 0\}$ is an $EIMTL$-filter of L and so it is fantastic filter. Hence by Theorem 2.9, $x\Delta y \in F$ and so $(x\Delta y)' = 0$. Conversely, let $(x\Delta y)' = 0$, for any $x, y \in L$ and F be an arbitrary $EIMTL$-filter of L. For any $x, y \in L$, $(x\Delta y)'' = 1 \in F$, so by Lemma 2.2, $x\Delta y \in F$. By Theorem 2.9, we conclude that F is a fantastic filter. □

Corollary 3.7. *If L is a BL-algebra, then $(x\Delta y)' = 0$, for any $x, y \in L$.*

Proof. It follows from [7, Lemma 1], and Theorem 3.6. □

Definition 3.3. Let F be a filter of L. Then F is called a *weak implicative filter* if $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z'' \in F$, for any $x, y, z \in F$.

By $x \leq x''$ and (R5), it can be obtained that any implicative filter of L is weak implicative.

Example 3.1. Let $(L = \{0, a, b, c, 1\}, \leq)$ be a partially ordered set such that $0 < a < b < c < 1$. Consider the following tables:

Table 1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Table 2

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Let $\vee$ and $\wedge$ be min and max on L, respectively. Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is an MTL-algebra and $\{1\}$ is a filter of L (see [2, Example 4.16]). Let $F = \{1\}$. It is easy to show that F is a weak implicative filter of L. Since $b \rightarrow (b \rightarrow a) = 1$ and $b \rightarrow a = b \notin F$, then F is not an implicative filter.

Proposition 3.8. *Let F be a filter of L. Then the following conditions are equivalent:*
(i) *F is a weak implicative filter,*
(ii) *$x \rightarrow (x^2)'' \in F$, for any $x \in L$,*
(iii) *$x \rightarrow (x \rightarrow y) \in F$ implies $x \rightarrow y'' \in F$.*

Proof. (i) ⇒ (ii) Let $x \in L$. Since $x \rightarrow (x \rightarrow x^2) = 1 \in F$ and $x \rightarrow x \in F$, we get $x \rightarrow (x^2)'' \in F$, by (i).

(ii)$\Rightarrow$ (iii) Let $x \to (x \to y) \in F$, for some $x, y \in L$. Since F is a filter, then by

$$(x^2)'' \to y'' \geq x^2 \to y = x \to (x \to y) \in F, \quad \text{by (R6)}$$

$$(x \to (x^2)'') \odot ((x^2)'' \to y'') \leq x \to y'', \quad \text{by (R7)}$$

we conclude that $x \to y'' \in F$.

(iii) $\Rightarrow$ (i) Let $x \to (y \to z) \in F$ and $x \to y \in F$. Since $y \to z \leq (x \to y) \to (x \to z)$, then $x \to (y \to z) \leq x \to ((x \to y) \to (x \to z))$ and so $x \to ((x \to y) \to (x \to z)) \in F$. Hence $x \to (x \to z) \in F$. Therefore, $x \to z'' \in F$. □

We know that any implicative filter of L is a strong filter (see [2]). We will prove that this result holds for weak implicative filters.

Proposition 3.9. *If F is a weak implicative filter of L, then F is a strong filter.*

Proof. Assume that F is a weak implicative filter of L and $x \in L$. Then by (R6),

$$\begin{aligned} (x'' \to x)' \to (x'' \to x)'' &\geq (x'' \to x)' \to (x'' \to x), \\ &\geq x' \to (x'' \to x), \\ &= (x' \odot x'') \to x = 0 \to x = 1 \in F. \end{aligned}$$

Since F is a weak implicative filter, then by Proposition 3.8, $(x'' \to x)' \to 0 \in F$. Therefore, F is a strong filter of L. □

Theorem 3.10. *Let F be an $EIMTL$-filter of L. Then F is a weak implicative filter if and only if F is an implicative filter.*

Proof. Assume that F is a weak implicative filter of L and $x \in L$. Since F is an $EIMTL$-filter, by Proposition 3.9, we conclude that F is an $IMTL$-filter. By Proposition 3.8, $x \to (x^2)'' \in F$. Since F is an $IMTL$-filter, then

$$(x \to (x^2)'') \to (x \to x^2) \geq (x^2)'' \to x^2 \in F, \quad \text{by (R6)}$$

and so $x \to x^2 \in F$. Therefore, by Theorem 2.8, F is an implicative filter of L. The converse is obvious. □

Corollary 3.11. *Let F be an filter of L. Then, F is a positive implicative filter if and only if F is an $EIMTL$ and weak implicative filter.*

Proof. It follows from Theorems 2.4 and 3.10. □

Definition 3.4. A filter F of L is called *weak positive implicative filter* of L if $(x \to y) \to x \in F$ implies $x'' \in F$, for all $x, y \in L$.

Clearly, if F is a positive implicative filter of L, then F is a weak positive implicative filter of L. Moreover, a filter F is a positive implicative if and only if it is an $EIMTL$ and weak positive implicative filter.

Example 3.2. Let $L = [0, 1]$ with ordinary partially order relation. Define $x \vee y = max\{x, y\}$, $x \wedge y = x \odot y = min\{x, y\}$ and

$$x \to y = \begin{cases} 1 & if\, x \leq y, \\ y & if\, y < x. \end{cases}$$

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is an MTL-algebra (see [8, Example 3.12]) and $F = [1/2, 1]$ is a filter of L. Let $(x \to y) \to x \in F$, for some $x, y \in L$.

(i) If $x \leq y$, then $x = 1 \to x = (x \to y) \to x \in F$ and so $x'' \in F$.

(ii) If $y < x$, then $0 < x$ and so $x'' = 1 \in F$.

Hence F is a weak positive implicative filter of L. But, it is not a positive implicative filter (since $(1/3 \to 0) \to 1/3 = 1$ and $1/3 \notin F$). Therefore, there exists a weak positive implicative filter, which is not positive implicative.

Theorem 3.12. *Let F be a filter of L. Then the following are equivalent:*
(i) *F is a weak positive implicative filter of L,*
(ii) *$x' \to x \in F$ implies $x'' \in F$, for any $x \in L$,*
(iii) *$((x' \to x) \to x)'' \in F$, for any $x \in L$,*
(iv) *$(x' \to x) \to x'' \in F$, for any $x \in L$,*
(v) *$x' \vee x'' \in F$, for any $x \in L$.*

Proof. (i) $\Rightarrow$ (ii) Straightforward.

(ii) $\Rightarrow$ (iii) Suppose that F is a weak positive implicative filter and $x \in L$.

$$\begin{aligned} ((x' \to x) \to x)' \to ((x' \to x) \to x) &\geq (x' \to x) \to (((x' \to x) \to x)' \to x), \text{ by (R4)} \\ &\geq ((x' \to x) \to x)' \to x', \text{ by (R6)} \\ &\geq x \to ((x' \to x) \to x), \text{ by (R6)} \\ &= 1. \text{ since } x \leq (x' \to x) \to x \end{aligned}$$

Since F is a weak positive implicative filter, then $((x' \to x) \to x)'' \in F$.

(iii) $\Rightarrow$ (iv) Let $x \in L$. Since F is a filter of L and

$$\begin{aligned} ((x' \to x) \to x)'' \to ((x' \to x) \to x'') &\geq (x' \to x) \to (((x' \to x) \to x)'' \to x''), \\ &\geq (x' \to x) \to (((x' \to x) \to x) \to x), \\ &= ((x' \to x) \to x) \to ((x' \to x) \to x), \\ &= 1 \in F. \end{aligned}$$

we get that $(x' \to x) \to x'' \in F$.

(iv) $\Rightarrow$ (i) Let $(x \to y) \to x \in F$, for some $x, y \in L$. Since $(x' \to x) \to x'' \in F$, $((x \to y) \to x \leq x' \to x$ and F is a filter, then we conclude that $x'' \in F$. Hence F is a weak positive implicative filter of L.

(v) $\Rightarrow$ (iv) Let $x \in L$. Since $x' \vee x'' \leq ((x' \to x'') \to x'') \wedge ((x'' \to x') \to x')$, then we get $(x' \to x'') \to x'' \in F$. Hence by $(R6)$,

$$((x' \to x'') \to x'') \to ((x' \to x) \to x'') \leq (x' \to x) \to (x' \to x'') \leq x \to x'' = 1 \in F$$

and so $(x' \to x) \to x'' \in F$.

(ii) $\Rightarrow$ (v) Let $x \in L$. Then by $(R9)$, we get

$$(x' \vee x'')' \to (x' \vee x'') = (x'' \wedge x') \to (x' \to x'') = 1 \in F.$$

Hence by (ii), $(x' \vee x'')'' \in F$. Since L is an MTL-algebra, then by $(R9)$ and $(M1)$, $x' \vee x'' \in F$. □

It has been known that, if F is a positive implicative filter of L, then L/F is a Boolean lattice. In the next theorem, we want to generalize this result for weak positive implicative filters. Note that, $L/F = L'/F$, for any positive implicative filter F of L.

Theorem 3.13. *Let $L' = \{x' | \ x \in L\}$, F be a weak positive implicative filter of L and L/F be the quotient MTL-algebra with respect to F. Then*
(i) *$x' \vee y' \in L'$, $x' \wedge y' \in L'$ and $x' \to y' \in L$, for any $x, y \in L$,*
(ii) *$L'/F = \{[x] | \ x \in L'\}$ is a Boolean algebra.*

Proof. (i) Let $x, y \in L$. Then by $(M1)$ and $(R9)$ we have $x' \vee y' = (x \wedge y)' \in L'$ and $x' \wedge y' = (x \vee y)' \in L'$. Moreover, $x' \to y' = x' \to (y \to 0) = (x' \odot y) \to 0 = (x' \odot y)' \in L'$, by $(R4)$.

(ii) Clearly, $[0], [1] \in L'/F$. Let $[a], [b] \in L'/F$. Then there exist $x, y \in L$ such that $a \equiv_F x'$ and $b \equiv_F y'$. Since by (i), $x' \vee y' \in L'$, then $[a] \vee [b] = [x'] \vee [y'] = [x' \vee y'] \in L'/F$. By the similar way, we can show that $[a] \wedge [b] \in L'/F$. Hence L'/F is a sublattice of L/F containing $[0]$ and $[1]$. Since L is a distributive lattice, then clearly, L'/F is distributive. It follows from Theorem 3.12 that $[a] \vee [a]' = [a] \vee [a'] = [x'] \vee [x''] = [x' \vee x''] = [1]$ and so by $(R8)$ and $(R9)$, we get $[a] \wedge [b] = [x' \wedge x''] = [x''' \wedge x''] = [(x'' \vee x')'] = [(x'' \vee x')]' = ([x''] \vee [x'])' = ([a'] \vee [a']) = [1]' = [0]$. Therefore, L'/F is a Boolean algebra. □

Theorem 3.14. *Every weak positive implicative filter of L is a weak implicative filter.*

Proof. Let F be a weak positive implicative filter of L and $x \to (x \to y) \in F$, for some $x, y \in L$. Since by (R6), $x \to (x \to y) \leq ((x \to y) \to y) \to (x \to y)$ and F is a weak positive implicative filter, then $(x \to y)'' \in F$. Moreover,

$$\begin{aligned}(x \to y)'' \to (x \to y'') &= x \to ((x \to y)'' \to y''), \text{ by } (R4)\\ &\geq x \to ((x \to y) \to y), \text{ by } (R5) \text{ and } (R8)\\ &= (x \to y) \to (x \to y) = 1.\end{aligned}$$

and so $x \to y'' \in F$. Therefore, by Proposition 3.8, F is a weak implicative filter. □

Corollary 3.15. *Every weak positive implicative filter of L is a strong filter.*

Proof. It follows from Proposition 3.9 and Theorem 3.14. □

Example 3.3. (i) Consider the MTL-algebra in Example 3.1. It can be easily obtained that $\{1\}$, $\{1, c\}$ are two prime filters of L. But, $\{1\}$ is not implicative and $\{1, c\}$ is not fantastic.

(ii) Let $(L = \{0, a, b, c, 1\}, \leq)$ be a partially ordered set such that $0 < a < b < c < 1$. Consider the following tables:

Table 3

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	a	a	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Table 4

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

Let $\vee$ and $\wedge$ be min and max on L, respectively. Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is an MTL-algebra (see [2]) and $\{1\}$ is a prime (an $EIMTL$)-filter of L. But, it is not strong filter. Moreover, $\{1, c\}$ is a prime filter of L. But, it is not $EIMTL$-filter.

In the next remark, we will verify the figure which appeared in [2] and attempt to improve and correct it.

Remark 3.1. *Let F be an associative filter of L. Then by $0 \to (1 \to 0) = 1 \in F$ and $0 \to 1 = 1 \in F$, we obtain $0 \in F$ and so $F = L$. Hence L does not have any proper associative filter. Therefore, Theorem 3.18, 3.19 and 3.20, which appeared in* [2], *are trivial.*

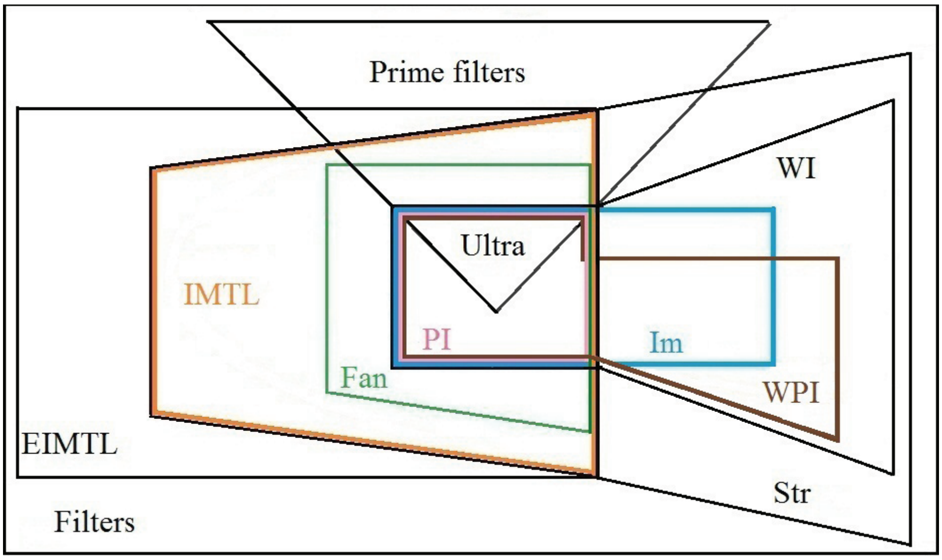

FIGURE 1. Relation between filters of MTL-algebras.

Remark 3.2. *Let* **EIMTL**, **IMTL**, **PI**, **Fan**, **Im**, **WI**, **WPI**, **Prime**, **Ultra** *and* **Str** *be the set of all $EIMTL$-, $IMTL$-, positive implicative, fantastic, implicative, weak implicative, weak positive implicative, Prime, ultra and strong filters of L, respectively. Then we have*

$$\mathbf{PI} \subseteq \mathbf{WPI} \subseteq \mathbf{WI} \subseteq \mathbf{Str}$$

$$\mathbf{PI} \subseteq \mathbf{Fan} \subseteq \mathbf{IMTL} \subseteq \mathbf{EIMTL}$$

$$\mathbf{PI} = \mathbf{IMTL} \cap \mathbf{WI} = \mathbf{EIMTL} \cap \mathbf{WI} = \mathbf{EIMTL} \cap \mathbf{Im} = \mathbf{IMTL} \cap \mathbf{Im}$$

$$\mathbf{EIMTL} \cap \mathbf{Str} = \mathbf{IMTL}$$

$$\mathbf{Ultra} = \mathbf{Prime} \cap \mathbf{PI}$$

In Figure 1, we try to depict the relation between filters in MTL-algebras.

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