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The univalence conditions for two integral operators

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Abstract. In this paper, we introduce two new integral operators. The main object of the present paper is to discuss some univalence conditions for these operators.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers.
Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$.

In this paper, we define two families of integral operators. The first family of integral operators is defined as follows:

$$F_n(f;g)(z) = \left(\sum_{i=1}^{n} \frac{1}{\gamma_i} \int_0^z t^{-1} \prod_{i=1}^{n} \left(f_i(t) e^{g_i(t)} \right)^{\frac{1}{\gamma_i}} dt \right)^{\frac{1}{\sum_{i=1}^{n} \frac{1}{\gamma_i}}} \tag{1.1}$$

$f_i, g_i \in \mathcal{A}$ and $\gamma_i \in \mathbb{C}$, $\gamma_i \neq 0$, for all $i \in \{1, 2, ..., n\}$.

Remark 1.1. For $n = 1$, $f_1 = f$, $\gamma_1 = \gamma$ and $g_1(z) = 0$ from (1.1) we obtain the integral operator

$$J_\gamma(z) = \left(\frac{1}{\gamma} \int_0^z t^{-1} \left(f(t) \right)^{\frac{1}{\gamma}} dt \right)^{\gamma}$$

studied in [4].

The second family of integral operators has the following form:

$$G_n(f;g)(z) = \left(\beta \int_0^z t^{\beta - 1} \prod_{i=1}^{n} \left(f_i'(t) e^{g_i(t)} \right)^{\alpha_i} dt \right)^{\frac{1}{\beta}} \tag{1.2}$$

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$$f_i, g_i \in \mathcal{A},\ \alpha_i \in \mathbb{C} \text{ and } \beta \in \mathbb{C} \setminus \{0\} \text{ for all } i \in \{1,2,...,n\}.$$

Remark 1.2. From (1.2) for $g_1 = g_2 = ... = g_n = 0$, we obtain the integral operator

$$F_{\alpha_1,...,\alpha_n,\beta}(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} (f_i'(t))^{\alpha_i} dt\right)^{\frac{1}{\beta}}$$

which was introduced by D. Breaz and N. Breaz in [1].

In particular, for $\beta = 1$, the integral operator $F_{\alpha_1,...,\alpha_n,\beta}(z)$ reduces to the integral operator $F_{\alpha_1,...,\alpha_n}(z)$ which was studied by Breaz *et al.* (see [2]). We observe also that, for $n = \beta = 1$, the integral operator $F(z)$ was introduced and studied by Pfaltzgraff (see [6]) and Kim and Merkes (see [3]).

In the present paper, we obtain some sufficient conditions for the integral operators $F_n(f;g)(z)$ and $G_n(f;g)(z)$ to be in the class $\mathcal{S}$.

In the proof of our main results we need:

Theorem 1.1. [5] *Let β be a complex number,* $\mathrm{Re}\beta > 0$ *and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad z \in \mathbb{U},$$

then the integral operator $F_\beta(z)$, defined by

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t) dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

2. Main Results

Our main univalence conditions for the integral operators $F_n(f;g)(z)$ and $G_n(f;g)(z)$ defined by (1.1), (1.2) are asserted by Theorem 2.1 and Theorem 2.3 below.

Theorem 2.1. *Let γ_i be complex numbers, $\gamma_i \neq 0$, $\sum_{i=1}^n \frac{1}{\gamma_i} = \beta$ and M_i, N_i are real positive numbers for all $i \in \{1,2,...,n\}$. Also, let the functions $f_i(z), g_i(z) \in \mathcal{A}$ satisfy the conditions*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad z \in \mathbb{U}, \quad \textit{and} \quad |g_i(z)| \leq N_i, \quad z \in \mathbb{U}, \tag{2.1}$$

for all $i \in \{1,2,...,n\}$. If

$$\left|\frac{zg_i'(z)}{g_i(z)}\right| \leq 1, \quad z \in \mathbb{U}, \tag{2.2}$$

and

$$\mathrm{Re}\beta \geq \sum_{i=1}^{n} \frac{M_i + N_i}{|\gamma_i|} \tag{2.3}$$

for all $i \in \{1,2,...,n\}$, then the integral operator $F_n(f;g)(z)$ defined by (1.1) is in the class $\mathcal{S}$.

Proof. We begin by observing that the integral operator $F_n(f;g)(z)$ in (1.1) can be rewritten as follows:

$$F_n(f;g)(z) = \left(\sum_{i=1}^{n} \frac{1}{\gamma_i} \int_0^z t^{\left(-1+\sum_{i=1}^n \frac{1}{\gamma_i}\right)} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} e^{g_i(t)}\right)^{\frac{1}{\gamma_i}} dt\right)^{\frac{1}{\sum_{i=1}^n \frac{1}{\gamma_i}}}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\frac{1}{\gamma_i}} dt, \quad f_i, g_i \in \mathcal{A},$$

for all $i \in \{1, 2, ..., n\}$.
The function $h(z)$ is regular in $\mathbb{U}$ and satisfy the following usual normalization condition:

$$h(0) = h'(0) - 1 = 0.$$

Now, calculating the derivatives of $h(z)$ of the first and second orders, we readily obtain

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z} e^{g_i(z)} \right)^{\frac{1}{\gamma_i}}$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(\left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \frac{zg_i'(z)}{g_i(z)} g_i(z) \right), \quad z \in \mathbb{U}.$$

Thus, we have

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left(\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \left| \frac{zg_i'(z)}{g_i(z)} \right| |g_i(z)| \right)$$

so

$$\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left(\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \left| \frac{zg_i'(z)}{g_i(z)} \right| |g_i(z)| \right). \quad (2.4)$$

From the hypothesis (2.1), (2.2) of Theorem 2.1 and from the inequality (2.4), we have

$$\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{M_i + N_i}{|\gamma_i|} \leq \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{M_i + N_i}{|\gamma_i|}$$

which in the lights of the hypothesis (2.3) of Theorem 2.1, we obtain

$$\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, \quad z \in \mathbb{U}.$$

Applying Theorem 1.1, we conclude that the integral operator $F_n(f; g)(z)$ defined by (1.1) is in the class $\mathcal{S}$. □

Setting $n = 1$ in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

Corollary 2.2. *Let γ be a complex number, $\gamma \neq 0$, and M, N are real positive numbers. Also, let the functions $f(z), g(z) \in \mathcal{A}$ satisfy the conditions*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq M, \quad z \in \mathbb{U}, \quad \textit{and} \quad |g(z)| \leq N, \quad z \in \mathbb{U}.$$

If

$$\left| \frac{zg'(z)}{g(z)} \right| \leq 1, \quad z \in \mathbb{U}, \quad \textit{and} \quad M + N \leq \frac{|\gamma|}{\mathrm{Re}\gamma}$$

then the integral operator

$$F(f;g)(z)=\left(\frac{1}{\gamma}\int_0^z t^{-1}\left(f(t)e^{g(t)}\right)^{\frac{1}{\gamma}}dt\right)^{\gamma}$$

is in the class $\mathcal{S}$.

Theorem 2.3. *Let* α_i, β *be complex numbers with* $\mathrm{Re}\beta>0$ *and* M_i, N_i *real positive numbers for all* $i\in\{1,2,...,n\}$. *Also, let the functions* $f_i(z),g_i(z)\in\mathcal{A}$ *satisfy the conditions*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right|\le M_i,\quad z\in\mathbb{U},\quad \text{and}\quad |g_i(z)|\le N_i,\quad z\in\mathbb{U} \tag{2.5}$$

for all $i\in\{1,2,...,n\}$. *If*

$$\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\le 1,\quad z\in\mathbb{U}, \tag{2.6}$$

and

$$\mathrm{Re}\beta\ge\sum_{i=1}^{n}|\alpha_i|\,(M_i+2N_i) \tag{2.7}$$

for all $i\in\{1,2,...,n\}$, *then the integral operator* $G_n(f;g)(z)$ *defined by (1.2) is in the class* $\mathcal{S}$.

Proof. Let us consider the function

$$h(z)=\int_0^z\prod_{i=1}^{n}\left(f_i'(t)e^{g_i(t)}\right)^{\alpha_i}dt. \tag{2.8}$$

The function $h(z)$ is regular in $\mathbb{U}$. From (2.8), we have

$$h'(z)=\prod_{i=1}^{n}\left(f_i'(z)e^{g_i(z)}\right)^{\alpha_i}$$

and

$$\begin{aligned}\frac{zh''(z)}{h'(z)}&=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}+zg_i'(z)\right)\\&=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}+\frac{zg_i'(z)}{g_i(z)}g_i(z)\right)\end{aligned}$$

which readily shows that

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right|&\le\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{zf_i''(z)}{f_i'(z)}\right|+\left|\frac{zg_i'(z)}{g_i(z)}\right||g_i(z)|\right)\\&\le\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{zf_i''(z)}{f_i'(z)}\right|+\left(\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+1\right)|g_i(z)|\right)\end{aligned} \tag{2.9}$$

From (2.9) and the conditions (2.5) and (2.6) of Theorem 2.3, we get

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\sum_{i=1}^{n}|\alpha_i|\left(M_i+2N_i\right)\\ &\leq \frac{1}{\mathrm{Re}\beta}\sum_{i=1}^{n}|\alpha_i|\left(M_i+2N_i\right)\\ &\leq 1\end{aligned}$$

where we have also used the hypothesis (2.7) of Theorem 2.3.
Finally, by applying Theorem 1.1, we conclude that the integral operator $G_n(f;g)(z)$ defined by (1.2) is in the class $\mathcal{S}$. □

Setting $n = 1$ in Theorem 2.3, we obtain the following consequence of Theorem 2.3.

Corollary 2.4. *Let α, β be complex numbers with $\mathrm{Re}\beta > 0$ and M, N real positive numbers. Also, let the functions $f(z), g(z) \in \mathcal{A}$ satisfy the conditions*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq M, \quad z \in \mathbb{U}, \quad \text{and} \quad |g(z)| \leq N, \quad z \in \mathbb{U}.$$

If

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leq 1, \quad z \in \mathbb{U},$$

and

$$\mathrm{Re}\beta \geq |\alpha|\left(M + 2N\right)$$

then the integral operator

$$G(f,g)(z) = \left(\beta \int_0^z t^{\beta-1}\left(f'(t)e^{g(t)}\right)^{\alpha} dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

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