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# On Jachymski's theorem

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Abstract. In this note we prove that a fixed point theorem (due to Jachymski in [5]) extends and subsumes some results in the fixed point theory.

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## 1. Introduction

The classical Banachs contraction principle is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

**Theorem 1.1.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ a strict contraction, i.e. a map satisfying*

$$d(Tx, Ty) \leqslant ad(x, y), \text{for all } x, y \in X, \tag{1}$$

*where $0 < a < 1$ is constant. Then:*
(1) *$T$ has a unique fixed point $x^*$ in $X$;*
(2) *The Picard iteration $(x_n)_{n \geqslant 0}$ defined by*

$$x_{n+1} = Tx_n, n = 0, 1, 2, ... \tag{2}$$

*converges to $x^*$, for any $x_0 \in X$.*

There is more than one way to generalize the Banach's contraction mapping principle. One of them is to get the subset $M$ of $X \times X$, the contraction condition to be satisfied only for $(x, y) \in M$ and still the operator is the operator is a Picard operator (abbr. PO). This article will prove that the Jachymski's theorem is one of the most general theorems of its kind.

Let $T$ be a selfmap of a metric space $(X, d)$. Following Petruşel and Rus [12], we say that $T$ is a *Picard operator* if $T$ has a unique fixed point $x^*$ and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$ and $T$ is a *weakly Picard operator* (abbr. WPO) if the sequence $(T^n x)_{n\in\mathbb{N}}$ converges, for all $x \in X$ and the limit (which may depend on $x$) is a fixed point of $T$. Thus any contraction on complete metric space is PO.

Let $(X, d)$ be a metric space. Let $\Delta$ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph $G$ such that the set $V(G)$ of its vertices coincides with $X$, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume $G$ has no parallel edges, so we can identify $G$ with the pair $(V(G), E(G))$. Moreover, we may treat $G$ as a weighted graph (see [[6], p. 309]) by assigning to each edge the

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distance between its vertices. By $G^{-1}$ we denote the conversion of graph $G$, i.e. the graph obtained from $G$ by reversing the direction of edges. Thus we have

$$E\left(G^{-1}\right)=\{(x,y)\mid(y,x)\in G\}.$$

The letter $\tilde{G}$ denotes the undirected graph obtained from $G$ by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E\left(\tilde{G}\right)=E\left(G\right)\cup E\left(G^{-1}\right) \tag{3}$$

We call $(V',E')$ a subgraph of $G$ if $V'\subseteq V\left(G\right)$, $E'\subseteq E\left(G\right)$ and for any edge $(x,y)\in E'$, $x,y\in V'$.

Now we recall a few basic notions concerning the connectivity of graphs. All of them can be found, e.g., in [6]. If $x$ and $y$ are vertices in a graph $G$, then *a path* in $G$ from $x$ to $y$ of length $N$ ($N\in\mathbb{N}$) is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0=x$, $x_N=y$ and $(x_{i-1},x_i)\in E\left(G\right)$ for $i=1,...,N$. A graph $G$ is *connected* if there is a path between any two vertices. $G$ is *weakly connected* if $\tilde{G}$ is connected. If $G$ is such that $E\left(G\right)$ is symmetric and $x$ is a vertex in $G$, then the subgraph $G_x$ consisting of all edges and vertices which are contained in some path beginning at $x$ is called the component of $G$ containing $x$. In this case $V\left(G_x\right)=[x]_G$, where $[x]_G$ is the equivalence class of the following relation $R$ defined on $V\left(G\right)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Clearly, $G_x$ is connected.

Recently, J. Jakhymski [5] was the first author who gave sufficient conditions for an operator to be a PO if $(X,d)$ is endowed with a graph and defined the next concept:

**Definition 1.1** ([5], Def. 2.1)**.** We say that a mapping $f:X\to X$ is a *Banach G-contraction* or simply *G-contraction* if $f$ preserves edges of $G$, i.e.,

$$\forall x,y\in X\left((x,y)\in E\left(G\right)\Rightarrow\left(f\left(x\right),f\left(y\right)\right)\in E\left(G\right)\right) \tag{4}$$

and $f$ decreases weights of edges of $G$ in the following way:

$$\exists\alpha\in(0,1),\forall x,y\in X\left((x,y)\in E\left(G\right)\Rightarrow d\left(f\left(x\right),f\left(y\right)\right)\leqslant\alpha d\left(x,y\right)\right). \tag{5}$$

The main theorem is:

**Theorem 1.2** ([5], Th 3.2)**.** *Let $(X,d)$ be complete, and let the triple $(X,d,G)$ have the following property:*

*(P) for any $(x_n)_{n\in\mathbb{N}}$ in $X$, if $x_n\to x$ and $(x_n,x_{n+1})\in E\left(G\right)$ for $n\in\mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n},x)\in E\left(G\right)$ for $n\in\mathbb{N}$.*

*Let $f:X\to X$ be a $G$-contraction, and $X_f=\{x\in X\,|(x,fx)\in E\left(G\right)\}$. Then the following statements hold.*

1. $\operatorname{card} Fix\, f=\operatorname{card}\{[x]_{\tilde{G}}\,|x\in X_f\}$.
2. $\operatorname{Fix} f\neq\emptyset$ iff $X_f\neq\emptyset$.
3. *$f$ has a unique fixed point iff there exists $x_0\in X_f$ such that $X_f\subseteq[x_0]_{\tilde{G}}$.*
4. *For any $x\in X_f$, $f\left|_{[x]_{\tilde{G}}}\right.$ is a PO.*
5. *If $X_f\neq\emptyset$ and $G$ is weakly connected, then $f$ is a PO.*
6. *If $X':=\cup\{[x]_{\tilde{G}}\,|x\in G\}$ then $f|_{X'}$ is a WPO.*
7. *If $f\subseteq E\left(G\right)$, then $f$ is a WPO.*

Here $Fixf$ denotes the set of fixed points of operator $f$.

Since then several authors have considered the problem of existence and uniqueness of a fixed point for contraction type operators in metric spaces endowed with a graph (see [1], [2], [3], [9]).

## 2. Main results

Using Jachymski's Theorem [5] we can give alternate proofs to some known results by choosing the right form of graph.

**Proposition 2.1.** *Banach's contraction principle is a consequence of Jakhymski's Theorem.*

*Proof.* Let the graph $G_0$ defined by $V(G) = X$ and $E(G_0) = X \times X$, which is a connected graph and the contraction $T$ by Theorem 1.1 is a $G_0$-Banach contraction. The property (P) is fulfilled so from Theorem 1.1 the operator $T$ is PO. □

In the following we prove that the fixed point theorems for contractions in partially ordered metric spaces are consequences of Jakhymski's Theorem.

**Theorem 2.2** (Ran and Reurings [14], Th 2.1)**.** *Let $(X, d)$ be a complete metric space endowed with a partial ordering "$\leq$" such that*

$$every\ pair\ of,\ elements\ of\ X\ has\ an\ upper\ and\ a\ lower\ bound. \tag{6}$$

*Let $T : X \to X$ be continuous and monotone, and such that*

$$\exists \alpha \in (0,1)\ \forall x, y \in X\ (x \leqslant y \Rightarrow d(Tx, Ty) \leqslant \alpha d(x, y)). \tag{7}$$

*If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$ or $Tx_0 \leqslant x_0$, then $T$ is a PO.*

*Proof.* Let $G$ be the graph defined by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X \,|x \leq y\}.$$

Because every pair of elements of $X$ has an upper and a lower bound the graph $G$ is weakly connected. The mapping $T : X \to X$ which satisfies (7) and is monotone is a $G$-Banach contraction. From the continuity of mapping $T$ we get that the property (P) is true so from Theorem 1.1 the operator $T$ is PO. □

Further improvements of the above results were found independently by Petruşel and Rus [12], and Nieto and Rodríguez-López [11]. Here we give a slightly more general version of these extensions taken from the paper by Nieto, Pouso and Rodríguez-López [10]. Following [12] we denote:

$$X_{\leq} := \{(x, y) \in X \times X \,|\, x \leq y \text{ or } y \leq x\}.$$

**Theorem 2.3.** *Let $(X, d)$ be a complete metric space endowed with a partial ordering "$\leq$" such that every pair of elements of $X$ has an upper or a lower bound. Let $T : X \to X$ be such that $T$ preserves comparable elements, i.e.*

$$for\ any\ x, y \in X, (x, y) \in X_{\leq}\ implies\ \ (Tx, Ty) \in X_{\leq}, \tag{8}$$

*and (7) holds. Assume that either $T$ is orbitally continuous or $(X, d, \leq)$ is such that*

$$\begin{gathered} for\ any\ \ (x_n)_{n\in\mathbb{N}},\ if\, x_n \to x\ and\ \ (x_n, x_{n+1}) \in X_{\leq},\ for\ n \in \mathbb{N},\ then \\ there\ is\ a\ subsequence\ (x_{k_n})_{n\in\mathbb{N}}\ such \quad (x_{k_n}, x) \in X_{\leq}\ \forall n \in \mathbb{N}. \end{gathered} \tag{9}$$

*If there exists $x_0 \in X$ with $(x_0, Tx_0) \in X_{\leq}$, then $T$ is a PO.*

*Proof.* Let $G$ be the graph defined by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X \,|x \leq y \text{ or } y \leq x\}.$$

Because every pair of elements of $X$ has an upper and a lower bound the graph $G$ is weakly connected. The mapping $T : X \to X$ which satisfies (7) and is monotone is a $G$-Banach contraction. From the orbitally continuity of mapping $T$ or by (9) we get the property (P) is true so from Theorem 1.1 the operator $T$ is PO. □

On the other hand, Theorem 1.1 yields directly the following well-known fixed point theorem which is quite different from the above results.

**Theorem 2.4** (Edelstein, [7]). *Let $(X, d)$ be complete and $\epsilon$-chainable for some $\epsilon > 0$, i.e., given $x, y \in X$, there is $N \in \mathbb{N}$ and a sequence $(x_i)_{i=0}^{N}$ such that $x_0 = x, x_N = y$ and $d(x_{i-1}, x_i) < \varepsilon$ for $i = 1, ..., N$. Let $T : X \to X$ be such that*

$$\exists \alpha \in (0,1), \forall x, y \in X \left(d(x,y) < \varepsilon \Rightarrow d(Tx, Ty) \leqslant \alpha d(x,y)\right). \tag{10}$$

*Then $T$ is a PO.*

*Proof.* Clearly, (10) implies $T$ is continuous. Consider the graph $G$ with $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X \,| d(x,y) < \varepsilon\}.$$

Then $\epsilon$-chainability of $(X, d)$ means $G$ is connected. If $(x, y) \in E(G)$, then

$$d(Tx, Ty) \leqslant \alpha d(x, y) \leqslant \alpha\varepsilon \leqslant \varepsilon.$$

Hence (4) and (5) hold, so $T$ is a $G$-contraction. By Theorem 1.1 , $T$ is a PO. □

In the following we show the fixed point theorem for cyclic contractions proved in [8] by W.A. Kirk, P.S. Srinivasan and P. Veeramani is a consequence of Theorem 1.1.

**Definition 2.1.** Let $p \in \mathbb{N}$, $p \geq 2$ and $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space $(X, d)$. An operator $T : \{A_i\}_{i=1}^{p} \to \{A_i\}_{i=1}^{p}$ is called a *cyclic operator* if the following condition is satisfied:

$$T(A_i) \subseteq A_{i+1} \; for \; all \; i \in \{1, 2, ..., p\}, \tag{11}$$

where $A_{p+1} = A_1$.

**Theorem 2.5** ([8]). *Let $p \in \mathbb{N}$, $p \geq 2$ and $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space $(X, d)$, and suppose $T : \{A_i\}_{i=1}^{p} \to \{A_i\}_{i=1}^{p}$ satisfies (11) and the following one*

$$d(Tx, Ty) \leqslant kd(x, y), \; for \;\; all\, x \in A_i, \; y \in A_{i+1}, 1 \leq i \leq p, \tag{12}$$

*where $A_{p+1} = A_1$. If $k \in [0, 1)$ then $T$ has an unique fixed point.*

*Proof.* Consider the graph $G$ with $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X \,| x \in A_i \text{ şi } y \in A_{i+1}, i = 1, ..., p\}$$

Because $T$ is a cyclic operator and $(x, y) \in E(G)$ then $x \in A_i$ and $y \in A_{i+1}$ so $Tx \in A_{i+1}$ and $Ty \in A_{i+2}$, which implies $(Tx, Ty) \in E(G)$. Using (12) we have $T$ is a $G$-contraction. From the definition of edges of $G$ we have that $G$ is weakly connected. Let $(x_n)_{n\in\mathbb{N}}$ in $X$, with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$. Then there is $j \in \{1, 2, ..., n\}$ such that $x \in A_j$. However in view of (11) the sequence $\{x_n\}$ has an infinite number of terms in each $A_i$, for all $i \in \{1, 2, ..., n\}$. The subsequence of the sequence $\{x_n\}$ formed by the terms which are in $A_{j-1}$ satisfies the condition (P) from Theorem 1.1. Obviously $X_T \neq \emptyset$ so $T$ is PO. □

The last consequence of Jachymski's Theorem which we present is The Alternative of Fixed Point, due to Diaz and Margolis [4]. Here we will give a slightly more general version of these extensions taken from the paper by V. Radu [13].

**Theorem 2.6** (The fixed point alternative)**.** *Suppose we are given a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with the Lipschitz constant $a$. Then, for each given element $x \in \Omega$, either*

$$d\left(T^n x,\, T^{n+1} x\right) = \infty,\ \forall n \geqslant 0,$$

*or there exists a natural number $n_0$ such that*

i. $d\left(T^n x,\, T^{n+1} x\right) < \infty$ *for all $n \geqslant n_0$;*
ii. *The sequence $\left(T^n x\right)_{n\geqslant 0}$ is convergent to a fixed point $y*$ of $T$;*
iii. *$y*$ is the unique fixed point of $T$ in the set $\Delta = \{y \in \Omega \,|\, d\left(T^{n_0} x, y\right) < \infty\}$;*
iv. $d\left(y,\, y*\right) \leqslant \frac{1}{1-a} d\left(y,\, Ty\right)$ *for all $y \in \Delta$.*

*Proof.* Let the graph $G$ with $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X \,|\, d(x, y) < \infty\}.$$

By the symmetry of distance we have $\tilde{G} = G$.

If $(x, y) \in E(G)$, so $d(x, y) < \infty$, then $d(Tx, Ty) \leqslant ad(x, y) < \infty$ in conclusion $(Tx, Ty) \in E(G)$, and $T$ being a strictly contractive mapping we have $T$ is a $G$-contraction.

Let $x \in X$ such that, there exists $n_0 \in \mathbb{N}$ with property $d\left(T^{n_0} x,\, T^{n_0+1} x\right) < \infty$, then $\left(T^{n_0} x,\, T^{n_0+1} x\right) \in E(G)$, that is $T^{n_0} x \in X_T$ so

$$X_T \neq \emptyset.$$

An easy induction shows

$$d\left(T^n x, T^{n+1} x\right) \leqslant a^{n-n_0} d\left(T^{n_0} x, T^{n_0+1} x\right), \text{ for all } n \geqslant n_0$$

so $d\left(T^n x,\, T^{n+1} x\right) < \infty$ for all $n \geqslant n_0$, therefore the relation *i.* is true.

For the same $x$ like the one above, we have:

$$\left[T^{n_0} x\right]_G = \{y \in X \,|\, \exists \text{ a path in } G \text{ from } x \text{ to } y\}.$$

If $y \in \left[T^{n_0} x\right]_G$ then there is a path $(x_i)_{i=0}^N$ in $G$ from $T^{n_0} x$ to $y$, that is, $x_0 = T^{n_0} x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., N$. Then

$$d\left(T^{n_0} x, y\right) \leqslant \sum_{i=1}^{N} d\left(x_{i-1}, x_i\right) < \infty.$$

Consequently $\left[T^{n_0} x\right]_G \subseteq \Delta$ and the relation $\Delta \subseteq \left[T^{n_0} x\right]_G$ is obvious, so $\Delta = \left[T^{n_0} x\right]_G$.

If $(x_n)_{n\in\mathbb{N}}$ converges to $x^* \in X$ with property $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ then for $\epsilon = 1$ there exists $n_\epsilon \in \mathbb{N}$ such that $d(x_n, x^*) < 1$ for all $n \geq n_\epsilon$, that is

$$(x_n, x^*) \in E(G),\ \forall n \geqslant n_\epsilon.$$

The subsequence $(x_n)_{n\geqslant n_0}$ satisfies the condition (P) from Theorem 1.2. Then from Theorem 1.2, *4.* we have $T|_{\left[T^{n_0} x\right]_G}$ is PO which implies *ii.* and *iii.* from The Fixed Point Alternative.

For *iv.*, let $y \in \Delta$ so $d\left(y, T^{n_0} x\right) < \infty$. Because $\left(T^n x\right)_{n\geqslant n_0}$ converges, then $d\left(T^{n_0} x, T^{n-1} x\right) < \infty$ for all $n > n_0$ and $d\left(y, T^{n-1} x\right) < \infty$, that is $\left(y, T^{n-1} x\right) \in E(G)$. By the triangle inequality and because $T$ is a $G$-contraction, we get:

$$d\left(y, T^n x\right) \leqslant d(y, Ty) + d\left(Ty, T^n x\right) \leqslant d(y, Ty) + ad\left(y, T^{n-1} x\right), \tag{13}$$

for all $n \geq n_0$.

Hence, letting $n$ tend to $\infty$ in (13) we conclude

$$d(y, y^*) \leqslant d(y, Ty) + ad(y, y^*)$$

that is $d(y, y*) \leqslant \frac{1}{1-a} d(y, Ty)$ for any $y \in \Delta$, which completes the proof. □


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