On Jachymski’s theorem

FLORIN BOJOR

ABSTRACT. In this note we prove that a fixed point theorem (due to Jachymski in [5]) extends and subsumes some results in the fixed point theory.

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1. Introduction

The classical Banach’s contraction principle is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

Theorem 1.1. Let \((X, d)\) be a complete metric space and \(T : X \rightarrow X\) a strict contraction, i.e. a map satisfying

\[d(Tx, Ty) \leq ad(x, y), \text{ for all } x, y \in X,\]

where \(0 < a < 1\) is constant. Then:

1. \(T\) has a unique fixed point \(x^*\) in \(X\);
2. The Picard iteration \((x_n)_{n \geq 0}\) defined by

\[x_{n+1} = Tx_n, n = 0, 1, 2, \ldots\]

converges to \(x^*\), for any \(x_0 \in X\).

There is more than one way to generalize the Banach’s contraction mapping principle. One of them is to get the subset \(M \subseteq X \times X\), the contraction condition to be satisfied only for \((x, y) \in M\) and still the operator is a Picard operator (abbr. PO). This article will prove that the Jachymski’s theorem is one of the most general theorems of its kind.

Let \(T\) be a selfmap of a metric space \((X, d)\). Following Petruşel and Rus [12], we say that \(T\) is a Picard operator if \(T\) has a unique fixed point \(x^*\) and \(\lim_{n \to \infty} T^n x = x^*\) for all \(x \in X\) and \(T\) is a weakly Picard operator (abbr. WPO) if the sequence \((T^n x)_{n \in \mathbb{N}}\) converges, for all \(x \in X\) and the limit (which may depend on \(x\)) is a fixed point of \(T\). Thus any contraction on complete metric space is PO.

Let \((X, d)\) be a metric space. Let \(\Delta\) denote the diagonal of the Cartesian product \(X \times X\). Consider a directed graph \(G\) such that the set \(V(G)\) of its vertices coincides with \(X\), and the set \(E(G)\) of its edges contains all loops, i.e., \(E(G) \supseteq \Delta\). We assume \(G\) has no parallel edges, so we can identify \(G\) with the pair \((V(G), E(G))\). Moreover, we may treat \(G\) as a weighted graph (see [6], p. 309) by assigning to each edge the...
We say that a mapping $f : X \to X$ is a Banach $G$-contraction or simply $G$-contraction if $f$ preserves edges of $G$, i.e.,
\[
\forall x, y \in X \ ((x, y) \in E(G) \Rightarrow f(x), f(y) \in E(G))
\]
and $f$ decreases weights of edges of $G$ in the following way:
\[
\exists \alpha \in (0, 1), \forall x, y \in X \ ((x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y)).
\]

The main theorem is:

**Theorem 1.2** ([5], Th 3.2). Let $(X, d)$ be complete, and let the triple $(X, d, G)$ have the following property:

(P) for any $(x_n)_{n \in \mathbb{N}}$ in $X$, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let $f : X \to X$ be a $G$-contraction, and $X_f = \{x \in X | (x, fx) \in E(G)\}$. Then the following statements hold.

1. $\dim Fix f = \dim \{[x]_{\tilde{G}} | x \in X_f\}$.
2. Fix $f \neq \emptyset$ iff $X_f \neq \emptyset$.
3. $f$ has a unique fixed point iff there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\tilde{G}}$.
4. For any $x \in X_f$, $f \mid [x]_{\tilde{G}}$ is a PO.
5. If $X_f \neq \emptyset$ and $G$ is weakly connected, then $f$ is a PO.
6. If $X' := \bigcup \{[x]_{\tilde{G}} | x \in G\}$ then $f \mid X'$ is a WPO.
7. If $f \subseteq E(G)$, then $f$ is a WPO.

Here $Fix f$ denotes the set of fixed points of operator $f$.

Since then several authors have considered the problem of existence and uniqueness of a fixed point for contraction type operators in metric spaces endowed with a graph (see [1], [2], [3], [9]).
2. Main results

Using Jachymski’s Theorem [5] we can give alternate proofs to some known results by choosing the right form of graph.

**Proposition 2.1.** Banach’s contraction principle is a consequence of Jakhymski’s Theorem.

*Proof.* Let the graph $G_0$ defined by $V(G) = X$ and $E(G_0) = X \times X$, which is a connected graph and the contraction $T$ by Theorem 1.1 is a $G_0$-Banach contraction. The property (P) is fulfilled so from Theorem 1.1 the operator $T$ is PO.

In the following we prove that the fixed point theorems for contractions in partially ordered metric spaces are consequences of Jakhymski’s Theorem.

**Theorem 2.2** (Ran and Reurings [14], Th 2.1). Let $(X, d)$ be a complete metric space endowed with a partial ordering “$\leq$” such that every pair of elements of $X$ has an upper and a lower bound.

Let $T : X \to X$ be continuous and monotone, and such that

$$\exists \alpha \in (0, 1) \forall x, y \in X (x \leq y \Rightarrow d(Tx, Ty) \leq \alpha d(x, y)).$$

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$ or $Tx_0 \leq x_0$, then $T$ is a PO.

*Proof.* Let $G$ be the graph defined by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X | x \leq y \}.$$

Because every pair of elements of $X$ has an upper and a lower bound the graph $G$ is weakly connected. The mapping $T : X \to X$ which satisfies (7) and is monotone is a $G$-Banach contraction. From the continuity of mapping $T$ we get that the property (P) is true so from Theorem 1.1 the operator $T$ is PO.

Further improvements of the above results were found independently by Petru¸ sel and Rus [12], and Nieto and Rodr´ ıguez-L´ opez [11]. Here we give a slightly more general version of these extensions taken from the paper by Nieto, Pouso and Rodríguez-López [10]. Following [12] we denote:

$$X_\leq := \{(x, y) \in X \times X | x \leq y \text{ or } y \leq x \}.$$

**Theorem 2.3.** Let $(X, d)$ be a complete metric space endowed with a partial ordering “$\leq$” such that every pair of elements of $X$ has an upper or a lower bound. Let $T : X \to X$ be such that $T$ preserves comparable elements, i.e.

for any $x, y \in X, (x, y) \in X_\leq$ implies $(Tx, Ty) \in X_\leq,$

and (7) holds. Assume that either $T$ is orbitally continuous or $(X, d, \leq)$ is such that

for any $(x_n)_{n\in\mathbb{N}},$ if $x_n \to x$ and $(x_n, x_{n+1}) \in X_\leq,$ for $n \in \mathbb{N},$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ such $(x_{k_n}, x) \in X_\leq \forall n \in \mathbb{N}.$

If there exists $x_0 \in X$ with $(x_0, Tx_0) \in X_\leq,$ then $T$ is a PO.

*Proof.* Let $G$ be the graph defined by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X | x \leq y \text{ or } y \leq x \}.$$

Because every pair of elements of $X$ has an upper and a lower bound the graph $G$ is weakly connected. The mapping $T : X \to X$ which satisfies (7) and is monotone is a $G$-Banach contraction. From the orbitally continuity of mapping $T$ or by (9) we get the property (P) is true so from Theorem 1.1 the operator $T$ is PO.
On the other hand, Theorem 1.1 yields directly the following well-known fixed point theorem which is quite different from the above results.

**Theorem 2.4** (Edelstein, [7]). Let \((X, d)\) be complete and \(\epsilon\)-chainable for some \(\epsilon > 0\), i.e., given \(x, y \in X\), there is \(N \in \mathbb{N}\) and a sequence \((x_i)_{i=0}^N\) such that \(x_0 = x, x_N = y\) and \(d(x_{i-1}, x_i) < \epsilon\) for \(i = 1, \ldots, N\). Let \(T : X \to X\) be such that

\[
\exists \alpha \in (0, 1), \forall x, y \in X \left( d(x, y) < \epsilon \Rightarrow d(Tx, Ty) \leq \alpha d(x, y) \right). \tag{10}
\]

Then \(T\) is a PO.

*Proof.* Clearly, (10) implies \(T\) is continuous. Consider the graph \(G\) with \(V(G) = X\) and

\[E(G) = \{(x, y) \in X \times X \mid d(x, y) < \epsilon\}.\]

Then \(\epsilon\)-chainability of \((X, d)\) means \(G\) is connected. If \((x, y) \in E(G)\), then

\[d(Tx, Ty) \leq \alpha d(x, y) \leq \alpha \epsilon \leq \epsilon.\]

Hence (4) and (5) hold, so \(T\) is a \(G\)-contraction. By Theorem 1.1, \(T\) is a PO. \(\square\)

In the following we show the fixed point theorem for cyclic contractions proved in [8] by W.A. Kirk, P.S. Srinivasan and P. Veeramani is a consequence of Theorem 1.1.

**Definition 2.1.** Let \(p \in \mathbb{N}\), \(p \geq 2\) and \(\{A_i\}_{i=1}^p\) be nonempty closed subsets of a complete metric space \((X, d)\). An operator \(T : \{A_i\}_{i=1}^p \to \{A_i\}_{i=1}^p\) is called a cyclic operator if the following condition is satisfied:

\[T(A_i) \subseteq A_{i+1} \text{ for all } i \in \{1, 2, \ldots, p\}, \tag{11}\]

where \(A_{p+1} = A_1\).

**Theorem 2.5** ([8]). Let \(p \in \mathbb{N}\), \(p \geq 2\) and \(\{A_i\}_{i=1}^p\) be nonempty closed subsets of a complete metric space \((X, d)\), and suppose \(T : \{A_i\}_{i=1}^p \to \{A_i\}_{i=1}^p\) satisfies (11) and the following one

\[d(Tx, Ty) \leq kd(x, y), \text{ for all } x \in A_i, y \in A_{i+1}, 1 \leq i \leq p, \tag{12}\]

where \(A_{p+1} = A_1\). If \(k \in [0, 1)\) then \(T\) has an unique fixed point.

*Proof.* Consider the graph \(G\) with \(V(G) = X\) and

\[E(G) = \{(x, y) \in X \times X \mid x \in A_i, y \in A_{i+1}, i = 1, \ldots, p\}\]

Because \(T\) is a cyclic operator and \((x, y) \in E(G)\) then \(x \in A_i\) and \(y \in A_{i+1}\) so \(Tx \in A_{i+1}\) and \(Ty \in A_{i+2}\), which implies \((Tx, Ty) \in E(G)\). Using (12) we have \(T\) is a \(G\)-contraction. From the definition of edges of \(G\) we have that \(G\) is weakly connected. Let \((x_n)_{n \in \mathbb{N}}\) in \(X\), with \(x_n \to x\) and \((x_n, x_{n+1}) \in E(G)\) for \(n \in \mathbb{N}\). Then there is \(j \in \{1, 2, \ldots, n\}\) such that \(x \in A_j\). However in view of (11) the sequence \(\{x_n\}\) has an infinite number of terms in each \(A_i\), for all \(i \in \{1, 2, \ldots, n\}\). The subsequence of the sequence \(\{x_n\}\) formed by the terms which are in \(A_{j-1}\) satisfies the condition (P) from Theorem 1.1. Obviously \(X_T \neq \emptyset\) so \(T\) is PO. \(\square\)

The last consequence of Jachymski’s Theorem which we present is The Alternative of Fixed Point, due to Díaz and Margolis [4]. Here we will give a slightly more general version of these extensions taken from the paper by V. Radu [13].
Theorem 2.6 (The fixed point alternative). Suppose we are given a complete generalized metric space \((\Omega, d)\) and a strictly contractive mapping \(T : \Omega \to \Omega\) with the Lipschitz constant \(a\). Then, for each given element \(x \in \Omega\), either
\[
d(T^n x, T^{n+1} x) = \infty, \ \forall n \geq 0,
\]
or there exists a natural number \(n_0\) such that
i. \(d(T^n x, T^{n+1} x) < \infty\) for all \(n \geq n_0\);
ii. The sequence \((T^n x)_{n \geq 0}\) is convergent to a fixed point \(y^*\) of \(T\);
iii. \(y^*\) is the unique fixed point of \(T\) in the set \(\Delta = \{y \in \Omega | d(T^m x, y) < \infty\}\);
iv. \(d(y, y^*) \leq \frac{1}{1-a} d(y, Ty)\) for all \(y \in \Delta\).

Proof. Let the graph \(G\) with \(V(G) = X\) and \(E(G) = \{(x, y) \in X \times X | d(x, y) < \infty\}\).

By the symmetry of distance we have \(\bar{G} = G\).

If \((x, y) \in E(G)\), so \(d(x, y) < \infty\), then \(d(Tx, Ty) \leq ad(x, y) < \infty\) in conclusion \((Tx, Ty) \in E(G)\), and \(T\) being a strictly contractive mapping we have \(T\) is a \(G\)-contraction.

Let \(x \in X\) such that, there exists \(n_0 \in \mathbb{N}\) with property \(d(T^{n_0} x, T^{n_0+1} x) < \infty\), then \((T^{n_0} x, T^{n_0+1} x) \in E(G)\), that is \(T^{n_0} x \in X_T\) so \(X_T \neq \emptyset\).

An easy induction shows
\[
d(T^n x, T^{n+1} x) \leq a^{n-n_0} d(T^{n_0} x, T^{n_0+1} x), \text{ for all } n \geq n_0
\]
so \(d(T^n x, T^{n+1} x) < \infty\) for all \(n \geq n_0\), therefore the relation i. is true.

For the same \(x\) like the one above, we have:
\[
[T^{n_0} x]_G = \{y \in X | \exists \text{ a path in } G \text{ from } x \to y\}.
\]

If \(y \in [T^{n_0} x]_G\) then there is a path \((x_i)_{i=0}^N\) in \(G\) from \(T^{n_0} x\) to \(y\), that is, \(x_0 = T^{n_0} x\), \(x_N = y\) and \((x_{i-1}, x_i) \in E(G)\) for \(i = 1, ..., N\). Then
\[
d(T^{n_0} x, y) \leq \sum_{i=1}^N d(x_{i-1}, x_i) < \infty.
\]

Consequently \([T^{n_0} x]_G \subseteq \Delta\) and the relation \(\Delta \subseteq [T^{n_0} x]_G\) is obvious, so \(\Delta = [T^{n_0} x]_G\).

If \((x_n)_{n \in \mathbb{N}}\) converges to \(x^* \in X\) with property \((x_n, x_{n+1}) \in E(G)\) for all \(n \in \mathbb{N}\) then for \(\epsilon = 1\) there exists \(n_\epsilon \in \mathbb{N}\) such that \(d(x_n, x^*) < 1\) for all \(n \geq n_\epsilon\), that is
\[
(x_n, x^*) \in E(G), \ \forall n \geq n_\epsilon.
\]

The subsequence \((x_n)_{n \geq n_0}\) satisfies the condition (P) from Theorem 1.2. Then from Theorem 1.2, 4, we have \(T|_{[T^{n_0} x]_G}\) is PO which implies ii. and iii. from The Fixed Point Alternative.

For iv., let \(y \in \Delta\) so \(d(y, T^{n_0} x) < \infty\). Because \((T^n x)_{n \geq n_0}\) converges, then \(d(T^n x, T^{n-1} x) < \infty\) for all \(n > n_0\) and \(d(y, T^{n-1} x) < \infty\), that is \((y, T^{n-1} x) \in E(G)\). By the triangle inequality and because \(T\) is a \(G\)-contraction, we get:
\[
d(y, T^n x) \leq d(y, Ty) + d(Ty, T^n x) < d(y, Ty) + ad(y, T^{n-1} x), \quad (13)
\]
for all \(n \geq n_0\).

Hence, letting \(n\) tend to \(\infty\) in (13) we conclude
\[
d(y, y^*) \leq d(y, Ty) + ad(y, y^*)
\]
that is \( d(y, y^*) \leq \frac{1}{1-a} d(y, Ty) \) for any \( y \in \Delta \), which completes the proof. 

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