Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 39(2), 2012, Pages 182–190
ISSN: 1223-6934

Uniqueness of strong solution for a 1D viscous bi-layer Shallow Water model

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Abstract. The aim of this paper is to prove the uniqueness of strong solution of a one dimensional viscous bilayer shallow water model. Our analysis is based on some new useful estimate namely BD entropy and on a method developed by Mellet and Vasseur in [14] to prove the existence and uniqueness on some compressible one dimensional Navier-Stokes system. Under suitable assumptions on the solutions and using Gronwall Lemma, we obtain the uniqueness of strong solution. We perform our analysis in periodic domain with periodic boundaries conditions.

2010 Mathematics Subject Classification. 35Q30.
Key words and phrases. Strong solutions, shallow water, viscous models, bilayer, stability, uniqueness.

1. Introduction

In this paper, we study the uniqueness of strong solution of the following viscous bilayer shallow water model :

$$\partial_t h_1 + \partial_x(h_1 v_1) = 0, \tag{1}$$

$$\partial_t(h_1 v_1) + \partial_x(h_1 v_1^2) + g h_1 \partial_x h_1 + r g h_1 \partial_x h_2 - \nu_1 \partial_x(h_1 \partial_x v_1) = 0, \tag{2}$$

$$\partial_t h_2 + \partial_x(h_2 v_2) = 0, \tag{3}$$

$$\partial_t(h_2 v_2) + \partial_x(h_2 v_2^2) + g h_2 \partial_x h_2 + g h_2 \partial_x h_1 - \nu_2 \partial_x(h_2 \partial_x v_2) = 0. \tag{4}$$

where $(t, x) \in (0, T) \times \Omega$, and Ω is a periodic domain in one dimension. We denote by ρ_1 and ρ_2 the densities of each layer of fluid, and r is their ratio $r = \rho_2/\rho_1 < 1$. The quantities ν_1 and ν_2 are the respective kinematic viscosity, that is $\nu_i = \mu_i/\rho_i$ where μ_i is the dynamic viscosity. These equations represent a system composed of two layers of immiscible fluids. Index 1 refers to the deeper layer and index 2 to the upper layer, see Figure 1.

In the two dimensional case, this model is formally derived in [17]. Such model appears naturally in geophysical flows, see [1, 6].

The existence result for the one dimensional Navier-Stokes equations which includes the shallow water equations has been studied by many authors.

When the viscosity coefficient is constant, there were a lot of investigations of the one dimensional Navier-Stokes equations. For instance, the one dimensional Navier-Stokes problem were investigated in [11] for a smooth data and in [8, 10, 18] for discontinuous data. The authors prove in these papers the global existence of smooth solutions. For the multidimensional case, one can see [9, 13].

The first global existence result for initial density which can vanish was shown in [13]. That result was later extended in [7] to the full Navier-Stokes equations.

Received November 5, 2012.

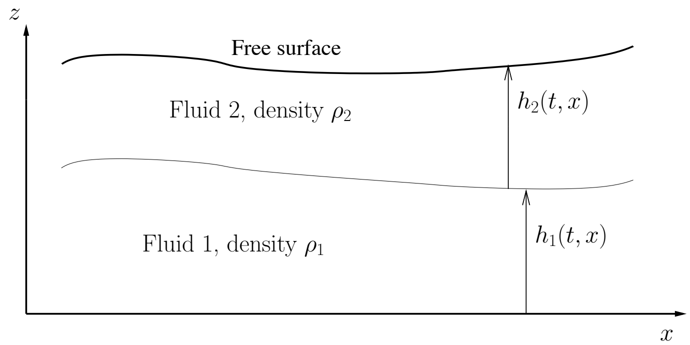

FIGURE 1. Notations for the bi-layer model

In [3, 4], the authors proved the existence of a global weak solution for a 2D shallow water system and a Korteweg system with a diffusion term of type $\operatorname{div}(hD(u))$. The key point that allows us to get this result is an entropy inequality namely BD entropy derived first in [3] and in [4]. Using also this inequality in [16, 20], the authors proved the existence of a global weak solution for viscous bilayer shallow water models in two dimensions.

Notice that in one dimension, the BD entropy gives control on some negative power of the density. This nice control was shown in [14]. Using techniques similar to those in [14], the authors proved in [19] that vacuum cannot arise if there is no vacuum at the initial time and obtained the existence of strong solutions of a bilayer shallow water model. This existence result was obtained thanks to a construction of approximate solutions following the work performed in [12].

An existence result concerning an one dimensional bi-layer shallow water model was studied in [15]. The authors obtained the existence, the uniqueness and some smoothness of weak solution under the assumption that the data are sufficiently small.

We consider in this paper, the system of bi-layer immiscible fluids obtained by derivation. The existence of strong solutions of such system was studied in [19]. In this paper, we will prove the uniqueness of strong solution under suitable conditions.

Our hypothesis on the initial data $h_{i|t=0}$, $v_{i|t=0}$ (for $i = 1, 2$) to define the strong solution in [19] are the following:

$$\begin{gathered} 0 < \underline{c}_0 \leq h_{i|t=0} = h_{i_0} \leq c_0, \quad h_i v_{i|t=0} = q_{i_0} \\ h_{i_0} \in H^1(\Omega), \quad v_{i_0} \in H^1(\Omega). \end{gathered} \tag{5}$$

The rest of the paper is organized as follows: in Section 2, we give the main uniqueness result. Next, in Section 3, we give some inequalities that will be useful to prove our main statement. We prove the result in Section 4 by using Gronwall Lemma. The last Section is dedicated to the proof of the physical energy and the BD entropy inequalities.

2. Main result

In this section, we give our main result. It can be written as follows:

Theorem 2.1. *Assume that*

$$\nu_1 > \frac{r(\nu_1 + \nu_2)}{2}, \quad \nu_2 > \frac{\nu_1 + \nu_2}{2}. \tag{6}$$

Let (h_1, v_1, h_2, v_2) and $(\widetilde{h_1}, \widetilde{v_1}, \widetilde{h_2}, \widetilde{v_2})$ be strong solutions of the system (1)–(4) *satisfying the inequalities* (8) *and* (10) *with initial data $(h_{1_0}, v_{1_0}, h_{2_0}, v_{2_0})$ which satisfy conditions* (5) *and verify the following bounds:*

$$\mathcal{E}_0 = \frac{1}{2}\int_\Omega h_{1_0}|v_{1_0}|^2 + \frac{r}{2}\int_\Omega h_{2_0}|v_{2_0}|^2 + \frac{1}{2}g(1-r)\int_\Omega |h_{1_0}|^2 + \frac{r}{2}g\int_\Omega |h_{1_0}+h_{2_0}|^2 \leq C$$

$$\mathcal{F}_0 = \frac{1}{2}\int_\Omega \left|\nu_1 \frac{\partial_x h_{1_0}}{h_{1_0}}\right|^2 + \frac{1}{2}\int_\Omega \left|\nu_2 \frac{\partial_x h_{2_0}}{h_{2_0}}\right|^2 \leq C.$$

Assume moreover that there exists a constant c such that the two solutions satisfy the following condition for $i = 1, 2$:

$$\int_\Omega \mathcal{E}(\widetilde{V}|V)dx \geq c \max\left(\left(\int_\Omega |h_i - \widetilde{h}_i|^2 dx\right)^{1/2}, \left(\int_\Omega \widetilde{h}_i (v_i - \widetilde{v}_i)^2 dx\right)^{1/2}\right) \qquad (7)$$

where $\mathcal{E}(\widetilde{V}|V)$ is defined by (12)*. Then*

$$h_i = \widetilde{h}_i, \quad v_i = \widetilde{v}_i, \quad \textit{for } i = 1, 2.$$

3. Energy inequalities

In this section, we give the physical energy inequality and the BD entropy for the solution of System (1)–(4).

Proposition 3.1. *If (h_1, h_2, v_1, v_2) is a smooth solution of* (1)-(4)*, then the following classical inequality holds :*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_1|v_1|^2 + \frac{r}{2}\frac{d}{dt}\int_\Omega h_2|v_2|^2 + \nu_1\int_\Omega h_1(\partial_x v_1)^2 + r\nu_2\int_\Omega h_2(\partial_x v_2)^2 + \frac{g(1-r)}{2}\frac{d}{dt}\int_\Omega |h_1|^2 + \frac{rg}{2}\frac{d}{dt}\int_\Omega |h_1+h_2|^2 \leq 0. \qquad (8)$$

Corollary 3.2. *The classical energy estimate gives the following uniform bounds:*

$$\begin{aligned} &\|h_i\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T); \quad \|\sqrt{h_i}v_i\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T);\\ &\|\sqrt{h_i}\partial_x v_i\|_{L^2(0,T;(L^2(\Omega))^2)} \leq C(T),\ \forall i = 1, 2. \end{aligned} \qquad (9)$$

It is well known that these bounds are not enough to obtain the existence of strong solutions for our system. We write the BD entropy to have more informations on the solutions.

Proposition 3.3. *Let (h_1, v_1, h_2, v_2) be a smooth solution of* (1)-(4)*, then the following mathematical BD entropy inequality holds :*

$$\begin{aligned} &\frac{1}{2}\int_\Omega h_1\left|v_1 + \nu_1\frac{\partial_x h_1}{h_1}\right|^2 + \frac{r}{2}\int_\Omega h_2\left|v_2 + \nu_2\frac{\partial_x h_2}{h_2}\right|^2 + \frac{g(1-r)}{2}\int_\Omega |h_1|^2\\ &+\frac{rg}{2}\int_\Omega |h_1+h_2|^2 + \frac{(2-r)\nu_1 - r\nu_2}{2}g\int_0^t\int_\Omega (\partial_x h_1)^2 + \frac{\nu_2-\nu_1}{2}rg\int_0^t\int_\Omega (\partial_x h_2)^2\\ &\quad\leq \frac{1}{2}\int_\Omega h_{1_0}\left|v_{1_0} + \nu_1\frac{\partial_x h_{1_0}}{h_{1_0}}\right|^2 + \frac{r}{2}\int_\Omega h_{2_0}\left|\ v_{2_0} + \nu_2\frac{\partial_x h_{2_0}}{h_{2_0}}\right|^2\\ &\qquad + \frac{g(1-r)}{2}\int_\Omega |h_{1_0}|^2 + \frac{r\,g}{2}\int_\Omega |h_{1_0}+h_{2_0}|^2 \end{aligned} \qquad (10)$$

for all $t \in [0, T]$.

This nice inequality was first derived in [2, 3, 5] in dimension 2 and 3.

Corollary 3.4. *The BD mathematical entropy inequality implies that:*

$$\left\| \frac{\partial_x h_i}{\sqrt{h_i}} \right\|_{L^\infty(0,T;L^2(\Omega))} = 2\|\partial_x \sqrt{h_i}\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T). \tag{11}$$

This bound is the key point to obtain the existence result in [19].

4. Proof of Theorem 2.1.

This part is devoted to the proof of Theorem 2.1.
Following the work performed in [14], we define some functions which will be useful to do the proof.
Assume that the state vector is V and let us define the following functions of V:

$$V = \begin{pmatrix} h_1 \\ h_1 v_1 \\ h_2 \\ h_2 v_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ q_1 \\ h_2 \\ q_2 \end{pmatrix}, \quad A(V) = \begin{pmatrix} q_1 \\ \frac{q_1^2}{2h_1} + \frac{1}{2} g h_1^2 \\ q_2 \\ \frac{q_2^2}{2h_2} + \frac{1}{2} g h_2^2 \end{pmatrix}$$

$$B(V) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \nu_1 h_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r}\nu_2 h_2 \end{pmatrix}, \quad C(V) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & rgh_1 & 0 \\ 0 & 0 & 0 & 0 \\ gh_2 & 0 & 0 & 0 \end{pmatrix},$$

$$F(V) = \frac{q_1^3}{2h_1^2} + gq_1h_1 + rgq_1h_2 + rgq_2h_1 + rgq_2h_2 + r\frac{q_2^3}{2h_2^2},$$

$$\mathcal{E}(V) = \frac{q_1^2}{2h_1} + \frac{1}{2}g(1-r)h_1^2 + \frac{1}{2}g(h_1+h_2)^2 + r\frac{q_2^2}{2h_2}$$

and

$$\mathcal{E}(\widetilde{V}|V) = \mathcal{E}(\widetilde{V}) - \mathcal{E}(V) - D\mathcal{E}(V)(\widetilde{V} - V),$$

where $D\mathcal{E}(V)$ is the Jacobian matrix of $\mathcal{E}(V)$.
We also have

$$\mathcal{E}(\widetilde{V}|V) = \frac{1}{2}\widetilde{h_1}(v_1 - \widetilde{v_1})^2 + \frac{1}{2}\widetilde{h_2}(v_2 - \widetilde{v_2})^2 + \frac{1}{2}g(1-r)(h_1 - \widetilde{h_1})^2 + \frac{r}{2}g(h_1 + h_2 - (\widetilde{h_1} + \widetilde{h_2}))^2. \tag{12}$$

Now, we can state the following proposition :

Proposition 4.1. *Suppose that* (h_1, v_1, h_2, v_2) *and* $(\widetilde{h_1}, \widetilde{h_2}, \widetilde{v_1}, \widetilde{v_2})$ *are two strong solutions of system* (1)–(4) *given in* [19] *and satisfying the physical entropy* (8) *and the*

BD entropy (10). *Then, the following inequality holds:*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathcal{E}(\widetilde{V}|V) + \nu_1\int_\Omega \widetilde{h_1}(\partial_x\widetilde{v_1} - \partial_x v_1)^2 + r\nu_2\int_\Omega \widetilde{h_2}(\partial_x\widetilde{v_2} - \partial_x v_2)^2 \\
&\quad \le rg\int_\Omega (\widetilde{h_1}u_1\partial_x\widetilde{h_2} - h_1u_1\partial_x h_2) + \int_\Omega (\widetilde{h_2}u_2\partial_x\widetilde{h_1} - h_2u_2\partial_x h_1) \\
&\quad -\nu_1\int_\Omega \partial_x v_1(\widetilde{h_1} - h_1)[\partial_x(\widetilde{v_1} - v_1)] - r\nu_2\int_\Omega \partial_x v_2(\widetilde{h_2} - h_2)[\partial_x(\widetilde{v_2} - v_2)] \qquad (13)\\
&+\nu_1\int_\Omega \frac{\partial_x(h_1\partial_x v_1)}{h_1}(v_1 - \widetilde{v_1})(\widetilde{h_1} - h_1) + \nu_2 r\int_\Omega \frac{\partial_x(h_2\partial_x v_2)}{h_2}(v_2 - \widetilde{v_2})(\widetilde{h_2} - h_2) \\
&+r\,g\int_\Omega (\widetilde{h_1}\partial_x h_2\cdot(\widetilde{v_1} - v_1) + \widetilde{h_2}\partial_x h_1\cdot(\widetilde{v_2} - v_2)) + C\int_\Omega (|\partial_x v_1| + |\partial_x v_2|)\mathcal{E}(\widetilde{V}|V).
\end{aligned}
$$

Proof. Thanks to a careful computation, we ensure that

$$
\begin{aligned}
\partial_t\mathcal{E}(\widetilde{V}|V) &= \partial_t\mathcal{E}(\widetilde{V}) + \partial_x F(\widetilde{V}) - \partial_x\left[B(\widetilde{V})\partial_x D\mathcal{E}(\widetilde{V})\right] D\mathcal{E}(\widetilde{V}) \\
&\quad - (\partial_t\mathcal{E}(V) + \partial_x F(V) - \partial_x\left[B(V)\partial_x D\mathcal{E}(V)\right] D\mathcal{E}(V)) \\
&\quad - \left(\partial_x F(\widetilde{V}) - \partial_x F(V)\right) + D\mathcal{E}(V)\partial_x A(\widetilde{V}|V) + \partial_x[DF(V)(\widetilde{V} - V)] \\
&\quad - D^2\mathcal{E}(V)\left[\partial_t V + \partial_x A(V) + C(V)\partial_x V - \partial_x(B(V)\partial_x D\mathcal{E}(V))\right]\cdot(\widetilde{V} - V) \\
&\quad + D\mathcal{E}(V)\left[\partial_t V + \partial_x A(V) + C(V)\partial_x V - \partial_x(B(V)\partial_x D\mathcal{E}(V))\right] \\
&\quad - D\mathcal{E}(V)\left[\partial_t\widetilde{V} + \partial_x A(\widetilde{V}) + C(\widetilde{V})\partial_x\widetilde{V} - \partial_x(B(\widetilde{V})\partial_x D\mathcal{E}(\widetilde{V}))\right] \\
&+ \partial_x\left[B(\widetilde{V})\partial_x D\mathcal{E}(\widetilde{V}) - B(V)\partial_x D\mathcal{E}(V)\right]\cdot[\mathcal{E}(\widetilde{V}) - \mathcal{E}(V)] + D^2\mathcal{E}(V)C(V)\partial_x V\cdot(\widetilde{V} - V) \\
&\quad + D\mathcal{E}(V)C(\widetilde{V})\partial_x\widetilde{V} - D\mathcal{E}(V)C(V)\partial_x V + \partial_x[B(V)\partial_x D\mathcal{E}(V)]D\mathcal{E}(\widetilde{V}|V). \qquad (14)
\end{aligned}
$$

Since $V = (h_1, q_1, h_2, q_2)$ and $\widetilde{V} = (\widetilde{h_1}, \widetilde{q_1}, \widetilde{h_2}, \widetilde{q_2})$ are solutions of system (1)–(4) satisfying the natural entropy equality (8), we deduce that

$$
\begin{aligned}
\partial_t\mathcal{E}(\widetilde{V}|V) &\le -\left(\partial_x F(\widetilde{V}) - \partial_x F(V)\right) + D\mathcal{E}(V)\partial_x A(\widetilde{V}|V) + \partial_x[DF(V)(\widetilde{V} - V)] \\
&\quad \partial_x\left[B(\widetilde{V})\partial_x D\mathcal{E}(\widetilde{V}) - B(V)\partial_x D\mathcal{E}(V)\right]\cdot[D\mathcal{E}(\widetilde{V}) - D\mathcal{E}(V)] \\
&\quad + D^2\mathcal{E}(V)C(V)\partial_x V\cdot(\widetilde{V} - V) + D\mathcal{E}(V)C(\widetilde{V})\partial_x\widetilde{V} \\
&\quad - D\mathcal{E}(V)C(V)\partial_x V + \partial_x[B(V)\partial_x D\mathcal{E}(V)]D\mathcal{E}(\widetilde{V}|V).
\end{aligned}
$$

Next, we integrate over Ω and use the boundary conditions to obtain the following inequality:

$$
\begin{aligned}
\partial_t\int_\Omega \mathcal{E}(\widetilde{V}|V)dx &\le \int_\Omega D\mathcal{E}(V)\partial_x A(\widetilde{V}|V)dx \\
&\quad + \int_\Omega \partial_x\left[B(\widetilde{V})\partial_x D\mathcal{E}(\widetilde{V}) - B(V)\partial_x D\mathcal{E}(V)\right]\cdot[D\mathcal{E}(\widetilde{V}) - D\mathcal{E}(V)]dx \\
&\quad + \int_\Omega D^2\mathcal{E}(V)C(V)\partial_x V\cdot(\widetilde{V} - V)dx + \int_\Omega D\mathcal{E}(V)C(\widetilde{V})\partial_x\widetilde{V}dx \\
&\quad - \int_\Omega D\mathcal{E}(V)C(V)\partial_x V dx + \int_\Omega \partial_x[B(V)\partial_x D\mathcal{E}(V)]D\mathcal{E}(\widetilde{V}|V)dx
\end{aligned}
$$

We study now every term in the right hand side. We integrate by parts the first term. This implies that

$$\int_\Omega D\mathcal{E}(V)\partial_x A(\widetilde{V}|V)dx = -\int_\Omega \partial_x[D\mathcal{E}(V)]\cdot A(\widetilde{V}|V)dx$$
$$= -\int_\Omega \{\partial_x v_1[\widetilde{h_1}(v_1-\widetilde{v_1})^2+\frac{1}{2}g(h_1-\widetilde{h_1})^2]+r\partial_x v_2[\widetilde{h_2}(v_2-\widetilde{v_2})^2+\frac{1}{2}g(h_2-\widetilde{h_2})^2]\}dx.$$

Since $|h_2-\widetilde{h_2}|\le|h_1-\widetilde{h_1}+h_2-\widetilde{h_2}|+|h_1-\widetilde{h_1}|$, it is clear that the quantities $\widetilde{h_1}(v_1-\widetilde{v_1})^2+\frac{1}{2}g(h_1-\widetilde{h_1})^2$ and $r\partial_x v_2[\widetilde{h_2}(v_2-\widetilde{v_2})^2+\frac{1}{2}g(h_2-\widetilde{h_2})^2]$ are less than some constant multiplied by $\mathcal{E}(\widetilde{V}|V)$. So,

$$\int_\Omega D\mathcal{E}(V)\partial_x A(\widetilde{V}|V)dx \quad\le\quad C\Big(\int_\Omega|\partial_x v_1|\mathcal{E}(\widetilde{V}|V)dx \;+\; \int_\Omega|\partial_x v_2|\mathcal{E}(\widetilde{V}|V)dx\Big).$$

Moreover, we have successively:

$$\begin{aligned}&\int_\Omega \partial_x\Big[B(\widetilde{V})\partial_x D\mathcal{E}(\widetilde{V})-B(V)\partial_x D\mathcal{E}(V)\Big]\cdot[D\mathcal{E}(\widetilde{V})-D\mathcal{E}(V)]dx\\ &\qquad= -\int_\Omega\Big[B(\widetilde{V})\partial_x D\mathcal{E}(\widetilde{V})-B(V)\partial_x D\mathcal{E}(V)\Big]\cdot\partial_x[D\mathcal{E}(\widetilde{V})-D\mathcal{E}(V)]dx\\ &\qquad\quad= -\int_\Omega\big\{\nu_1\big(\widetilde{h_1}(\partial_x\widetilde{v_1}-\partial_x v_1)^2+\partial_x v_1\cdot(\widetilde{h_1}-h_1)\partial_x(\widetilde{v_1}-v_1)\big)\\ &\qquad\qquad\qquad+r\nu_2\big(\widetilde{h_2}(\partial_x\widetilde{v_2}-\partial_x v_2)^2+\partial_x v_2\cdot(\widetilde{h_2}-h_2)\partial_x(\widetilde{v_2}-v_2)\big)\big\}dx,\end{aligned}$$

$$\begin{aligned}&\int_\Omega D^2\mathcal{E}(V)C(V)\partial_x V\cdot(\widetilde{V}-V)dx\\ &\qquad\qquad= rg\int_\Omega\Big(\widetilde{h_1}\partial_x h_2\cdot(\widetilde{v_1}-v_1)+\widetilde{h_2}\partial_x h_1\cdot(\widetilde{v_2}-v_2)\Big)dx,\end{aligned}$$

$$\begin{aligned}\int_\Omega D\mathcal{E}(V)C(\widetilde{V})\partial_x\widetilde{V}dx &= rg\int_\Omega\big(\widetilde{h_1}u_1\partial_x\widetilde{h_2}+\widetilde{h_2}u_2\partial_x\widetilde{h_1}\big)dx,\\ \int_\Omega D\mathcal{E}(V)C(V)\partial_x Vdx &= rg\int_\Omega\big(h_1u_1\partial_x h_2+h_2u_2\partial_x h_1\big)dx,\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\partial_x[B(V)\partial_x D\mathcal{E}(V)]D\mathcal{E}(\widetilde{V}|V)dx\\ &\qquad\qquad=\nu_1\int_\Omega\frac{\partial_x(h_1\partial_x v_1)}{h_1}(v_1-\widetilde{v_1})(\widetilde{h_1}-h_1)dx\\ &\qquad\qquad\qquad+\nu_2 r\int_\Omega\frac{\partial_x(h_2\partial_x v_2)}{h_2}(v_2-\widetilde{v_2})(\widetilde{h_2}-h_2)dx.\end{aligned}$$

Substituting all these terms, we obtain the proclaimed result. □

Let us estimate the right hand side of the inequality (13). First, we have

$$\begin{aligned}&rg\int_\Omega(\widetilde{h_1}v_1\partial_x\widetilde{h_2}-h_1v_1\partial_x h_2)dx\\ &= rg\int_\Omega(\widetilde{h_1}-h_1)v_1\partial_x\widetilde{h_2}dx-rg\int_\Omega(\widetilde{h_2}-h_2)h_1\partial_x v_1dx-rg\int_\Omega(\widetilde{h_2}-h_2)v_1\partial_x h_1dx.\end{aligned}$$

Notice that

$$\begin{aligned}\int_\Omega (\widetilde{h_1}-h_1)v_1\partial_x\widetilde{h_2}dx &\leq \int_\Omega |\widetilde{h_1}-h_1)||v_1||\partial_x\widetilde{h_2}|dx\\ &\leq \|v_1\|_{L^\infty(\Omega)}\Big(\int_\Omega |\widetilde{h_1}-h_1)|^2dx\Big)^{1/2}\Big(\int_\Omega |\partial_x\widetilde{h_2}|^2dx\Big)^{1/2}\\ &\leq C(t)\int_\Omega \mathcal{E}(\widetilde{V}|V)dx.\end{aligned}$$

In the same way, we establish that: $\int_\Omega (\widetilde{h_2}-h_2)h_1\partial_x v_1 dx \leq C(t)\int_\Omega \mathcal{E}(\widetilde{V}|V)dx$ and $\int_\Omega (\widetilde{h_2}-h_2)v_1\partial_x h_1 dx \leq C(t)\int_\Omega \mathcal{E}(\widetilde{V}|V)dx$. Finally, the following estimate holds:

$$rg\int_\Omega (\widetilde{h_1}v_1\partial_x\widetilde{h_2} - h_1v_1\partial_x h_2)dx + rg\int_\Omega (\widetilde{h_2}v_2\partial_x\widetilde{h_1} - h_2v_2\partial_x h_1)dx \leq C(t)\int_\Omega \mathcal{E}(\widetilde{V}|V)dx.$$

Secondly,

$$\begin{aligned}&\int_\Omega \partial_x v_i(\widetilde{h_i}-h_i)[\partial_x(\widetilde{v_i}-v_i)]dx\\ &\qquad\leq C\|\partial_x v_i\|_{L^\infty(\Omega)}\|\partial_x(\widetilde{v_i}-v_i)\|_{L^\infty(\Omega)}\Big(\int_\Omega |\widetilde{h_i}-h_i|^2dx\Big)^{1/2}\\ &\qquad\leq C(t)\int_\Omega \mathcal{E}(\widetilde{V}|V)dx, \quad \text{for } i=1,\,2.\end{aligned}$$

Besides, we check that:

$$\begin{aligned}&rg\int_\Omega \Big(\widetilde{h_1}\partial_x h_2\cdot(\widetilde{v_1}-v_1) + \widetilde{h_2}\partial_x h_1\cdot(\widetilde{v_2}-v_2)\Big)dx\\ &\qquad\leq rg\|\partial_x h_2\|_{L^\infty(\Omega)}\Big(\int_\Omega \widetilde{h_1}(\widetilde{v_1}-v_1)^2dx\Big)^{1/2}\Big(\int_\Omega |\widetilde{h_1}|dx\Big)^{1/2}\\ &+ rg\|\partial_x h_1\|_{L^\infty(\Omega)}\Big(\int_\Omega \widetilde{h_2}(\widetilde{v_2}-v_2)^2dx\Big)^{1/2}\Big(\int_\Omega |\widetilde{h_2}|dx\Big)^{1/2} \leq C(t)\int_\Omega \mathcal{E}(\widetilde{V}|V)dx,\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \frac{\partial_x(h_i\partial_x v_i)}{h_i}(v_i-\widetilde{v_i})(\widetilde{h_i}-h_i)dx\\ &\qquad\leq C(t)\|\widetilde{h_i}-h_i\|_{L^2(\Omega)}\|\partial_x(h_i\partial_x v_i)\|_{L^2(\Omega)}\|v_i-\widetilde{v_i}\|_{L^\infty(\Omega)}\\ &\qquad\leq C(t)\int_\Omega \mathcal{E}(\widetilde{V}|V)dx,\end{aligned}$$

for $i = 1, 2$.

Gathering all these results, we end up with:

$$\frac{d}{dt}\int_\Omega \mathcal{E}(\widetilde{V}|V)dx \leq C(t)\int_\Omega \mathcal{E}(\widetilde{V}|V)dx.$$

In the fact that $\mathcal{E}(\widetilde{V}|V)(t=0)=0$, Gronwall Lemma allows us to write that $\int_\Omega \mathcal{E}(\widetilde{V}|V)dx = 0$. So, we conclude that

$$h_i = \widetilde{h_i}, \quad v_i = \widetilde{v_i}, \text{ for } i = 1,\,2.$$

5. Proof of propositions 3.1 and 3.3

We multiply both momentum equations (2) and (4) respectively by v_1 and v_2 and integrate by parts. We obtain for $i = 1, 2$:

$$\int_\Omega (\partial_t(h_1 v_1) + \partial_x(h_1 v_1^2))v_1 dx + g\int_\Omega (h_1\partial_x h_1 + rh_1\partial_x h_2)v_1 dx \\ - \nu_1 \int_\Omega \partial_x(h_1\partial_x v_1)v_1 dx = 0 \quad (15)$$

and

$$\int_\Omega (\partial_t(h_2 v_2) + \partial_x(h_2 v_2^2))v_2 dx + g\int_\Omega (h_2\partial_x h_2 + h_2\partial_x h_1)v_1 dx \\ - \nu_2 \int_\Omega \partial_x(h_1\partial_x v_2)v_2 dx = 0. \quad (16)$$

We can reformulate some terms, namely (for $i = 1, 2$)

$$\int_\Omega \left(\partial_t(h_i v_i) + \partial_x(h_i v_i^2)\right) v_i \, dx = \frac{1}{2}\frac{d}{dt}\int_\Omega h_i |v_i|^2 \, dx, \quad (17)$$

$$-\nu_i \int_\Omega \partial_x\left(h_i \partial_x v_i\right) v_i \, dx = \nu_i \int_\Omega h_i (\partial_x v_i)^2 \, dx. \quad (18)$$

To obtain the energy inequality, we add (15) to (16) multiplied by r. We remark that:

$$g\int_\Omega h_1\partial_x h_1 v_1 \, dx + rg\int_\Omega h_1\partial_x h_2 v_1 \, dx + rg\int_\Omega h_2\partial_x h_2 v_2 \, dx + rg\int_\Omega h_2 \partial_x h_1 v_2 \, dx \\ = \frac{g(1-r)}{2}\frac{d}{dt}\int_\Omega |h_1|^2 dx + \frac{rg}{2}\frac{d}{dt}\int_\Omega |h_1 + h_2|^2 dx.$$

Our next concern will be the proof of the BD entropy (10):
Differentiating the mass equations with respect to x; we get

$$\partial_t\partial_x h_i + v_i\partial_x^2 h_i + \partial_x v_i \partial_x h_i + h_i \partial_x^2 v_i + \partial_x v_i \partial_x h_i = 0.$$

We introduce the corresponding viscosity coefficient and obtain:

$$\partial_t\left(h_i\nu_i\frac{\partial_x h_i}{h_i}\right) + \partial_x\left(h_i\nu_i\frac{\partial_x h_i}{h_i}v_i\right) + \nu_i\partial_x(h_i\partial_x v_i) = 0.$$

Adding after the momentum equation (equation (2) for $i = 1$ and equation (4) for $i = 2$), we deduce the following equalities:

$$\partial_t\left(h_1 v_1 + h_1\nu_1\frac{\partial_x h_1}{h_1}\right) + \partial_x\left(h_1 v_1^2 + h_1\nu_1\frac{\partial_x h_1}{h_1}v_1\right) + gh_1\partial_x h_1 + rgh_1\partial_x h_2 = 0, \quad (19)$$

$$\partial_t\left(h_2 v_2 + h_2\nu_2\frac{\partial_x h_2}{h_2}\right) + \partial_x\left(h_2 v_2^2 + h_2\nu_2\frac{\partial_x h_2}{h_2}v_2\right) + gh_2\partial_x h_2 + gh_2\partial_x h_1 = 0. \quad (20)$$

We add Equation (19) multiplied by $\left(v_1 + \nu_1\frac{\partial_x h_1}{h_1}\right)$ to Equation (20) multiplied by $r\left(v_2 + \nu_2\frac{\partial_x h_2}{h_2}\right)$. To end, it suffices to integrate over Ω and use the fact that $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$.

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