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Durrmeyer operators of King-type

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Abstract. A class of linear and positive operators which generalizes the classical Durrmeyer's operators in the King sense is constructed. For these operators, uniform convergence results, error estimations in terms of first modulus of continuity and Voronovskaja's type theorems are established.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In 1967, J.L. Durrmeyer introduced in [5] a class of linear and positive operators $(D_m)_{m \in \mathbb{N}_0}$ defined for any $f \in L_1([0,1])$, any $x \in [0,1]$ and $m \in \mathbb{N}_0$ by

$$(D_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt, \tag{1.1}$$

where $p_{m,k}$ are the Berstein's polynomials defined by

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{1.2}$$

for any $k \in \{0, 1, ..., m\}$. Regarding the operators from (1.1), are well known the following results:

Theorem 1.1. (see [9]) *For any $f \in C([0,1])$*

$$\lim_{m \longrightarrow \infty} D_m f = f \tag{1.3}$$

uniform on $[0,1]$ and

$$|(D_m f)(x) - f(x)| \leq 2\omega\left(f; \frac{1}{\sqrt{2m+6}}\right) \tag{1.4}$$

for any $x \in [0,1], f \in C([0,1))$ and $m \in \mathbb{N}_0, m \geq 3$.

Theorem 1.2. (see [4]) *Let $f \in L_1([0,1])$ be a bounded function on $[0,1]$. If f is two times differentiable in $x \in [0,1]$, then*

$$\lim_{m \longrightarrow \infty} m\left((D_m f)(x) - f(x)\right) = (1-2x)f^{(1)}(x) + x(1-x)f^{(2)}(x). \tag{1.5}$$

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Here $f^{(s)}(x)$ denotes the s-order derivative of f with respect to x.

We recall the following results from [8].

Let $I \subset \mathbb{R}$ be an interval of real axis, a, b, a', b' be real numbers such that $a < b, a' < b', [a, b] \subset I, [a', b'] \subset I$ and $[a, b] \cap [a', b'] \neq \emptyset$. Like usually, $E(I)$ denotes the set of real valued functions defined on I, $C(I) = \{f \in E(I) | f - \textit{continuous on } I\}, B(I) = \{f \in E(I) | f - \textit{bounded on } I\}, C_B(I) = \{f \in C(I) | f - \textit{bounded on } I\}$.

For any $m \in \mathbb{N}$ consider the functions $\varphi_{m,k} : I \longrightarrow \mathbb{R}$ having the property $\varphi_{m,k} \geq 0$ for any $x \in [a', b'], k \in \{0, 1, ..., m\}$ and the linear positive functionals $A_{m,k} : E([a, b]) \longrightarrow \mathbb{R}, k \in \{0, 1, ..., m\}$. For $m \in \mathbb{N}$ let $L_m : E([a, b]) \longrightarrow E(I)$ be the operator defined by

$$(L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f) \tag{1.6}$$

for any $f \in E([a, b])$ and $x \in I$. It is immediately that the operators (1.4) are linear and positive on $[a, b] \cap [a', b']$.

For $m \in \mathbb{N}, i \in \mathbb{N}_0$, let us to define $T_{m,i}$ by $(T_{m,i} L_m)(x) = m^i (L_m \psi_x^i)(x)$, thus

$$(T_{m,i} L_m)(x) = m^i \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi_x^i) \tag{1.7}$$

for any $x \in [a, b] \cap [a', b']$, where

$$\psi_x(t) = t - x, \tag{1.8}$$

for any $t \in I$. Next, let $s \in \mathbb{N}_0$ be fixed and s even. We suppose that the operators $(L_m)_{m \geq 1}$ verify the condition: there exists the smallest $\alpha_s, \alpha_{s+2} \in [0, \infty)$ so that the following:

$$\lim_{m \longrightarrow \infty} \frac{(T_{m,j} L_m)(x)}{m^{\alpha j}} = B_j(x) \in \mathbb{R} \tag{1.9}$$

holds for any $x \in [a, b] \cap [a', b'], j \in \{s, s + 2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{1.10}$$

Theorem 1.3. *Let $f \in C([a, b])$ be given. If $x \in [a, b] \cap [a', b']$ and f is s-times differentiable in x, having the s-order derivate $f^{(s)}$ continuous in x, then the following identity:*

$$\lim_{m \longrightarrow \infty} m^{s - \alpha_s} \left\{ (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \right\} = 0 \tag{1.11}$$

holds.

If f is s-times differentiable on $[a, b]$, the function $f^{(s)}$ is continuous on $[a, b]$ and there exists $m(s) \in \mathbb{N}, k_j \in \mathbb{R}$ such that for any $m \in \mathbb{N}, m \geq m(s), x \in [a, b] \cap [a', b'], j \in \{s, s + 2\}$ the inequality

$$\frac{(T_{m,j} L_m)(x)}{m^{\alpha j}} \leq k_j \tag{1.12}$$

holds, then the convergence from (1.11) is uniform on $[a, b] \cap [a', b']$ and

$$\begin{aligned} m^{s - \alpha_s} & \left| (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,j} L_m)(x) \right| \\ & \leq \frac{1}{s!} (k_s + k_{s+2}) \omega \left(f^{(s)}; \frac{1}{\sqrt{m^{2 + \alpha_s - \alpha_{s+2}}}} \right) \end{aligned} \tag{1.13}$$

for any $x \in [a, b] \cap [a', b']$ and $m \in \mathbb{N}, m \geq m(s)$.

In [6], J.P. King constructed a sequence $(L_m)_{m\in\mathbb{N}_0}$ of positive linear operators defined on $C([0,1])$ which preserve the test functions e_0 and e_2, i.e. $L_m(e_0)=e_0$, $L_m(e_2)=e_2$, where $e_i:[0,1]\longrightarrow\mathbb{R}, e_i(x)=x^i$, for $i\in\{0,1,2\}$.

In what follows, King-type operator means a linear positive operator which preserve exactly two test functions from the set $\{e_0,e_1,e_2\}$.

In the second section, we define the general form of Durrmeyer operators. In the following sections of this paper, we construct Durrmeyer operators of King-type. We study, in each case, the uniform convergence, the approximation order in terms of first modulus of continuity ω and a Voronovskaja's type theorem of these operators.

2. The general form of Durrmeyer operators

For $m,k,p\in\mathbb{N}_0, k\le m$, is well known the following result (see [9])

$$\int_0^1 p_{m,k}(t)t^p dt=\frac{(k+p)!}{k!}\frac{m!}{(m+p+1)!}. \tag{2.1}$$

Let $I\subset[0,1]$ be an interval, $m_0\in\mathbb{N}_0$, $m_0\ge 2$ fixed, $\mathbb{N}_1=\{m\in\mathbb{N}|m\ge m_0\}$, the functions $\alpha_m,\beta_m:I\longrightarrow\mathbb{R},\alpha_m(x)\ge 0,\beta_m(x)\ge 0$, for any $x\in I$ and $m\in\mathbb{N}_1$.

By using the idea of Durrmeyer operators construction, we consider operators of a general form defined by

$$(Q_mf)(x)=(m+1)\sum_{k=0}^{m}\binom{m}{k}(\alpha_m(x))^k(\beta_m(x))^{m-k}\int_0^1 p_{m,k}(t)f(t)dt, \tag{2.2}$$

where $x\in I, m\in\mathbb{N}_1$ and $f\in L_1([0,1])$.

The operators $(Q_m)_{m\in\mathbb{N}_1}$ are called *Durrmeyer type operators.*

Remark 2.1. *The operators $Q_m, m\in\mathbb{N}_1$, are linear and positive.*

Remark 2.2. *Consider $[a,b]=[0,1]$ and $[a',b']=I$,*

$$\varphi_{m,k}(x)=(m+1)\binom{m}{k}(\alpha_m(x))^k(\beta_m(x))^{m-k} \tag{2.3}$$

and

$$A_{m,k}(f)=\int_0^1 p_{m,k}(t)f(t)dt, \tag{2.4}$$

where $x\in I$, $m\in\mathbb{N}_1, k\in\{0,1,...,m\}$ and $f\in L_1([0,1])$. Then

$$(Q_me_0)(x)=(\alpha_m(x)+\beta_m(x))^m, \tag{2.5}$$

$$(Q_me_1)(x)=\frac{(\alpha_m(x)+\beta_m(x))^{m-1}}{m+2}(m\alpha_m(x)+\alpha_m(x)+\beta_m(x)), \tag{2.6}$$

and

$$\begin{aligned}(Q_me_2)(x)=&\frac{(\alpha_m(x)+\beta_m(x))^{m-2}}{(m+2)(m+3)}\cdot\left((m(m-1)(\alpha_m(x))^2+4m\alpha_m(x)\cdot\right.\\&\left.\cdot(\alpha_m(x)+\beta_m(x))+2(\alpha_m(x)+\beta_m(x))^2\right).\end{aligned} \tag{2.7}$$

3. Durrmeyer operators preserving the test functions e_0 and e_1

In this section we construct a sequence of Durrmeyer operators as defined in section 2, which preserve the test functions e_0 and e_1.

Imposing the conditions $(Q_m e_0)(x) = e_0(x)$ and $(Q_m e_1)(x) = e_1(x)$, for any $x \in I$ and $m \in \mathbb{N}_1$ and taking into account (2.5) and (2.6), we obtain

$$\alpha_m(x) = \frac{(m+2)x-1}{m}, \tag{3.1}$$

$$\beta_m(x) = \frac{m+1-(m+2)x}{m}, \tag{3.2}$$

for any $x \in I$ and $m \in \mathbb{N}_1$.

The conditions $\alpha_m(x) \geq 0$ and $\beta_m(x) \geq 0$, for any $x \in I$ and $m \in \mathbb{N}_1$, imply

$$\frac{1}{m+2} \leq x \leq \frac{m+1}{m+2}. \tag{3.3}$$

Lemma 3.1. *The following*

$$\left[\frac{1}{m_0+2}, \frac{m_0+1}{m_0+2}\right] \subset \left[\frac{1}{m+2}, \frac{m+1}{m+2}\right] \tag{3.4}$$

holds for any $m \in \mathbb{N}_1$.

Proof. Because the function $\frac{1}{m+2}$ is decreasing and the function $\frac{m+1}{m+2}$ is increasing, relation (3.4) follows. □

Remark 3.1. *In this case $I = \left[\frac{1}{m_0+2}, \frac{m_0+1}{m_0+2}\right]$, so for remaining of this section we shall consider $I = \left[\frac{1}{m_0+2}, \frac{m_0+1}{m_0+2}\right]$. Thus, for α_m, β_m defined by (3.1) and (3.2) we have $\alpha_m(x) \geq 0$ and $\beta_m(x) \geq 0$, for any $x \in I$ and $m \in \mathbb{N}_1$.*

Taking into account the above remarks, we construct the sequence of Durrmeyer operators $(Q_{1,m})_{m \geq m_0}$ as follows. If $m \in \mathbb{N}_1$, we define the operator

$$(Q_{1,m}f)(x) = \frac{m+1}{m^m} \sum_{k=0}^{m} \binom{m}{k} ((m+2)x-1)^k (m+1-(m+2)x)^{m-k} \cdot \tag{3.5}$$

$$\cdot \int_0^1 p_{m,k}(t) f(t) dt$$

for any $x \in \left[\frac{1}{m_0+2}, \frac{m_0+1}{m_0+2}\right]$.

Lemma 3.2. *We have*

$$(Q_{1,m}e_0)(x) = 1, \tag{3.6}$$

$$(Q_{1,m}e_1)(x) = x, \tag{3.7}$$

and

$$(Q_{1,m}e_2)(x) = \frac{(m-1)(m+2)}{m(m+3)} x^2 + \frac{2(m+1)}{m(m+3)} x - \frac{m+1}{m(m+2)(m+3)} \tag{3.8}$$

for any $x \in I$ and $m \in \mathbb{N}_1$.

Proof. Results immediately from the definition above and (2.7). □

Lemma 3.3. *The following identities*

$$(T_{m,0}Q_{1,m})(x) = 1, \tag{3.9}$$

$$(T_{m,1}Q_{1,m})(x) = 0, \tag{3.10}$$

and

$$(T_{m,2}Q_{1,m})(x) = \frac{m^3(-2x^2+2x)+m^2(-6x^2+6x-1)+m(-4x^2+4x-1)}{(m+2)(m+3)} \tag{3.11}$$

hold, for any $x \in I$ and $m \in \mathbb{N}_1$.

Proof. By using Lemma 3.2 and relation (1.7) we have

$$(T_{m,0}Q_{1,m})(x) = (Q_{1,m}e_0)(x) = 1,$$
$$(T_{m,1}Q_{1,m})(x) = m(Q_{1,m}\psi_x)(x) = m((Q_{1,m}e_1)(x) - x(Q_{1,m}e_0)(x)) = 0,$$

and

$$\begin{aligned}(T_{m,2}Q_{1,m})(x) &= m^2(Q_{1,m}\psi_x^2)(x)\\ &= m^2((Q_{1,m}e_2)(x) - 2x(Q_{1,m}e_1)(x) + x^2(Q_{1,m}e_0)(x))\\ &= m^2\left(\frac{(m-1)(m+2)}{m(m+3)}x^2 + \frac{2(m+1)}{m(m+3)}x - \frac{m+1}{m(m+2)(m+3)} - 2x^2 + x^2\right),\end{aligned}$$

from where (3.11) follows. □

Lemma 3.4. *We have that*

$$\lim_{m\longrightarrow\infty}(T_{m,0}Q_{1,m})(x) = 1, \tag{3.12}$$

$$\lim_{m\longrightarrow\infty}\frac{(T_{m,2}Q_{1,m})(x)}{m} = 2x(1-x), \tag{3.13}$$

for any $x \in I$, and there exists $m(0) \in \mathbb{N}$ such that

$$\frac{(T_{m,2}Q_{1,m})(x)}{m} \le \frac{3}{2}, \tag{3.14}$$

for any $x \in I$ and $m \in \mathbb{N}_1,\ m \ge m(0)$.

Proof. The relations (3.12) and (3.13) result taking (3.9) and (3.11) into account. By using the definition of limit of a function and because $x(1-x) \le \frac{1}{4}$ for any $x \in [0,1]$, from (3.13) the relation (3.14) is obtained. □

Theorem 3.1. *Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous function on $[0,1]$. Then*

$$\lim_{m\longrightarrow\infty} Q_{1,m}f = f \tag{3.15}$$

uniformly on I and there exists $m(0) \in \mathbb{N}$ such that

$$|(Q_{1,m}f)(x) - f(x)| \le \frac{5}{2}\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{3.16}$$

for any $x \in I$ and $m \in \mathbb{N}_1, m \ge m(0)$.

Proof. Theorem 3.1 is a corollary of Theorem 1.3, for $s = 0, \alpha_0 = 0, \alpha_2 = 1$, $k_0 = 1$ and $k_2 = \frac{3}{2}$. □

Theorem 3.2. *If $f \in C([0,1]), x \in I, f$ is two times differentiable in x and $f^{(2)}$ is continuous on I, then*

$$\lim_{m\longrightarrow\infty} m((Q_{1,m}f)(x) - f(x)) = x(1-x)f^{(2)}(x). \tag{3.17}$$

Proof. We apply Theorem 1.3 for $s = 2$. □

4. Durrmeyer operator preserving the test function e_0 and e_2

In this section, we construct a sequence of Durrmeyer operators as defined in section 2, which preserve the test functions e_0 and e_2.

Imposing the conditions $(Q_m e_0)(x) = e_0(x)$ and $(Q_m e_2)(x) = e_2(x)$ for any $x \in I$ and $m \in \mathbb{N}_1$. Then, taking (2.5) and (2.7) into account, we obtain

$$\alpha_m(x) + \beta_m(x) = 1, \tag{4.1}$$
$$m(m-1)(\alpha_m(x))^2 + 4m\alpha_m(x) + 2 - (m+2)(m+3)x^2 = 0.$$

for any $x \in I$ and $m \in \mathbb{N}_1$. Since we are interest only in the positive valued functions α_m, β_m, an elementary computation leads to a unique solution for system (4.1), namely

$$\alpha_m(x) = \frac{-2m + \sqrt{\delta_m(x)}}{m(m-1)}, \quad \beta_m(x) = \frac{m^2 + m - \sqrt{\delta_m(x)}}{m(m-1)}, \tag{4.2}$$

where

$$\delta_m(x) = m(2m + 2 + (m-1)(m+2)(m+3)x^2), \tag{4.3}$$

for $x \in I$ and $m \in \mathbb{N}_1$.

Since $4\delta_m(x)$ is the discriminant of the second equation from (4.1), the above solutions exist and are positive if

$$x \geq \sqrt{\frac{2}{(m+2)(m+3)}}. \tag{4.4}$$

Lemma 4.1. *Let $m \in \mathbb{N}_1$. Then $\beta_m(x) \geq 0, x \geq 0$ if and only if*

$$0 \leq x \leq \sqrt{\frac{m+1}{m+3}}. \tag{4.5}$$

Proof. From $\beta_m(x) \geq 0$ we have $m^2 + m \geq \sqrt{\delta_m(x)}$, equivalent after calculus to $m + 1 \geq (m+3)x^2$, from where (4.5) follows. □

Lemma 4.2. *Let $m \in \mathbb{N}_1$. If $x \in \left[\sqrt{\frac{2}{(m+2)(m+3)}}, \sqrt{\frac{m+1}{m+3}}\right]$, then $\alpha_m(x) \geq 0$ and $\beta_m(x) \geq 0$.*

Proof. Results immediately from (4.4) and (4.5). □

Lemma 4.3. *The following inclusions*

$$\left[\sqrt{\frac{2}{(m_0+2)(m_0+3)}}, \sqrt{\frac{m_0+1}{m_0+3}}\right] \subset \left[\sqrt{\frac{2}{(m+2)(m+3)}}, \sqrt{\frac{m+1}{m+3}}\right] \subset [0,1] \tag{4.6}$$

hold, for any $m \in \mathbb{N}_1$.

Proof. By using that the function $\sqrt{\frac{2}{(m+2)(m+3)}}$ is decreasing and the function $\sqrt{\frac{m+1}{m+3}}$ is increasing, relations (4.6) follows. □

Remark 4.1. *For the remaining of this section we shall consider $I = \left[\sqrt{\frac{2}{(m_0+2)(m_0+3)}}, \sqrt{\frac{m_0+1}{m_0+3}}\right]$. Thus, for α_m, β_m defined by (4.2) we have $\alpha_m(x) \geq 0$ and $\beta_m(x) \geq 0$, for any $x \in I$ and $m \in \mathbb{N}_1$.*

If $m \in \mathbb{N}_1$ and $f \in L_1([0,1])$, we define the operator

$$(Q_{2,m}f)(x) = \frac{m+1}{(m(m-1))^m} \sum_{k=0}^{m} \binom{m}{k} \left(-2m + \sqrt{\delta_m(x)}\right)^k \cdot \\ \cdot \left(m^2 + m - \sqrt{\delta_m(x)}\right)^{m-k} \cdot \int_0^1 p_{m,k}(t) f(t) dt \tag{4.7}$$

for any $x \in I$.

Lemma 4.4. *We have*

$$(Q_{2,m}e_0)(x) = 1, \tag{4.8}$$

$$(Q_{2,m}e_1)(x) = \frac{\sqrt{\delta_m(x)} - m - 1}{(m-1)(m+2)}, \tag{4.9}$$

$$(Q_{2,m}e_2)(x) = x^2, \tag{4.10}$$

for any $x \in I$ and $m \in \mathbb{N}_1$.

Proof. Results immediately from the condition above and (2.6). □

Lemma 4.5. *The following identities*

$$(T_{m,0}Q_{2,m})(x) = 1, \tag{4.11}$$

$$(T_{m,1}Q_{2,m})(x) = m\left(\frac{\sqrt{\delta_m(x)} - m - 1}{(m-1)(m+2)} - x\right), \tag{4.12}$$

and

$$(T_{m,2}Q_{2,m})(x) = 2m^2 x\left(x - \frac{1}{m+2}\left(1 + \frac{\sqrt{\delta_m(x)} - 2m}{m-1}\right)\right) \tag{4.13}$$

hold, for any $x \in I$ and $m \in \mathbb{N}_1$.

Proof. By using Lemma 4.4 and relation (1.7), the relations (4.11)-(4.13) follows. □

Lemma 4.6. *The following identity*

$$\lim_{m \longrightarrow \infty} m\left(\frac{\sqrt{\delta_m(x)} - m - 1}{(m-1)(m+2)} - x\right) = x - 1 \tag{4.14}$$

holds for any $x \in I$.

Proof. We have

$$\lim_{m \longrightarrow \infty} m\left(\frac{\sqrt{\delta_m(x)} - m - 1}{(m-1)(m+2)} - x\right) = \\ = \lim_{m \longrightarrow \infty}\left(\frac{m^2}{(m-1)(m+2)} \cdot \frac{\sqrt{\delta_m(x)} - (m-1)(m+2)x}{m} - \frac{m(m+1)}{(m-1)(m+2)}\right) \\ = -1 + \lim_{m \longrightarrow \infty} \frac{\sqrt{\delta_m(x)} - (m-1)(m+2)x}{m}.$$

and after calculus, identity (4.14) results. □

Lemma 4.7. *We have that*

$$\lim_{m\longrightarrow\infty}(T_{m,0}Q_{2,m})(x)=1, \tag{4.15}$$

$$\lim_{m\longrightarrow\infty}\frac{(T_{m,2}Q_{2,m})(x)}{m}=2x(1-x), \tag{4.16}$$

for any $x\in I$, and there exists $m(0)\in\mathbb{N}$ such that

$$\frac{(T_{m,2}Q_{2,m})(x)}{m}\leq\frac{3}{2} \tag{4.17}$$

for any $x\in I$ and $m\in\mathbb{N}_1,\ m\geq m(0)$.

Proof. The relations (4.15) and (4.16) result taking (4.11), (4.13) and (4.14) into account. By using the definition of the limit of a function and because $x(1-x)\leq\frac{1}{4}$ for any $x\in[0,1]$, from (4.16) the relation (4.17) is obtained. □

Theorem 4.1. *Let $f:[0,1]\longrightarrow\mathbb{R}$ be a continuous function on [0,1]. Then*

$$\lim_{m\longrightarrow\infty}Q_{2,m}f=f \tag{4.18}$$

uniformly on I and there exists $m(0)\in\mathbb{N}$ such that

$$|(Q_{2,m}f)(x)-f(x)|\leq\frac{5}{2}\omega\left(f;\frac{1}{\sqrt{m}}\right) \tag{4.19}$$

for any $x\in I$ and $m\in\mathbb{N}_1,m\geq m(0)$.

Proof. Theorem 4.1 is a corollary of Theorem 1.3 for $s=0$, $\alpha_0=0$, $\alpha_2=1$, $k_0=1$ and $k_2=\frac{3}{2}$. □

Theorem 4.2. *If $f\in C([0,1]),x\in I$, f is two times differentiable in x and $f^{(2)}$ is continuous on I, then*

$$\lim_{m\longrightarrow\infty}m((Q_{2,m}f)(x)-f(x))=(x-1)f^{(1)}(x)+x(1-x)f^{(2)}(x). \tag{4.20}$$

Proof. Taking Lemma 4.6 into account and applying Theorem 1.3 for $s=2$ we obtain relation (4.20). □

5. Durrmeyer operator preserving the test functions e_1 and e_2

In this section we construct a sequence of Durrmeyer operators as defined in section 2, which preserve the test functions e_1 and e_2.

Imposing the conditions $(Q_me_1)(x)=e_1(x)$ and $(Q_me_2)(x)=e_2(x)$, for any $x\in I$ and any $m\in\mathbb{N}_1$ and taking (2.6) and (2.7) into account, we have

$$\frac{(\alpha_m(x)+\beta_m(x))^{m-1}}{m+2}\left((m+1)\alpha_m(x)+\beta_m(x)\right)=x \tag{5.1}$$

and

$$\frac{(\alpha_m(x)+\beta_m(x))^{m-2}}{(m+2)(m+3)}\big(m(m-1)\alpha_m^2(x)+4m\alpha_m(x)(\alpha_m(x)+\beta_m(x))+ \tag{5.2}$$

$$+2(\alpha_m(x)+\beta_m(x))^2\big)=x^2.$$

We note $t_m(x)=\alpha_m(x)+\beta_m(x)$ and from (5.1) and (5.2) we obtain that

$$\alpha_m(x)=\frac{(m+2)}{m}\frac{x}{t_m^{m-1}(x)}-\frac{1}{m}t_m^m(x),\ \beta_m(x)=t_m(x)-\alpha_m(x), \tag{5.3}$$

and the equation in $t_m^m(x)$

$$(m+1)t_m^{2m}(x)+(m(m+3)(m+2)x^2-2(m+1)(m+2)x)t_m^m(x)+(1-m)(m+2)^2x^2=0. \tag{5.4}$$

The discriminant of the equation (5.4) is

$$\Delta_m=(m+2)^2mx^2(8(m+1)-4(m+1)(m+3)x+m(m+3)^2x^2)\geq 0, \tag{5.5}$$

for any $x\in I$ and $m\in\mathbb{N}_1$.

We note $\delta_m(x)=m\left(8(m+1)-4(m+1)(m+3)x+m(m+3)^2x^2\right)$, for any $x\in I$ and $m\in\mathbb{N}_1$.

Because $m\geq 2$, it results

$$(1-m)(m+2)^2x^2\leq 0 \tag{5.6}$$

and then equation (5.4) has exactly one positive solution

$$t_m^m(x)=\frac{(m+2)x\left(2(m+1)-m(m+3)x+\sqrt{\delta_m}\right)}{2(m+1)}. \tag{5.7}$$

Lemma 5.1. *Let $m\in\mathbb{N}_1$. Then $\beta_m(x)\geq 0$ if and only if*

$$0\leq x\leq\frac{m+2}{m+3}. \tag{5.8}$$

Proof. From $\beta_m(x)\geq 0$ we have $m(m+3)x-2m\leq\sqrt{\delta_m}$, equivalent after some calculus with (5.8). □

Lemma 5.2. *Let $m\in\mathbb{N}_1$. Then $\alpha_m(x)\geq 0$ if and only if*

$$x\geq\frac{2}{m+3}. \tag{5.9}$$

Proof. From $\alpha_m(x)\geq 0$ we have $m(m+3)x\geq\sqrt{\delta_m}$, equivalent after calculus with (5.9). □

Lemma 5.3. *Let $m\in\mathbb{N}_1$. If $x\in\left[\frac{2}{m+3},\frac{m+2}{m+3}\right]$ then $\alpha_m(x)\geq 0$ and $\beta_m(x)\geq 0$.*

Proof. Results immediately from (5.8) and (5.9). □

Lemma 5.4. *The following inclusions*

$$\left[\frac{2}{m_0+3},\frac{m_0+2}{m_0+3}\right]\subset\left[\frac{2}{m+3},\frac{m+2}{m+3}\right]\subset[0,1] \tag{5.10}$$

hold for any $m\in\mathbb{N}_1$.

Proof. By using that the function $\frac{2}{m+3}$ is decreasing and the function $\frac{m+2}{m+3}$ is increasing, the relation (5.10) follows. □

If $m\in\mathbb{N}_1$ and $f\in L_1([0,1])$ we define the operator

$$(Q_{3,m}f)(x)=(m+1)\sum_{k=0}^{m}\binom{m}{k}(\alpha_m(x))^k(\beta_m(x))^{m-k}\int_0^1 p_{m,k}(t)f(t)dt, \tag{5.11}$$

where $\alpha_m(x)$ and $\beta_m(x)$ are given by the relations (5.3), for any $x\in\left[\frac{2}{m_0+3},\frac{m_0+2}{m_0+3}\right]$.

Remark 5.1. *In this section we note $I=\left[\frac{2}{m_0+3},\frac{m_0+2}{m_0+3}\right]$.*

Lemma 5.5. *We have*

$$(Q_{3,m}e_0)(x) = \frac{(m+2)x(2(m+1) - m(m+3)x + \sqrt{\delta_m(x)})}{2(m+1)}, \tag{5.12}$$

$$(Q_{3,m}e_1)(x) = x, \tag{5.13}$$

$$(Q_{3,m}e_2)(x) = x^2, \tag{5.14}$$

for any $x \in I$ and any $m \in \mathbb{N}_1$.

Proof. Results immediately from the condition above and (2.5). □

Lemma 5.6. *The following identities*

$$(T_{m,0}Q_{3,m})(x) = t_m^m(x), \tag{5.15}$$

$$(T_{m,1}Q_{3,m})(x) = mx(1 - t_m^m(x)), \tag{5.16}$$

$$(T_{m,2}Q_{3,m})(x) = -m^2x^2(1 - t_m^m(x)) \tag{5.17}$$

hold for any $x \in I$ and for any $m \in \mathbb{N}_1$.

Proof. By using Lemma 5.5 and the relation (1.7) the relations (5.15)-(5.17) follow. □

Lemma 5.7. *The following identities*

$$\lim_{m\longrightarrow\infty} t_m^m(x) = 1 \tag{5.18}$$

and

$$\lim_{m\longrightarrow\infty} m(1 - t_m^m(x)) = \frac{2(x-1)}{x} \tag{5.19}$$

hold.

Proof. We have

$$\lim_{m\longrightarrow\infty} t_m^m(x) = \frac{x}{2}\lim_{m\longrightarrow\infty} \frac{\delta_m(x) - (m(m+3)x - 2(m+1))^2}{\sqrt{\delta_m(x)} + m(m+3)x - 2(m+1)} \tag{5.20}$$

and after calculus, (5.18) follows. Taking (5.7) into account, similarly (5.19) is obtained. □

Lemma 5.8. *We have that*

$$\lim_{m\longrightarrow\infty}(T_{m,0}Q_{3,m})(x) = 1, \tag{5.21}$$

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,2}Q_{3,m})(x)}{m} = 2x(1-x), \tag{5.22}$$

for any $x \in I$ and there exists $m(0) \in \mathbb{N}$ such that

$$(T_{m,0}Q_{3,m})(x) \leq 2 \tag{5.23}$$

and

$$\frac{(T_{m,2}Q_{3,m})(x)}{m} \leq \frac{3}{2} \tag{5.24}$$

for any $x \in I$ and any $m \in \mathbb{N}_1, m \geq m(0)$.

Proof. The relations (5.21) and (5.22) result taking (5.15), (5.18) and (5.19) into account. By using the definition of the limit of a function and because $x(1-x) \leq \frac{1}{4}$ for any $x \in [0,1]$, from (5.20) and (5.21) the relations (5.22) and (5.23) are obtained. □

Theorem 5.1. *Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous function on $[0,1]$, then*

$$\lim_{m \longrightarrow \infty} (Q_{3,m} f)(x) = f(x) \tag{5.25}$$

uniformly on I and there exists $m(0) \in \mathbb{N}$ such that

$$|(Q_{3,m} f)(x) - f(x)| \leq \frac{7}{2} \omega \left(f; \frac{1}{\sqrt{m}} \right), \tag{5.26}$$

for any $x \in I$ and $m \in \mathbb{N}_1, m \geq m(0)$.

Proof. Theorem 5.1 is a corollary of Theorem 1.3 for $s = 0$, $\alpha_0 = 0$, $\alpha_1 = 1$, $k_0 = 2$ and $k_2 = \frac{3}{2}$. □

Theorem 5.2. *If $f \in C([0,1]), x \in I, f$ is two times differentiable in x and $f^{(2)}$ is continuous on I then*

$$\lim_{m \longrightarrow \infty} m((Q_{3,m} f)(x) - f(x)) = \frac{2(1-x)}{x} f(x) + 2(x-1) f^{(1)}(x) + x(x-1) f^{(2)}(x). \tag{5.27}$$

Proof. Taking Lemma 5.8 into account and applying Theorem 1.3 for $s = 2$, we obtain the relation (5.27). □

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