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Category of soft sets

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Abstract. The aim of this paper is to introduce a category, whose objects are soft sets, and obtain some basic results of this category, such as existence of product and coproduct. Then we introduce a subcategory of the category of soft sets, whose objects are soft BCK/BCI-algebras and develop the theory of soft ideals and soft BCK/BCI-subalgebras.

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Introduction

Molodtsov initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties which traditional mathematical tools can not handle. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Later other authors like Maji et al. [11, 12, 13], have further studied the theory of soft sets and used this theory to solve some decision making problems. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [15]. In 1991, the fuzzy set theory was applied to BCK-algebras [8]. Then Y. B. Jun et al. applied and studied the fuzzy set theory to BCK-algebras, soft BCK-algebras [8, 9, 6], BCC-algebras [2], MTL-algebras [10]. Later, other authors such as Y. B. Jun, C. H. Park, H. Aktas et al. study the theory of soft set in some kind of algebras such as groups, BCK-algebras and BCI-algebras (see [1, 5, 7]).

In this paper, we define the concept of soft morphism and introduce new category, which is called soft set category. Then verify some properties of a subcategory of soft set category, in the last section.

1. Preliminaries

Definition 1.1. [14] Let U be a initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U and $A \subseteq E$. A pair (f, A) is called a *soft set over* U, where $f : A \to P(U)$ is a map. If (f, A) is a soft set over U, we denote it by $(f, A)_U$, briefly.

Definition 1.2. Let $\{A_i\}_{i\in I}$ be a family of sets. The map $\pi_i : \prod_{i\in I} A_i \to A_i$, defined by $\pi_i((x_i)_{i\in I}) = x_i$ is called the $i-th$ *canonical projection map*, for any $i \in I$.

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Definition 1.3. [3, 4] A *BCI-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:
$(BCI1)$ $((x * y) * (x * z)) * (z * y) = 0$;
$(BCI2)$ $x * 0 = x$;
$(BCI3)$ $x * y = 0$ and $y * x = 0$ imply $y = x$.

We will also use the following notation in brevity: $x * y^n = (...(x * \overbrace{y) * ...) * y}^{n\ times}$, where $x, y \in X$ and $n \in \mathbb{N}$. A BCI-algebra X is called a *BCK-algebra* if $0 * x = 0$, for all $x \in X$. A nonempty subset S of BCK/BCI-algebra $(X, *, 0)$ is called a *subalgebra* of X if $x * y \in S$, for all $x, y \in X$. A subset I of X is called an *ideal* of X if (i) $0 \in I$; (ii) $x * y \in I$ and $x \in I$ imply $y \in I$, for all $x, y \in X$. Let $(x, *, 0)$ and $(y, *, 0)$ be two BCK/BCI-algebras. The map $f : X \to Y$ is called a *BCK/BCI-homomorphism* if $f(x*y) = f(x)*f(y)$, for all $x, y \in X$. If $f : X \to Y$ is a BCK/BCI-homomorphism, then $\ker(f) = f^{-1}(0)$ is an ideal of X. For any element $x \in X$, we define the order of x, denoted by $o(x)$, as $o(x) = min\{n \in \mathbb{N} |\ 0 * x^n = 0\}$.

Proposition 1.1. [16] *Let $(X, *, 0)$ be a BCI-algebra and $f : X \to X$ be a map defined by $f(x) = 0 * x$, for all $x \in X$. Then*
(i) *f is a BCK/BCI-homomorphism.*
(ii) *$o(x) = o(f(x))$, for all $x \in X$.*

Proposition 1.2. [16] *Let $(X, *, 0)$ and $(Y, *, 0)$ be two BCK/BCI-algebras and $f : X \to Y$ be a BCK/BCI-homomorphism.*
(i) *If S is a subalgebra of X, then $f(S)$ is a subalgebra of Y.*
(ii) *If T is a subalgebra of Y, then $f^{-1}(T)$ is a subalgebra of X.*
(iii) *If I is an ideal of Y, then $f^{-1}(I)$ is an ideal of X.*
(iv) *If f is onto and I be an ideal of X, then $f(I)$ is an ideal of Y.*

Definition 1.4. [7] Let $(X, *, 0)$ be a BCK/BCI-algebra and $(f, A)_X$ be a soft set. Then $(f, A)_X$ is called a *soft BCK/BCI-algebra* if $f(x)$ is a BCK/BCI-subalgebra of X, for all $x \in X$.

Definition 1.5. [7] Let $(X, *, 0)$ be a BCK/BCI-algebra and $(f, A)_X$, $(g, B)_X$ be two soft BCK/BCI-algebras. Then $(f, A)_X$ is called a *soft subalgebra* of $(g, B)_X$ if it satisfies:
(i) $A \subseteq B$,
(ii) $f(x)$ is a BCK/BCI-subalgebra of $g(x)$, for all $x \in A$.

Definition 1.6. [5] Let $(X, *, 0)$ be a BCI/BCK-algebra and $(f, A)_X$ be a soft BCK/BCI-algebras. A soft set $(g, B)_X$ is called a *soft ideal* of $(f, A)_X$, if it satisfies:
(i) $B \subseteq A$,
(ii) $g(x)$ is an ideal of $f(x)$, for all $x \in B$.

Definition 1.7. [7] Let $(X, *, 0)$ be a BCK/BCI-algebra and $(f, A)_X$ be a soft set. Then $(f, A)_X$ is called an *idealistic soft BCK/BCI-algebra* over X if $f(a)$ is an ideal of X, for any $a \in A$.

Remark 1.1. [16] Let A be an ideal of a BCI-algebra X. Define a binary relation θ on X as follows: $(x, y) \in \theta$ if and only if $x * y, y * x \in A$, for all $x, y \in X$. Then θ is a congruence relation and it is called the congruence relation induced by A. Let $[x] = \{y \in X |\ (x, y) \in \theta\}$ and $X/A = \{[x] |\ x \in X\}$. Then $(X/A, *, [0])$ is a BCI-algebra, where $[x] * [y] = [x * y]$, for all $x, y \in X$.

2. Category of soft sets

From now on, in this paper, we use Id_A to denote the identity map from A to A, for all nonempty set A, unless otherwise stated.

Definition 2.1. Let $(f,A)_U$ and $(g,B)_V$ be two soft sets. If $\alpha : A \to B$ and $\beta : P(U) \to P(V)$ are two maps such that $\beta \circ f = g \circ \alpha$, then (α, β) is called a *soft morphism* or briefly morphism from $(f,A)_U$ to $(g,B)_V$ and we write $(\alpha,\beta) : (f,A)_U \to (g,B)_V$. Therefore, $(\alpha,\beta) : (f,A)_U \to (g,B)_V$ is a soft morphism, if and only if the following diagram is commutative.

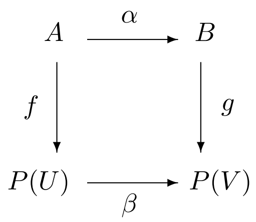

Figure 1. Diagram of soft morphism

If $(f,A)_U$ and $(g,B)_V$ are two soft sets, then we use $Hom((f,A)_U,(g,B)_V)$ to denote the set of all morphisms from $(f,A)_U$ to $(g,B)_V$.

Definition 2.2. Let X be a set and $f : X \to P(X)$, defined by $f(x) = \{x\}$, for any $x \in X$. Then the soft set $(f,X)_X$ is called a *simple soft set generated by* X.

Example 2.1. Let A and B be two sets, $\alpha : A \to B$ be a map, $(f,A)_A$ and $(g,B)_B$ be simple soft sets generated by A and B, respectively. Then clearly, $(\alpha,\beta) : (f,A)_A \to (g,B)_B$ is a soft morphism, where $\beta(S) = \{\alpha(s)|\ s \in S\}$, for all nonempty subset $S \in P(A)$ and $\beta(\emptyset) = \emptyset$.

Definition 2.3. Let $(\alpha,\beta),(\lambda,\mu) \in Hom((f,A)_U,(g,B)_V)$.
(i) We use $(\alpha,\beta) = (\lambda,\mu)$ to denote $\alpha = \lambda$ and $\beta = \mu$.
(ii) We use $(\alpha,\beta) \sim (\lambda,\mu)$ to denote $g \circ \alpha = g \circ \lambda$ (or equivalently, $\beta \circ f = \mu \circ f$).

Theorem 2.1. *The class of all soft sets together with the class of all soft morphisms form a category. It is called a* soft set category *and is denoted by* $\mathfrak{SS}$.

Proof. Let $(f,A)_U,(g,B)_V$ and $(h,C)_W$ be soft sets, $(\alpha,\beta) \in Hom((f,A)_U,(g,B)_V)$ and $(\gamma,\lambda) \in Hom((g,B)_V,(h,C)_W)$. Then $g \circ \alpha = \beta \circ f$ and $h \circ \gamma = \lambda \circ g$ and so $h \circ (\gamma \circ \alpha) = (h \circ \gamma) \circ \alpha = (\lambda \circ g) \circ \alpha = \lambda \circ (g \circ \alpha) = \lambda \circ (\beta \circ f)$. It follows that

$$(\gamma \circ \alpha, \lambda \circ \beta) \in Hom((f,A)_U,(h,C)_W).$$

Let $(k,D)_Z$ and $(k',D')_{Z'}$ be two soft sets and $(\varphi,\psi) \in Hom((f,A)_U,(k,D)_Z)$, $(\varphi',\psi') \in Hom((k',D')_{Z'},(f,A)_U)$. Clearly, $(Id_A, Id_{P(U)}) : (f,A)_U \to (f,A)_U$ is a morphism, $(Id_A, Id_{P(U)}) \circ (\varphi,\psi) = (\varphi,\psi)$ and $(\varphi',\psi') \circ (Id_A, Id_{P(U)}) = (\varphi',\psi')$. Also, $((\alpha,\beta)\circ(\gamma,\lambda))\circ(\mu,\nu) = (\alpha,\beta)\circ((\gamma,\lambda)\circ(\mu,\nu))$, for any soft set $(d,E)_Y$ and any morphism $(\mu,\nu) \in Hom((h,C)_W,(d,E)_Y)$, so these classes of morphisms and objects form a category. □

Definition 2.4. Let $(g,B)_V,(f,A)_U$ be two soft sets and $(\alpha,\beta) : (f,A)_U \to (g,B)_V$ be a morphism.

(i) (α, β) is called a *weak monic* if $(\alpha, \beta) \circ (\gamma, \eta) \sim (\alpha, \beta) \circ (\gamma', \eta')$, implies $(\gamma, \eta) \sim (\gamma', \eta')$, for any $(k, D)_Z$ and morphisms $(\gamma, \eta), (\gamma', \eta') \in Hom((k, D)_Z, (f, A)_U)$.
(ii) (α, β) is called a *weak epic* if $(\lambda, \mu) \circ (\alpha, \beta) \sim (\lambda', \mu') \circ (\alpha, \beta)$, implies $(\lambda, \mu) \sim (\lambda', \mu')$, for any $(h, C)_W$ and morphisms $(\lambda, \mu), (\lambda', \mu') \in Hom((g, B)_V, (h, C)_W)$.

Proposition 2.2. *Let $(g, B)_V, (f, A)_U$ be two soft sets and $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be a morphism.*
(i) *If β is an one to one map, then (α, β) is a weak monic.*
(ii) *If α and β are one to one maps, then (α, β) is a monic.*
(iii) *If α is an onto map, then (α, β) is a weak epic.*
(iv) *If α and β are onto maps, then (α, β) is an epic.*
(v) *(α, β) is an isomorphism if and only if α and β are one to one and onto maps.*

Proof. (i) Let $(\alpha, \beta) \circ (\gamma, \eta) \sim (\alpha, \beta) \circ (\gamma', \eta')$, for some soft set $(h, C)_W$ and some morphisms $(\gamma, \eta), (\gamma', \eta') \in Hom((h, C)_W, (f, A)_U)$. Then $(\alpha \circ \gamma, \beta \circ \eta) \sim (\alpha \circ \gamma', \beta \circ \eta')$ and so $(\beta \circ \eta) \circ h = (\beta \circ \eta') \circ h$. Since β is an one to one map, then $\eta \circ h = \eta' \circ h$. Hence $(\gamma, \eta) \sim (\gamma', \eta')$. Therefore, (α, β) is a weak monic.
(ii) Let $(\alpha, \beta) \circ (\gamma, \eta) = (\alpha, \beta) \circ (\gamma', \eta')$, for some soft set $(h, C)_W$ and some morphisms $(\gamma, \eta), (\gamma', \eta') \in Hom((h, C)_W, (f, A)_U)$. Then $\alpha \circ \gamma = \alpha \circ \gamma'$ and $\beta \circ \eta = \beta \circ \eta'$. Since α and β are one to one maps, we have $\gamma = \gamma'$ and $\eta = \eta'$ and so (α, β) is a monic.
(iii) Let $(\varphi, \chi) \circ (\alpha, \beta) = (\varphi', \chi') \circ (\alpha, \beta)$, for some soft set $(k, D)_Z$ and some morphisms $(\varphi, \chi), (\varphi', \chi') \in Hom((g, B)_V, (k, D)_Z)$. Then $(\varphi \circ \alpha, \chi \circ \beta) \sim (\varphi' \circ \alpha, \chi' \circ \beta)$ and so $k \circ (\varphi \circ \alpha) = k \circ (\varphi' \circ \alpha)$. Since α is onto, then we obtain $k \circ \varphi = k \circ \varphi'$. Hence $(\varphi, \chi) \sim (\varphi', \chi')$ and so (α, β) is a weak epic.
(iv) Straightforward.
(v) Let (α, β) be an isomorphism. Then there exists a morphism $(\lambda, \mu) : (g, B)_V \to (f, A)_U$ such that $(\alpha, \beta) \circ (\lambda, \mu) = (Id_B, Id_{P(V)})$ and $(\lambda, \mu) \circ (\alpha, \beta) = (Id_A, Id_{P(U)})$. Hence $\alpha o \lambda = Id_B$ and $\lambda \circ \alpha = Id_A$ and so α is one to one and onto. In a similar way, we obtain β is one to one and onto. Conversely, let α and β be two one to one and onto maps. Then there are $\lambda : B \to A$ and $\mu : P(V) \to P(U)$ such that $\alpha \circ \lambda = Id_B$, $\lambda \circ \alpha = Id_A$, $\beta \circ \mu = Id_{P(V)}$ and $\mu \circ \beta = Id_{P(U)}$. We show that $(\lambda, \mu) : (g, B)_V \to (f, A)_U$ is a morphism. Let $b \in B$. Then there exists an element $a \in A$ such that $b = \alpha(a)$ and so $f(\lambda(b)) = f(a) = \mu(\beta(f(a))) = \mu(g(\alpha(a))) = \mu(g(b))$. Hence (λ, μ) is a morphism. Clearly, $(\alpha, \beta) \circ (\lambda, \mu) = (Id_B, Id_{P(V)})$ and $(\lambda, \mu) \circ (\alpha, \beta) = (Id_A, Id_{P(U)})$. Therefore, (α, β) is an isomorphism. □

Remark 2.1. Let $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be a morphism, $C = Im(\alpha)$ and $h = g|_C$. Then $(h, C)_V$ is a soft set and $(\alpha, \beta) : (f, A)_U \to (h, c)_V$ is a soft morphism and so by Proposition 2.2(iii), $(\alpha, \beta) : (f, A)_U \to (h, c)_V$ is a weak epic.

Example 2.2. (i) Let $f : [0, 1] \to P\{[0, 1]\}$ and $g : [0, 1] \to P\{[0, 2]\}$ by defined by $f(x) = \{u \in [0, 1] |\ u \leq x\}$ and $g(x) = \{u \in [0, 2] |\ u \leq 2x\}$, for all $x \in [0, 1]$. Then $(f, [0, 1])_{[0,1]}$ and $(g, [0, 1])_{[0,2]}$ are two soft sets. Define $\alpha : [0, 1] \to [0, 1]$ by $\alpha(x) = x$ and $\beta : P(\{[0, 1]\}) \to P(\{[0, 2]\})$, by $\beta(S) = \{u \in [0, 2] |\ u \leq 2x\ ,\ \exists x \in S\}$. It is easy to see that $(\alpha, \beta) : (f, [0, 1])_{[0,1]} \to (g, [0, 1])_{[0,2]}$ is a soft morphism. Since β is one to one, then by Proposition 2.2(i), (α, β) is a weak monic.
(ii) Let $f : \mathbb{N} \to P(\{1\})$ and $g : \mathbb{N} \to P(\{1\})$ be two maps was defined by

$$f(a) = \begin{cases} \emptyset & \text{if } a \text{ is even ,} \\ \{1\} & \text{if } a \text{ is odd .} \end{cases} \qquad g(a) = \begin{cases} \{1\} & \text{if } a \text{ is even ,} \\ \emptyset & \text{if } a \text{ is odd .} \end{cases}$$

Then $(Id_{\mathbb{N}}, \beta) : (f, \mathbb{N})_{\{1\}} \to (g, \mathbb{N})_{\{1\}}$ is a soft morphism, where $\beta : P(\{1\}) \to P(\{1\})$ is a map defined by $\beta(\emptyset) = \{1\}$ and $\beta(\{1\}) = \emptyset$. Since $Id_{\mathbb{N}}$ is onto, then by Proposition 2.2(iii), $(Id_{\mathbb{N}}, \beta)$ is a weak epic.

Theorem 2.3. *Let $(g, B)_V$ and $(f, A)_U$ be two soft sets and $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be a monic such that β be one to one. Then α is one to one, too.*

Proof. Let $\alpha(x) = \alpha(y)$, for some $x, y \in A$. Let $\gamma : A \to A$ and $\gamma' : A \to A$ be two maps defined by

$$\gamma(a) = \begin{cases} a & \text{if } a \in A - \{x, y\}, \\ \{y\} & \text{if } a \in \{x, y\}. \end{cases} \qquad \gamma'(a) = \begin{cases} a & \text{if } a \in A - \{x, y\}, \\ \{x\} & \text{if } a \in \{x, y\}. \end{cases}$$

We show that $(\gamma, Id_{P(U)}) : (f, A)_U \to (f, A)_U$ and $(\gamma', Id_{P(U)}) : (f, A)_U \to (f, A)_U$ are two morphisms. Let $a \in A$. If $a \in A - \{x, y\}$, then $(f \circ \gamma)(a) = f(a) = (Id_{P(U)} \circ f)(a)$. Since $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ is a morphism, we have $g \circ \alpha = \beta \circ f$ and so $\beta(f(x)) = g(\alpha(x)) = g(\alpha(y)) = \beta(f(y))$. Since β is an one to one map, then $f(x) = f(y)$. Now, let $a \in \{x, y\}$. Then $(f \circ \gamma)(a) = f(y) = f(a) = Id_{P(U)} \circ f(a)$. Therefore, $(\gamma, Id_{P(U)}) \in Hom((f, A)_U, (f, A)_U)$. By the similar way, we obtain $(\gamma', Id_{P(U)}) \in Hom((f, A)_U, (f, A)_U)$. Moreover,

$$\begin{aligned} (\alpha, \beta) \circ (\gamma, Id_{P(U)}) &= (\alpha \circ \gamma, \beta \circ Id_{P(U)}) \\ &= (\alpha \circ \gamma', \beta \circ Id_{P(U)}), \quad \text{since } \alpha(x) = \alpha(y) \\ &= (\alpha, \beta) \circ (\gamma', Id_{P(U)}) \end{aligned}$$

Since (α, β) is a monic, then $(\gamma, Id_{P(U)}) = (\gamma', Id_{P(U)})$ and so $y = \gamma(x) = \gamma'(x) = x$. Therefore, α is an onto map. □

Theorem 2.4. *Let $(g, B)_V$ and $(f, A)_U$ be two soft sets and $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be an epic such that α is onto. Then β is onto, too.*

Proof. Let $a \in P(V) - Im(\beta)$ and $\mu' = Id_{P(V)}$. Let $\mu : P(V) \to P(V)$ be a map defined by

$$\mu(x) = \begin{cases} x & \text{if } x \in Im(\beta), \\ a & \text{if } x \in P(V) - Im(\beta). \end{cases}$$

We show that $(Id_B, \mu), (Id_B, \mu') \in Hom((g, B)_V, (g, B)_V)$. Since $g \circ \alpha = \beta \circ f$ and α is onto, then $Im(g) \subseteq Im(\beta)$, it follows that $(\mu \circ g)(b) = \mu(g(b)) = g(b)$, for any $b \in B$. Hence $g \circ Id_B = \mu \circ g$ and so $(Id_B, \mu) \in Hom((g, B)_V, (g, B)_V)$. Also, $\mu'(g(b)) = g(b) = (Id_B \circ g)(b)$, for any $b \in B$, so $(Id_B, \mu') \in Hom((g, B)_V, (g, B)_V)$. Now, let $b \in B$. Then $\mu'(\beta(b)) = \beta(b) = \mu(\beta(b))$ and so $(Id_B, \mu) \circ (\alpha, \beta) = (Id_B, \mu') \circ (\alpha, \beta)$. Since (α, β) is an epic, we have $(Id_B, \mu) = (Id_B, \mu')$ consequently, $\mu = \mu'$, which is impossible. Therefore, there is not any $a \in P(V) - Im(\beta)$ and β is onto. □

Definition 2.5. A morphism $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ is called *onto morphism*, if α and β are onto maps.

Let $f : X \to Y$ be a maps and $\ker(f) = \{(x, y) \in A \times A |\ f(x) = f(y)\}$. Then $\ker(f)$ is a equivalence relation on X. The set of all equivalent classes of X with respect to $\ker(f)$, is denoted by $X/\ker(f)$. Clearly, the map $\overline{f} : X/\ker(f) \to Im(f)$, was defined by $\overline{f}([x]) = f(x)$, for any $x \in X$ is a one to one and onto map.

Theorem 2.5. *Let $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be an onto morphism. Then the soft set $(\overline{f}, A/\ker(f))_V$ is isomorphic to $(g, B)_V$, where $\overline{f}([x]) = \beta(f(x))$, for any $x \in X$.*

Proof. Let $\alpha(x) = \alpha(y)$, for some $x, y \in A$. Then $\beta(f(x)) = g(\alpha(x)) = g(\alpha(y)) = \beta(f(y))$. Hence $\overline{f}$ is well defined. Moreover, $g(\overline{\alpha}([x])) = g(\alpha(x)) = \beta(f(x)) = Id_{P(V)}(\overline{f}([x]))$, for all $x \in A$ and so $(\overline{\alpha}, Id_{P(V)}) : (\overline{f}, A/\ker(f))_V \to (g, B)_V$ is a morphism. Since $\overline{\alpha}$ and $Id_{P(V)}$ are one to one and onto, then by Proposition 2.2(v), $(\overline{\alpha}, Id_{P(V)})$ is an isomorphism. □

Note that, if $\{A_i\}_{i\in I}$ is a family of sets, then $\prod_{i\in I} A_i = \{(x_i)_{i\in I} |\ x_i \in A_i\ ,\ \forall i \in I\}$ is the product of this family of sets. In the next theorem, we use this fact to find the product of $\{(f_i, A_i)_{U_i}\}_{i\in I}$ in $\mathfrak{SS}$.

Theorem 2.6. *Let $\{(f_i, A_i)_{U_i}\}_{i\in I}$ be a family of soft sets. Then $(f, A)_W$ is the product of this family, where $A = \prod_{i\in I} A_i$, $W = \bigcup_{i\in I}(U_i \times \{i\})$ and $f((x_i)_{i\in I}) = \bigcup_{i\in I}(f_i(x_i) \times \{i\})$, for any $(x_i)_{i\in I} \in A$.*

Proof. Let $\alpha_i : A \to A_i$ be the $i-$th canonical projection maps, for all $i \in I$ and $W_i = U_i \times \{i\}$. Define $\beta_i : P(W) \to P(U_i)$, by $\beta_i(Y) = \{u \in U_i |\ (u, i) \in W_i \cap Y\}$, for all $Y \subseteq W$ and $i \in I$. We show that $(\alpha_i, \beta_i) \in Hom((f, A)_W, (f_i, A_i)_{U_i})$, for all $i \in I$. Let $(a_i)_{i\in I} \in A$ and $j \in I$. Then $f_j(\alpha_j((a_i)_{i\in I})) = f_j(a_j)$ and $\beta_j(f((a_i)_{i\in I})) = \beta_j(\bigcup_{i\in I}(f_i(a_i) \times \{i\})) = f_j(a_j)$ and so $\beta_j \circ f((a_i)_{i\in I}) = f_j(\alpha((a_i)_{i\in I}))$. Hence (α_i, β_i) is a morphism, for all $i \in I$. Let $(g, B)_V$ be a soft set and $(\lambda_i, \mu_i) : (g, B)_V \to (f_i, A_i)_{U_i}$ be a morphism, for all $i \in I$. Then we define $\varphi : B \to A$ and $\chi : P(V) \to P(W)$, by $\varphi(b) = (\lambda_i(b))_{i\in I}$ and $\chi(X) = \bigcup_{i\in I}(\mu_i(X) \times \{i\})$, for all $b \in B$ and $X \subseteq V$. Clearly, χ and φ are well defined. Let $b \in B$. Then $f(\varphi(b)) = f((\lambda_i(b))_{i\in I}) = \bigcup_{i\in I}(f_i(\lambda_i(b))\times\{i\})$. Since $(\lambda_i, \mu_i) \in Hom((g, B)_V, (f_i, A_i)_{U_i})$, then $f_i \circ \lambda_i = \mu_i \circ g$, for all $i \in I$ and so $f(\varphi(b)) = \bigcup_{i\in I}(\mu_i \circ g(b) \times \{i\}) = \chi \circ g(b)$. Hence $(\varphi, \psi) \in Hom((g, B)_V, (f, A)_W)$. Assume that $x \in B$, $Y \in P(V)$ and $i \in I$. Then we have

$$
\begin{aligned}
(\alpha_i, \beta_i) \circ (\varphi, \chi)(x, Y) = (\alpha_i(\varphi(x)), \beta_i(\chi(Y))) &= (\lambda_i(x), \beta_i(\bigcup_{i\in I}(\mu_i(Y) \times \{i\}))) \\
&= (\lambda_i(x), \mu_i(Y)).
\end{aligned}
$$

Hence $(\alpha_i, \beta_i) \circ (\varphi, \chi) = (\lambda_i, \mu_i)$, for all $i \in I$. Now, we show that (φ, χ) is unique. Let $(\varphi', \chi') : (g, B)_V \to (f, A)_W$ be a morphism such that $(\alpha_i, \beta_i) \circ (\varphi', \chi') = (\lambda_i, \mu_i)$, for all $i \in I$. Then $\alpha_i(\varphi'(x)) = \lambda_i(x) = \alpha_i(\varphi(x))$, for all $x \in B$ and $i \in I$ and so by definition of α_i we get $\varphi = \varphi'$. Let $Y \in P(V)$. Since $\chi'(Y), \chi(Y) \subseteq W$, $W = \bigcup_{i\in I} W_i$ and $W_i \cap W_j = \emptyset$, for all distinct elements $i, j \in I$, then $\chi'(Y) = \chi(Y)$ if and only if $\chi'(Y) \cap W_i = \chi(Y) \cap W_i$, for all $i \in I$. Let $i \in I$ and $(a, i) \in W_i$. Then

$$
\begin{aligned}
(a, i) \in \chi'(Y) \cap W_i &\Leftrightarrow a \in \beta_i(\chi'(Y)), \quad \text{by definition of } \beta_i \\
&\Leftrightarrow a \in \beta_i(\chi(Y)), \quad \text{since } \beta_i \circ \chi = \beta_i \circ \chi' \\
&\Leftrightarrow (a, i) \in \chi(Y) \cap W_i, \quad \text{by definition of } \beta_i
\end{aligned}
$$

Hence $\chi'(Y) \cap W_i = \chi(Y) \cap W_i$ and so $\chi'(Y) = \chi(Y)$. Therefore, $(\varphi, \chi) = (\varphi', \chi')$ and so $(f, A)_W$ is the product of $\{(f_i, A_i)_{U_i}\}_{i\in I}$. □

Corollary 2.7. *The category* $\mathfrak{SS}$ *has arbitrary products.*

Definition 2.6. A weak *coproduct* for the family $\{(f_i, A_i)_{U_i}\}_{i\in I}$ of soft set is a soft set $(g, B)_V$ together with a family of morphisms $\{(\alpha_i, \beta_i) : (f_i, A_i)_{U_i} \to (g, B)_V\}_{i\in I}$ such that for any object $(h, C)_W$ and family of morphisms $\{(\varphi_i, \chi_i) : (f_i, A_i)_{U_i} \to (h, C)_W\}_{i\in I}$, there exists a morphism $(\varphi, \chi) : (g, B)_V \to (h, C)_W$ such that for any $i \in I$, $(\varphi, \chi) \circ (\alpha_i, \beta_i) = (\varphi_i, \chi_i)$. Moreover, if $(\varphi', \chi') : (g, B)_V \to (h, C)_W$ be another morphism such that $(\varphi', \chi') \circ (\alpha_i, \beta_i) = (\varphi_i, \chi_i)$, for any $i \in I$, then $(\varphi, \chi) \sim (\varphi', \chi')$.

Theorem 2.8. *Let* $\{(f_i, A_i)_{U_i}\}_{i\in I}$ *be a family of soft sets. Then this family has a weak coproduct in* $\mathfrak{SS}$.

Proof. Let $U'_i = U_i \times \{i\}$, $B_i = A_i \times \{i\}$, for any $i \in I$, $C = \bigcup_{i\in I} B_i$ and $U' = \bigcup_{i\in I} U'_i$. Let $g_i : B_i \to P(U_i)$ defined by $g_i(x, i) = f_i(x)$, for any $x \in A_i$ and $i \in I$. We define $g : C \to P(U')$ by $g((x, i)) = g_i((x, i)) \times \{i\}$, $\alpha_i : A_i \to C$, by $\alpha(y) = (y, i)$ and $\beta_i : P(U_i) \to P(U')$, by $\beta_i(Y) = Y \times \{i\}$, for any $(x, i) \in C$, $y \in A_i$, $Y \subseteq U'$ and $i \in I$. Clearly, g is well defined. We show that $(\alpha_i, \beta_i) : (f_i, A_i)_{U_i} \to (g, C)_{U'}$ is a morphism, for any $i \in I$. Let $i \in I$ and $a \in A_i$. Then $\beta_i(f_i(a)) = f_i(a) \times \{i\} = g_i((a, i)) \times \{i\} = g((a, i)) = g(\alpha_i(a))$. Hence $\beta_i \circ f_i = g \circ \alpha_i$, for any $i \in I$. Now, let $(h, X)_V$ be a soft set and $(\lambda_i, \mu_i) : (f_i, A_i)_{U_i} \to (h, X)_V$ be a morphism, for any $i \in I$. Then we define two maps $\varphi : C \to X$ and $\chi : P(U') \to P(V)$ by

$$\begin{aligned}
\varphi((x, i)) &= \lambda_i(x), \quad \text{for all } (x, i) \in C,\ i \in I \\
\chi(Y) &= \bigcup_{i\in I} \{\mu_i(X_i)|\ X_i = \{x|\ (x, i) \in Y \cap U_i\}, \quad \text{for all } Y \subseteq U' \ \}
\end{aligned}$$

Let $(x, i) \in C$. Then $h(\varphi((x, i))) = h(\lambda_i(x)) = \mu_i(f_i(x)) = \chi(g_i((x, i)) \times \{i\}) = \chi(g((x, i)))$. Hence $h \circ \varphi = \chi \circ g$ and so $(\varphi, \chi) \in Hom((g, C)_{U'}, (h, X)_V)$. Moreover,

$$(\varphi, \chi)\circ(\alpha_i, \beta_i)(a, Y) = (\varphi(\alpha_i(a)), \chi(\beta_i(Y))) = (\varphi((a, i)), \chi(Y \times \{i\})) = (\lambda_i(a), \mu_i(Y)),$$

for any $a \in A_i$, $Y \subseteq U_i$ and $i \in I$. Therefore, $(\varphi, \chi) \circ (\alpha_i, \beta_i) = (\lambda_i, \mu_i)$, for any $i \in I$. Now, let $(\varphi', \chi') : (g, C)_{U'} \to (h, X)_V$ be a morphism, such that $(\varphi', \chi') \circ (\alpha_i, \beta_i) = (\lambda_i, \mu_i)$, for any $i \in I$. Let $x \in C$. Then there exists $i \in I$ and $y \in A_i$ such that $x = (y, i)$. Hence $\varphi'(x) = \varphi'((y, i)) = \varphi'(\alpha_i(y)) = \lambda_i(y) = \varphi(\alpha_i(y)) = \varphi((y, i)) = \varphi(x)$. Hence $\varphi = \varphi'$. Since $(\varphi', \chi') \in Hom((g, C)_{U'}, (h, X)_V)$, then $\chi' \circ g = h \circ \varphi'$ and so $\chi' \circ g = h \circ \varphi' = h \circ \varphi = \chi \circ g$, whence $(\varphi, \chi) \sim (\varphi', \chi')$. Therefore, $(g, C)_{U'}$ is a weak coproduct of the family $\{(f_i, A_i)_{U_i}\}_{i\in I}$ in $\mathfrak{SS}$. ☐

Note that, in the proof of the last theorem, if $U_i \cap U_j = \emptyset$, $(A_i \cap A_j = \emptyset)$, for any $i, j \in I$, then we can use U_i (A_i) instead of U'_i (B_i).
In Example 2.3, we show that $(g, C)_{U'}$ is not coproduct of the family $\{(f_i, A_i)_{U_i}\}_{i\in I}$ in general, where $(g, c)_{U'}$ is a soft set, defined in the Theorem 2.8.

Remark 2.2. Let $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be a morphism. If $f' : A \to P(P(U))$, $g' : B \to P(P(V))$ and $\beta' : P(P(U)) \to P(P(V))$ defined by $f(a) = \{f(a)\}$, $g(b) = \{g(b)\}$ and $\beta'(Y) = \{\beta(y)|\ y \in Y\}$, for any $a \in A$, $b \in B$ and $Y \subseteq P(U)$. Then $\beta'(f'(a)) = \beta'(\{f(a)\} = \{\beta(f(a))\} = \{g\circ\alpha(a)\} = g'(\alpha(a))$, for any $a \in A$. Therefore, $(\alpha, \beta') \in Hom((f', A)_{P(U)}, (g', B)_{P(V)})$.

Example 2.3. Let $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5, 6\}$, $U_1 = \{a, b, c\}$, $U_2 = \{e, f, g\}$, $C = \{1, 2, 3, 4, 5, 6\}$ and $U = \{a, b, c, e, f, g\}$. Define the maps $f_1 : A_1 \to P(U_1)$,

$f_2 : A_2 \to P(U_2)$ and $g : C \to P(U)$ by

$$f_1(x) = \begin{cases} \{a\} & x = 1, \\ \{b\} & x = 2, \\ \{c\} & x = 3. \end{cases} \quad f_2(x) = \begin{cases} \{e\} & x = 4, \\ \{f\} & x = 5, \\ \{g\} & x = 6. \end{cases} \quad g(x) = \begin{cases} f_1(x) & x \in A_1, \\ f_2(x) & x \in A_2. \end{cases}$$

Let $\alpha_1 : A_1 \to C$, $\alpha_2 : A_2 \to C$, $\beta_1 : P(U_1) \to P(U)$ and $\beta_2 : P(U_2) \to P(U)$ be the inclusion maps. Then it is easy to check that $(\alpha_1, \beta_1) \in Hom((f_1, A_1)_{U_1}, (g, B)_C)$ and $(\alpha_2, \beta_2) \in Hom((f_2, A_2)_{U_2}, (g, B)_C)$. Let $X = \{7, 8, 9\}$, $W = \{m, n, o\}$ and $h : X \to P(W)$, $\lambda_1 : A_1 \to X$, $\lambda_2 : A_2 \to X$, $\mu_1 : U_1 \to W$ and $\mu_2 : U_2 \to W$ defined by

$$h(x) = \begin{cases} \{m\} & x = 7, \\ \{n\} & x = 8, \\ \{o\} & x = 9. \end{cases} \quad \lambda_1(x) = \begin{cases} 7 & x = 1, \\ 8 & x = 2, \\ 9 & x = 3. \end{cases} \quad \lambda_2(x) = \begin{cases} 7 & x = 4, \\ 8 & x = 5, \\ 9 & x = 6. \end{cases}$$

$$\mu_1(x) = \begin{cases} m & x = a, \\ n & x = b, \\ o & x = c. \end{cases} \quad \mu_2(x) = \begin{cases} m & x = e, \\ n & x = f, \\ o & x = g. \end{cases}$$

Then clearly, $(\lambda_i, \overline{\mu}_i) \in Hom((f_i, A_i)_{U_i}, (h, X)_W)$, for any $i \in \{1, 2\}$, where $\overline{\mu}_i(S) = \{\mu_i(s) |\ s \in S\}$, for all $S \subseteq U_i$, $i \in \{1, 2\}$. Let $\varphi : C \to X$, $\chi : P(U) \to P(W)$, $\varphi' : C \to X$ and $\chi' : P(U) \to P(W)$, were defined by

$$\varphi(x) = \begin{cases} 7 & x \in \{1, 4\}, \\ 8 & x \in \{2, 5\}, \\ 9 & x \in \{3, 6\}. \end{cases} \quad \chi(x) = \begin{cases} \{m\} & x \in \{\{a\}, \{e\}\}, \\ \{n\} & x \in \{\{b\}, \{f\}\}, \\ \{o\} & x \in \{\{c\}, \{g\}\}, \\ \{m, n\} & \text{otherwise.} \end{cases}$$

$$\chi'(x) = \begin{cases} \{m\} & x \in \{\{a\}, \{e\}\}, \\ \{n\} & x \in \{\{b\}, \{f\}\}, \\ \{o\} & x \in \{\{c\}, \{g\}\}, \\ \{o, n\} & \text{otherwise.} \end{cases}$$

and $\varphi(x) = \varphi'(x)$, for any $x \in C$. Then $(\varphi, \chi), (\varphi', \chi') \in Hom((g, C)_U, (h, C)_W)$ and $(\varphi, \chi) \circ (\alpha_i, \beta_i) = (\lambda_i, \mu_i)$, for any $i \in \{1, 2\}$. But $(\varphi, \chi) \neq (\varphi', \chi')$. Therefore, $(g, C)_U$ may not be coproduct of the family $\{(f_i, A_i)_{U_i}\}_{i \in \{1,2\}}$.

Corollary 2.9. *The category $\mathfrak{SS}$ has arbitrary weak coproducts.*

Proposition 2.10. *The category $\mathfrak{SS}$ has a terminal object.*

Proof. Let $A = \{a\}$, $U = \emptyset$ and $f : A \to P(U)$ defined by $f(a) = \emptyset$. Then $(f, A)_U$ is a soft set. Now, let $(g, B)_V$ be an other soft set. Define $\alpha : B \to A$ and $\beta : P(V) \to P(U)$ by $\alpha(b) = a$ and $\beta(Y) = \emptyset$, for all $b \in B$ and $Y \in P(V)$. Clearly, $(\alpha, \beta) : (g, B)_V \to (f, A)_U$ is a morphism. It is easy to see that (α, β) is a terminal object. □

Definition 2.7. Let $(f, A)_U$ and $(g, B)_V$ be two soft sets. The soft set $(f, A)_U$ is called a *soft subset* of $(g, B)_V$ if the following conditions are satisfied:

(i) $U \subseteq V$;
(ii) $A \subseteq B$;
(iii) $f(x) \subseteq g(x)$, for any $x \in A$.

Proposition 2.11. *Let $F : P(U) \to P(V)$ be a map and $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be a morphism such that β preserve the inclusion. Then*

(i) *$(F \circ f, A)_V$ is a soft set too.*
(ii) *$(Id_A, F) : (f, A)_U \to (F \circ f, A)_V$ is a morphism.*

(iii) *If* $(f', A')_{U'}$ *is a soft subset of* $(f, A)_U$, *then* $(g', \alpha(A'))_{\beta(U')}$ *is a soft subset of* $(g, B)_V$, *where* $g'(b) = \cup\{\beta(f'(a))|\ \alpha(a) = b)\}$, *for any* $b \in \alpha(A')$.

Proof. The proof of (i) and (ii) are straightforward.
(iii) Clearly, the maps $g' : \alpha(A') \to P(V)$ is well defined. Let $b \in \alpha(A')$. Then there is $a \in A'$ such that $\alpha(a) = b$. Since $f'(a) \subseteq f(a)$ and β preserve the inclusion, then $g(b) = g(\alpha(a)) = \beta(f(a)) \supseteq \beta(f'(a))$. Hence $g'(b) = \cup\{\beta(f'(a))|\ \alpha(a) = b)\} \subseteq g(b)$. Moreover, $f'(a) \subseteq U'$, for any $a \in A'$ and β preserve the inclusion and so $\beta(f'(a)) \subseteq \beta(U')$, for any $a \in A'$. Hence $g'(b) \subseteq U'$. Therefore, $(g', \alpha(A'))_{\beta(U')}$ is a soft subset of $(g, B)_V$. □

Definition 2.8. Let $(f, A)_U$ and $(g, B)_V$ be two soft sets. If $\alpha : A \to B$ and $\beta : U \to V$ are two maps such that $\beta \circ f = g \circ \alpha$, then (α, β) is called *normal soft morphism* or briefly normal morphism from $(f, A)_U$ to $(g, B)_V$ and we write $(\alpha, \beta) : (f, A)_U \to (g, B)_V$, where $\beta(X) = \{\beta(x)|\ x \in X\}$, for all $X \in P(U)$.

Proposition 2.12. *The class of all soft sets and all normal morphisms forms a subcategory of* $\mathfrak{SS}$. *It is denoted by* $\mathfrak{NSS}$.

Proof. Similar to the proof of Theorem 2.1. □

Theorem 2.13. $\mathfrak{NSS}$ *has arbitrary coporoducts.*

Proof. Let $\{(f_i, A_i)_{U_i}\}_{i\in I}$ be a family of soft sets and $(g, C)_{U'}$ be the soft set defined in the proof of Theorem 2.8. Define $\alpha_i : A_i \to C$, by $\alpha(x) = (x, i)$ and $\beta_i : U_i \to U'$, by $\beta_i(y) = (y, i)$, for any $x \in A_i$, $y \in U'$ and $i \in I$, where B_i is a set defined in the proof of Theorem 2.8. Then clearly, $(\alpha_i, \beta_i) : (f_i, A_i)_{U_i} \to (g, C)_{U'}$ is a normal soft morphism. Now, let $(h, X)_V$ be a soft set and $(\lambda_i, \mu_i) : (f_i, A_i)_{U_i} \to (h, X)_V$ be a normal soft morphism, for any $i \in I$. Then we define two maps $\varphi : C \to X$ and $\chi : U' \to V$ by

$$\begin{aligned} \varphi((x, i)) &= \lambda_i(x), \quad \text{for all } (x, i) \in C,\ i \in I \\ \chi((y, i)) &= \mu_i(x), \quad \text{for all } (y, i) \in U',\ i \in I \end{aligned}$$

Let $(x, j) \in C$, for some $j \in I$. Then

$$\begin{aligned} h(\varphi((x, j))) = h(\lambda_j(x)) &= \mu_j(f_j(x)), \quad (\lambda_j, \mu_j) : (f_j, A_j)_{U_j} \to (h, X)_V \text{ is a morphism} \\ &= \{\mu_j(u)|\ u \in f_j(x)\} \\ &= \{\chi(x, j)|\ u \in f_j(x)\}, \quad \text{by definition of } \chi \\ &= \chi(f_j(x) \times \{j\}) \\ &= \chi(g_j((x, j)) \times \{j\}) = \chi(g((x, j))). \end{aligned}$$

Hence $h \circ \varphi = \chi \circ g$ and so $(\varphi, \chi) : (g, C)_{U'} \to (h, X)_V$ is a normal soft morphism. Moreover,

$$(\varphi, \chi)\circ(\alpha_i, \beta_i)(a, Y) = (\varphi(\alpha_i(a)), \chi(\beta_i(Y))) = (\varphi((a, i)), \chi(Y\times\{i\})) = (\lambda_i(a), \mu_i(Y)),$$

for any $a \in A_i$, $Y \subseteq U_i$ and $i \in I$. Therefore, $(\varphi, \chi) \circ (\alpha_i, \beta_i) = (\lambda_i, \mu_i)$, for any $i \in I$. Now, let $(\varphi', \chi') : (g, C)_{U'} \to (h, X)_V$ be a normal soft morphism, such that $(\varphi', \chi') \circ (\alpha_i, \beta_i) = (\lambda_i, \mu_i)$, for any $i \in I$. Similar to the proof of Theorem 2.8, we can show that $\varphi = \varphi'$. Let $y \in U'$. Then there are $i \in I$ and $x \in U_i$ such that $y = (x, i)$.

$$\chi(y) = \chi(x, i) = \chi(\beta_i(x)) = \mu_i(x) = \chi'(\beta_i(x)) = \chi'(x, i) = \chi'(y).$$

Hence $\chi = \chi'$ and so $(\varphi, \chi) = (\varphi', \chi')$. Therefore, $(g, C)_{U'}$ is a coproduct of the family $\{(f_i, A_i)_{U_i}\}_{i\in I}$ in $\mathfrak{NSS}$. □

In the next section we want to verify a subcategory of $\mathfrak{SS}$, which its morphism are normal.

3. Some results on soft BCK/BCI-morphisms

In this section, we define the concept of soft BCK/BCI-morphism and give some results about soft subalgebras, soft ideals and soft BCK/BCI-morphisms.
From now on, in this paper, U and V will denote two BCK/BCI-algebras, unless otherwise stated.

Definition 3.1. Let $(f, A)_U$ and $(g, B)_V$ be two soft BCK/BCI-algebras. If $\alpha : A \to B$ and $\beta : U \to V$ are two maps such that β is a BCK/BCI-homomorphism and $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ is a normal soft morphism, then (α, β) is called a s*oft BCK/BCI-morphism.*

Example 3.1. Let U, V be two BCK-algebras and $\alpha : U \to V$ be a BCK/BCI-homomorphism. If $f(u) = \{0, u\}$ and $g(v) = \{0, v\}$, then $(f, U)_U$ and $(g, V)_V$ are soft BCK/BCI-algebras and $(\alpha, \beta) : (f, U)_U \to (g, V)_V$ is a soft BCK/BCI-morphism, where $\beta : P(U) \to P(V)$, was defined by $\beta(X) = \{\alpha(x) | \; x \in X\}$, for any $X \in P(U)$. Therefore, for any BCI-homomorphism we can obtain a soft BCK/BCI-morphism.

Example 3.2. Let $A = X = \{0, a, b, c, d, e, f, g\}$ and consider the following table:

Table 1

$*$	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Then $(X, *, 0)$ is a BCI-algebra. Let $(f, A)_X$ be a soft set, where $f : A \to P(X)$ is a map defined by $f(a) = \{0\} \cup \{b \in A | \; o(a) = o(b)\}$, for all $a \in A$ (see [5, Example 4.4]). Let $\beta : X \to X$ be a map defined by $\beta(x) = 0 * x$, for all $x \in X$. Then by Proposition 1.1(i) and (ii), β is a BCK/BCI-homomorphism and $f(Id_A(a)) = \beta(f(a))$ (note that, by [16, Proposition 1.3.9], $o(x) = o(0 * x)$, for all $x \in X$). Therefore, $(Id_A, \beta) : (f, A)_X \to (f, A)_X$ is a soft BCK/BCI-morphism.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ and $Y = \{0, 1\}$ and consider the following tables:

Table 2

$*_1$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	1	0

Table 3

$*_2$	0	1
0	0	0
1	0	1

Then $(X, *_1, 0)$ and $(Y, *_2, 0)$ are two BCK-algebras (see [16, Appendix A]). Define a map $\alpha : X \to Y$, by $\alpha(0) = \alpha(1) = 0$ and $\alpha(2) = \alpha(3) = 1$. It is easy to check that α is a BCK/BCI-homomorphism. Let $f : X \to P(X)$ and $g : Y \to P(Y)$ be two maps defined by $f(x) = \{u \in X | \; u * x = 0\}$ and $g(y) = \{u \in Y | \; u * y = 0\}$, for all $x \in X$ and $y \in Y$. Routine calculations show that $(f, X)_X$ and $(g, Y)_Y$ are soft BCK/BCI-algebras. Moreover, $g(\alpha(1)) = g(\alpha(0)) = \{0\} = \alpha(f(0)) = \alpha(f(1))$ and $g(\alpha(2)) = g(\alpha(3)) = \{0, 1\} = \alpha(f(2)) = \alpha(f(3))$. Therefore, $(\alpha, \alpha) : (f, X)_X \to (g, Y)_Y$ is a soft BCK/BCI-morphism.

Theorem 3.1. *The class of all soft BCK/BCI-algebras with the class of all soft BCK/BCI-morphisms form a subcategory of $\mathfrak{SS}$, which is called* category of soft BCK/BCI-algebras.

Proof. The proof is straightforward. In fact, if $(f, A)_U$, $(g, B)_V$ and $(h, C)_W$ are three soft BCK/BCI-algebras and $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ and $(\gamma, \lambda) : (g, B)_V \to (h, C)_W$ are soft BCK/BCI-morphisms, then by Theorem 2.1,

$$(\gamma, \lambda) \circ (\alpha, \beta) = (\gamma \circ \alpha, \lambda \circ \beta) \in Hom((f, A)_U, (h, C)_W).$$

Since $\lambda \circ \beta$ is a BCK/BCI-homomorphism, then $(\gamma \circ \alpha, \lambda \circ \beta)$ is a soft BCK/BCI-morphism. Moreover, $(Id_A, Id_{P(U)}) : (f, A)_U \to (f, A)_U$ is a soft BCK/BCI-morphism. □

Proposition 3.2. *Let $(f, A)_U$ and $(g, B)_V$ be two soft BCK/BCI-algebras and $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be a soft BCK/BCI-morphism.*

(i) *$(\beta^{-1} \circ \beta \circ f, A)_U$ is a soft BCK/BCI-algebra and $(f, A)_U$ is a soft subalgebra of $(\beta^{-1} \circ \beta \circ f, A)_U$.*

(ii) *If $(f', A')_U$ is a soft subalgebra of $(f, A)_U$, then $(g', B')_V$ is a soft subalgebra of $(g, B)_V$, where $B' = \alpha(A)$ and $g'(\alpha(a)) = \beta(f'(a))$, for all $a \in A$. $(g', B')_V$ is called the image of $(f', A')_U$ with respect to (α, β). Moreover, if β is an onto map and $(f', A')_U$ is a soft ideal of $(f, A)_U$, then the $(g', B')_V$ is a soft ideal of $(g, B)_V$.*

(iii) *If β is an one to one map and $(g', B')_V$ is a soft subalgebra of $(g, B)_V$, then $(f', A')_U$ is a soft subalgebra of $(f, A)_U$, where $A' = \alpha^{-1}(B')$ and $f'(x) = \beta^{-1}(g'(\alpha(a)))$, for all $x \in A'$.*

(iv) *If β is an one to one map and $(g', B')_V$ is a soft ideal of $(g, B)_V$, then $(f', A')_U$ is a soft ideal of $(f, A)_V$, where $A' = \alpha^{-1}(B')$ and $f'(x) = \beta^{-1}(g(\alpha(a)))$, for all $x \in A'$.*

Proof. (i) It is suffices to show that $\beta^{-1}(\beta(f(x)))$ is a BCK/BCI subalgebra of U, for all $x \in A$. Let $x \in A$. Since $(f, A)_U$ is a soft BCK/BCI-algebra, then $f(x)$ is a BCK/BCI-algebra of U and so by Proposition 1.2(i) and (ii), $\beta^{-1}(\beta(f(x)))$ is a BCK/BCI subalgebra of U. Clearly, $f(x) \subseteq \beta^{-1}(\beta(f(x))) \subseteq U$ and $f(x)$ is a BCK/BCI-subalgebra of U, for all $x \in U$. Hence $(f, A)_U$ is a soft subalgebra of $(\beta^{-1} \circ \beta \circ f, A)_U$.
(ii) By (α, β) is a soft BCK/BCI-morphism, it follows that $g \circ \alpha = \beta \circ f$, so

$$g'(\alpha(a)) = \beta(f'(a)) \subseteq \beta(f(a)) = g(\alpha(a)).$$

Since $f'(a)$ is a subalgebra of $f(a)$, then by Proposition 1.2(i), we have $\beta(f'(a))$ is a subalgebra of $\beta(f(a))$ and so $(g', B')_V$ is a soft subalgebra of $(g, B)_V$. Now, let β be an onto map, $(f', A')_U$ be a soft ideal of $(f, A)_U$ and $y \in B'$. Then there is $x \in A'$ such that $y = \alpha(x)$. By the first step of the proof, $g'(y) \subseteq g(y)$. Since $(f', A')_U$ is a soft ideal of $(f, A)_U$, then $f'(x)$ is an ideal of $f(x)$ and so by Proposition 1.2(iv), $\beta(f'(x))$ is an ideal of $\beta(f(x)) = g(\alpha(x)) = g(y)$ and so $g'(y)$ is an ideal of $g(y)$. Therefore, $(g', B')_V$ is a soft ideal of $(g, B)_V$.
(iii) Let $x \in A'$. Then $\alpha(x) \in B'$. Since $(g', B')_V$ is a soft subalgebra of $(g, B)_V$, then $g'(\alpha(x))$ is a soft subalgebra of $g(\alpha(x)) = \beta(f(x))$. Hence by Proposition 1.2(ii),

$\beta^{-1}(g'(\alpha(x)))$ is a subalgebra of U. Moreover, by $g'(\alpha(x)) \subseteq g(\alpha(x)) = \beta(f(x))$, we conclude that $f'(x) = \beta^{-1}(g'(\alpha(x))) \subseteq \beta^{-1}(\beta(f(x)))$. Since β is an one to one map, we get $f'(x) \subseteq f(x)$, whence $(f', A')_U$ is a soft subalgebra of $(f, A)_U$.
(iv) Similar to the proof of (ii), we can show that $f'(x) \subseteq f(x)$, for all $x \in A' = \alpha^{-1}(B')$. Let $x \in A'$. Then $\alpha(x) \in B'$ and so $g'(\alpha(x))$ is an ideal of $g(\alpha(x))$. Since $(g, B)_V$ is a soft BCK/BCI-algebra, then by Proposition 1.2(iii), we conclude that $\beta^{-1}(g'(\alpha(x)))$ is a ideal of $\beta^{-1}(g(\alpha(x))) = \beta^{-1}(\beta(f(x))) = f(x)$. Therefore, $(f', A')_U$ is a soft ideal of $(f, A)_U$. □

Proposition 3.3. *Let $(f, A)_U$ and $(g, B)_V$ be two soft sets and $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be a normal soft morphism such that β is a BCK/BCI-algebra homomorphism.*

(i) *If $(g, B)_V$ is an idealistic soft BCK/BCI-algebra over V, then $(\beta^{-1}\circ\beta\circ f, \alpha^{-1}(B))_U$ is an idealistic soft BCK/BCI-algebra over U. Moreover, if β is one to one, then $(f, \alpha^{-1}(B))_U$ is an idealistic soft BCK/BCI-algebra over U.*
(ii) *If β is onto and $(f, A)_U$ is an idealistic soft BCK/BCI-algebra over U, then $(g, \alpha(A))_V$ is an idealistic soft BCK/BCI-algebra over V.*

Proof. (i) Let $(g, B)_V$ is an idealistic soft BCK/BCI-algebra over V and $a \in \alpha^{-1}(B)$. Then $\alpha(a) \in B$ and so $\beta(f(a)) = g(\alpha(a))$ is an ideal of V. Since β is a BCK/BCI-homomorphism, then by Proposition 1.2(iii), $\beta^{-1}(\beta(f(a)))$ is an ideal of U. Hence $(\beta^{-1} \circ \beta \circ f, \alpha^{-1}(B))_U$ is an idealistic soft BCK/BCI-algebra over U. Now, let β be one to one map. Then $\beta^{-1} \circ \beta \circ f = f$. Therefore, $(f, \alpha^{-1}(B))_U$ is an idealistic soft BCK/BCI-algebra over U.
(ii) Let β is onto, $(f, A)_U$ is an idealistic soft BCK/BCI-algebra over U and $b \in \alpha(A)$. Then there exists $a \in A$ such that $b = \alpha(a)$. Since $f(a)$ is an ideal of U and β is a BCK/BCI-homomorphism, then by Proposition 1.2(iv), $\beta(f(a))$ is an ideal of V and so $g(\alpha(a)) = g(b)$ is an ideal of V. □

Let θ be an equivalence relation on a set X. Then we use $[x]_\theta$ to denote $\{u \in X|\ u\theta x\}$. Moreover, if A is a subset of X, the $[A]_\theta = \{[a]_\theta|\ a \in A\}$.

Theorem 3.4. *Let $(\alpha, \beta) : (f, A)_U \to (g, B)_V$ be a soft BCK/BCI-morphism, $\theta = \{(x, y) \in A \times A|\ \alpha(x) = \alpha(y)\}$ and ϕ be a congruence relation induced by $\ker(\beta)$. Then the following hold:*

(i) *θ is an equivalence relation on A, ϕ is a congruence relation on U and $(\overline{f}, \overline{A})_{\overline{U}}$ is a soft BCK/BCI-algebra, where $\overline{A} = \{[a]_\theta|\ a \in A\}$, $\overline{U} = \{[u]_\phi|\ u \in U\}$ and $\overline{f}([a]_\theta) = [f(a)]_\phi$, for all $a \in A$.*
(ii) *if $\overline{\alpha} : \overline{A} \to B$ and $\overline{B} : \overline{U} \to V$ are two maps defined by $\overline{\alpha}([x]) = \alpha(x)$ and $\overline{\beta}([u]_\phi) = \beta(u)$, for all $x \in A$ and $u \in U$, then $(\overline{\alpha}, \overline{\beta}) : (\overline{f}, \overline{A})_{\overline{U}} \to (g, B)_V$ is a soft BCK/BCI-monic.*

Proof. Clearly, θ is an equivalence relation on A and ϕ is a congruence relation on U. Let $[a] = [b]$, for some $a, b \in A$. Then $\alpha(a) = \alpha(b)$ and so $\beta(f(a)) = g(\alpha(a)) = g(\alpha(b)) = \beta(f(b))$. Hence $[f(a)]_\phi = [f(b)]_\phi$ whence $\overline{f}$ is well defined. By Remark 1.1, $U/\ker(\beta)$ is a BCI-algebra. Let $a \in A$. Since $f(a)$ is a subalgebra of U, then $0 \in f(a)$ and so $[0]_\phi \in [f(a)]_\phi = \overline{f}([a]_\theta)$. Now, let $[x]_\phi, [y]_\phi \in \overline{f}([a]_\theta)$. Then there exist $s, t \in f(a)$ such that $(x, s) \in \phi$ and $(y, t) \in \phi$. Since $f(a)$ is a subalgebra of U and ϕ is a congruence relation on U, then $s * t \in f(a)$ and $(x * y, s * t) \in \phi$ and so $[x]_\phi * [y]_\phi = [x * y]_\phi \in [f(a)]_\phi = \overline{f}([a]_\theta)$. Therefore, $\overline{f}([a]_\theta)$ is a subalgebra of $\overline{U}$ and $(\overline{f}, \overline{A})_{\overline{U}}$ is a soft BCK/BCI-algebra.

(ii) Clearly, α is well defined. Since

$$(x, y) \in U \Leftrightarrow x*y, y*x \in \beta^{-1}(0) \Leftrightarrow \beta(x*y) = \beta(y*x) = 0 \Leftrightarrow \beta(x)*\beta(y) = 0 = \beta(y)*\beta(x)$$

then by $BCI3$, we get $\beta(x) = \beta(y)$. Hence $U = \{(x, y) \in U \times U|\ \beta(x) = \beta(y)\}$. Hence $\overline{\beta}$ is well defined. Moreover, $\overline{B}([x]_\phi * [y]_\phi) = \overline{\beta}([x * y]_\phi) = \beta(x * y) = \beta(x) * \beta(y) = \overline{\beta}([x]_\phi) * \overline{\beta}([y]_\phi)$ and so $\overline{\beta}$ is a BCK/BCI-homomorphism. Finally, we show that $g(\overline{\alpha}([a]_\theta)) = \overline{\beta}(\overline{f}([a]_\theta))$, for all $[a]_\theta \in \overline{A}$. Let $[a]_\theta \in \overline{A}$. Then $g(\overline{\alpha}([a]_\theta)) = g(\alpha(a)) = \beta(f(a)) = \overline{\beta}([f(a)]_\phi) = \overline{\beta}(\overline{f}([a]_\theta))$. Hence, $(\overline{\alpha}, \overline{\beta}) : (\overline{f}, \overline{A})_{\overline{U}} \to (g, B)_V$ is a soft BCK/BCI-morphism. Clearly, $\overline{\alpha}$ and $\overline{\beta}$ are one to one maps. Therefore, by Proposition 2.2(ii), $(\overline{\alpha}, \overline{\beta}) : (\overline{f}, \overline{A})_{\overline{U}} \to (g, B)_V$ is a soft BCK/BCI-monic. □

4. Conclusions and future works

In this paper, we introduced category of soft sets and presented some properties of this category, such as existence of product, terminal object and coproduct. Then we verified category of soft BCK/BCI-algebras, and obtained some theorems about soft BCK/BCI ideals and soft subalgebras. For future research, we can study the category of soft BCK/BCI-algebras, in more details and find limits, colimits and other special elements of this category.

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