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Properties of the Closure Operator in Peano Algebras

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Abstract. A Peano σ-algebra generated by a set M, denoted by $\overline{M}$, is a set of words over the alphabet $M \cup \{\sigma\}$ satisfying some rules. The set M is the support set of $\overline{M}$. The symbol σ is a distinguished symbol to build these words. In this paper we study several properties of the closure operator in Peano algebras. If M_0 is a subset of the support set M then the closure operator f is defined by $f(M_0) = \overline{M_0}$. The main results show that f is a monotone operator under inclusion and satisfies the morphism property under intersection. Several properties concerning the layers of the Peano σ-algebras are also presented.

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1. Introduction

The concept of Peano σ-algebra is a basic one to define two mathematical structures for knowledge representation: the *stratified graphs* and the *semantic schemas*. We mention that these concepts are obtained by incorporating the concept of *labeled graph* into an algebraic environment given by a tuple of components, which are obtained applying several concepts of Peano algebras. These concepts were introduced in [11] and [16] respectively. Various algebraic properties are presented in [12], [13], [14] and [15] for stratified graphs. The concept of semantic schema was implied into various research papers treating several aspects of distributed computing ([17], [21]) and cooperating structures based on maximal paths, maximal graphs and master-slave cooperation [18], [19], [20], [22].

By a *labeled graph* we understand a tuple $G = (S, L_0, T_0, f_0)$, where S is a finite set of nodes, L_0 is a set of elements named *labels*, T_0 is a set of binary relations on S and $f_0 : L_0 \longrightarrow T_0$ is a surjective function.

Let M be an arbitrary nonempty set. We consider the set B given by

$$B = \bigcup_{n \geq 0} B_n \tag{1}$$

where

$$\begin{cases} B_0 = M, \\ B_{n+1} = B_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in B_n \times B_n\}, \ n \geq 0, \end{cases} \tag{2}$$

and $\sigma(x_1, x_2)$ is the word $\sigma x_1 x_2$ over the alphabet $\{\sigma\} \cup M$. The pair $\overline{M} = (B, \sigma)$ is a Peano σ-algebra over M ([1], [2], [3], [7]). Two Peano σ-algebras over the same set M are isomorphic algebras ([2], [5], [6]). For this reason in this paper we use the Peano algebra given by (1). Everywhere in this paper we suppose that the set

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M does not contain words containing the symbol σ. Particularly the set does not contain elements of the form $\sigma(u, v)$.

The Peano σ-algebra $\overline{M}$ contains several disjoint layers: $B_0 = M$ gives the layer of order 0, $B_1 \setminus B_0$ is the layer of order 1 and so on. The set M is the *support set* of the Peano algebra $\overline{M}$. The symbol σ gives the *type* of the Peano algebra $\overline{M}$.

We consider the Peano σ-algebra $\overline{M} = (B, \sigma)$. The main problems treated in this paper can be summarized as follows:

- If $K \subseteq M$ then $\overline{K} \subseteq \overline{M}$.
- The previous property allows to consider the operator $f : (\mathcal{P}(M), \cap) \longrightarrow (\mathcal{P}(\overline{M}), \cap)$ defined by $f(K) = \overline{K}$.
- The operator $f : (\mathcal{P}(M), \subseteq) \longrightarrow (\mathcal{P}(\overline{M}), \subseteq)$ is monotone.
- The operator $f : (\mathcal{P}(M), \cap) \longrightarrow (\mathcal{P}(\overline{M}), \cap)$ is a morphism of universal algebras, i.e. $f(L_{01} \cap L_{02}) = f(L_{01}) \cap f(L_{02})$.

The results presented in this paper can be viewed as independent results from the domain of the universal algebras. But these results are useful to prove several properties to study the closure properties of the family of languages generated by stratified graphs ([9], [10]).

2. The closure operator is monotone with respect to inclusion

In this section we define the closure operator in Peano algebras.

Proposition 2.1. *Suppose that $M_0 \subseteq N_0$. If we consider the sets*

$M_{n+1} = M_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in M_n \times M_n\},\ n \geq 0,$

$N_{n+1} = N_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in N_n \times N_n\},\ n \geq 0,$

then for every $n \geq 0$ we have

$$\begin{cases} M_n \subseteq N_n, \\ M_{n+1} \setminus M_n \subseteq N_{n+1} \setminus N_n. \end{cases} \tag{3}$$

As a consequence $\overline{M_0} \subseteq \overline{N_0}$.

Proof. If $M_0 = \emptyset$ then $M_n = \emptyset$ for every $n \geq 0$. In this case the proposition is true. Suppose $M_0 \neq \emptyset$. We prove (3) by induction on $n \geq 0$.

For $n = 0$ we have to verify the property

$$\begin{cases} M_0 \subseteq N_0, \\ M_1 \setminus M_0 \subseteq N_1 \setminus N_0. \end{cases}$$

Really, we have $M_0 \subseteq N_0$. On the other hand if $\sigma(\alpha, \beta) \in M_1 \setminus M_0$, $\alpha \in \overline{M_0}$, $\beta \in \overline{M_0}$ then $\alpha \in M_0$, $\beta \in M_0$. It follows that $\alpha \in N_0$, $\beta \in N_0$ therefore $\sigma(\alpha, \beta) \in N_1 \setminus N_0$. Thus (3) is true for $n = 0$.

Suppose that (3) is true for some $n \geq 0$ and we prove now that

$$\begin{cases} M_{n+1} \subseteq N_{n+1}, \\ M_{n+2} \setminus M_{n+1} \subseteq N_{n+2} \setminus N_{n+1}. \end{cases} \tag{4}$$

But $M_n \subseteq M_{n+1}$ and $N_n \subseteq N_{n+1}$, therefore we have $M_{n+1} = (M_{n+1} \setminus M_n) \cup M_n$ and $N_{n+1} = (N_{n+1} \setminus N_n) \cup N_n$. We use (3) and obtain $M_{n+1} \subseteq (N_{n+1} \setminus N_n) \cup N_n = N_{n+1}$. It follows that

$$M_{n+1} \subseteq N_{n+1}. \tag{5}$$

This is the first relation of (4).

Now consider $\sigma(\alpha, \beta) \in M_{n+2} \setminus M_{n+1}$. We have the following two cases:

(1) $\alpha \in M_{n+1} \setminus M_n$ and $\beta \in M_{n+1}$.
From (3) we obtain $\alpha \in N_{n+1} \setminus N_n$ and from (5) we obtain $\beta \in N_{n+1}$. Thus $\sigma(\alpha, \beta) \in N_{n+2} \setminus N_{n+1}$.

(2) $\alpha \in M_{n+1}$ and $\beta \in M_{n+1} \setminus M_n$.
From (5) we have $\alpha \in N_{n+1}$ and from (3) we have $\beta \in N_{n+1} \setminus N_n$. It follows that $\sigma(\alpha, \beta) \in N_{n+2} \setminus N_{n+1}$.

Thus the second relation of (4) is proved. □

Corollary 2.1. *Using the same notations as in Proposition 2.1 we have*

$$(M_{n+1} \setminus M_n) \cap \bigcup_{k \geq 0, k \neq n} (N_{k+1} \setminus N_k) = \emptyset. \tag{6}$$

Proof. Immediate from Proposition 2.1. Really,

$$(M_{n+1} \setminus M_n) \cap \bigcup_{k \geq 0, k \neq n} (N_{k+1} \setminus N_k) \subseteq (N_{n+1} \setminus N_n) \cap \bigcup_{k \geq 0, k \neq n} (N_{k+1} \setminus N_k)$$

and $(N_{n+1} \setminus N_n) \cap \bigcup_{k \geq 0, k \neq n} (N_{k+1} \setminus N_k) = \emptyset$ because $(N_{n+1} \setminus N_n) \cap (N_{k+1} \setminus N_k) = \emptyset$ for $k \neq n$. □

Proposition 2.2. *Using the same notations as in Proposition 2.1 we have*

$$(M_{n+1} \setminus M_n) \cap (N_{j+1} \setminus N_j) \neq \emptyset \Longleftrightarrow j = n.$$

Proof. We prove the direct and the converse implication as follows:

- Suppose that $(M_{n+1} \setminus M_n) \cap (N_{j+1} \setminus N_j) \neq \emptyset$.
By contrary, suppose that $j \neq n$. If this is the case then by Proposition 2.1 we obtain $(M_{n+1} \setminus M_n) \cap (N_{j+1} \setminus N_j) \subseteq (N_{n+1} \setminus N_n) \cap (N_{j+1} \setminus N_j) = \emptyset$, therefore $(M_{n+1} \setminus M_n) \cap (N_{j+1} \setminus N_j) = \emptyset$, which is not true because we supposed $(M_{n+1} \setminus M_n) \cap (N_{j+1} \setminus N_j) \neq \emptyset$. It follows that $j = n$.
- Suppose $j = n$. By Proposition 2.1 we have $(M_{n+1} \setminus M_n) \cap (N_{j+1} \setminus N_j) = (M_{n+1} \setminus M_n) \cap (N_{n+1} \setminus N_n) = M_{n+1} \setminus M_n$. But $M_{n+1} \setminus M_n \neq \emptyset$, therefore $(M_{n+1} \setminus M_n) \cap (N_{j+1} \setminus N_j) \neq \emptyset$.

□

Based on Proposition 2.1 we can define the operator

$$\begin{cases} f : \mathcal{P}(M) \longrightarrow \mathcal{P}(\overline{M}), \\ f(K) = \overline{K}. \end{cases} \tag{7}$$

Really, if $k \subseteq M$ then $\overline{K} \subseteq \overline{M}$, therefore $f(K) \in \mathcal{P}(\overline{M})$ for every $K \in \mathcal{P}(M)$.

Definition 2.1. *The operator $f : \mathcal{P}(M) \longrightarrow \mathcal{P}(\overline{M})$ defined by (7) is the **closure operator** of the Peano algebra $\overline{M}$.*

Proposition 2.3. *The closure operator $f : (\mathcal{P}(M), \subseteq) \longrightarrow (\mathcal{P}(\overline{M}), \subseteq)$ is monotone.*

Proof. If $M_0 \subseteq N_0 \subseteq M$, by Proposition 2.1 we obtain $f(M_0) \subseteq f(N_0)$. □

3. The closure operator is a morphism of universal algebras

We consider two label sets L_{01} and L_{02} of two labeled graphs and the Peano σ-algebras generated by L_{01}, L_{02}, $L_{01} \cup L_{02}$ and $L_{01} \cap L_{02}$:

- We denote $\overline{B_0} = \bigcup_{n \geq 0} B_n$, where

$$\begin{cases} B_0 = L_{01}, \\ B_{n+1} = B_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in B_n \times B_n\}, \; n \geq 0 \end{cases}. \tag{8}$$

- We denote $\overline{H_0} = \bigcup_{n\geq 0} H_n$, where

$$\begin{cases} H_0 = L_{02}, \\ H_{n+1} = H_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in H_n \times H_n\},\ n \geq 0. \end{cases} \tag{9}$$

- We denote $\overline{\Gamma_0} = \bigcup_{n\geq 0} \Gamma_n$, where

$$\begin{cases} \Gamma_0 = L_{01} \cup L_{02}, \\ \Gamma_{n+1} = \Gamma_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in \Gamma_n \times \Gamma_n\},\ n \geq 0. \end{cases} \tag{10}$$

- We denote $\overline{K_0} = \bigcup_{n\geq 0} K_n$, where

$$\begin{cases} K_0 = L_{01} \cap L_{02}, \\ K_{n+1} = K_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in K_n \times K_n\},\ n \geq 0. \end{cases} \tag{11}$$

Proposition 3.1. *For every $n \geq 0$ we have*

$$\begin{cases} K_n \subseteq B_n \cap H_n, \\ K_{n+1} \setminus K_n \subseteq (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n). \end{cases} \tag{12}$$

Proof. We apply Proposition 2.1 and obtain:

$$\begin{cases} K_n \subseteq B_n, \\ K_{n+1} \setminus K_n \subseteq B_{n+1} \setminus B_n, \end{cases} \tag{13}$$

and

$$\begin{cases} K_n \subseteq H_n, \\ K_{n+1} \setminus K_n \subseteq H_{n+1} \setminus H_n. \end{cases} \tag{14}$$

Now from (13) and (14) we obtain (12). □

Proposition 3.2. *Suppose that $L_{01} \cap L_{02} \neq \emptyset$. Then*

$$(B_{j+1} \setminus B_j) \cap (H_{n+1} \setminus H_n) \neq \emptyset \Longleftrightarrow j = n.$$

Proof. Suppose that $j = n$. Then $(B_{j+1} \setminus B_j) \cap (H_{n+1} \setminus H_n) = (B_{j+1} \setminus B_j) \cap (H_{j+1} \setminus H_j)$. By Proposition 3.1 we have $(B_{j+1} \setminus B_j) \cap (H_{j+1} \setminus H_j) \supseteq K_{j+1} \setminus K_j$. But $K_{j+1} \setminus K_j \neq \emptyset$ because $L_{01} \cap L_{02} \neq \emptyset$. It follows that $(B_{j+1} \setminus B_j) \cap (H_{n+1} \setminus H_n) \neq \emptyset$. The implication from right to left is proved.
Let us prove the implication from left to right. We suppose that $(B_{j+1} \setminus B_j) \cap (H_{n+1} \setminus H_n) \neq \emptyset$. We suppose by contrary, $j \neq n$. By Proposition 2.1 we have $B_{j+1} \setminus B_j \subseteq \Gamma_{j+1} \setminus \Gamma_j$ and $H_{n+1} \setminus H_n \subseteq \Gamma_{n+1} \setminus \Gamma_n$. It follows that $(B_{j+1} \setminus B_j) \cap (H_{n+1} \setminus H_n) \subseteq (\Gamma_{j+1} \setminus \Gamma_j) \cap (\Gamma_{n+1} \setminus \Gamma_n)$. But $(\Gamma_{j+1} \setminus \Gamma_j) \cap (\Gamma_{n+1} \setminus \Gamma_n) = \emptyset$, therefore $(B_{j+1} \setminus B_j) \cap (H_{n+1} \setminus H_n) = \emptyset$, which is not true. Our assumption that $j \neq n$ is false, therefore $j = n$. □

Proposition 3.3. *Suppose $L_{01} \cap L_{02} \neq \emptyset$. Consider the sequences $\{B_i\}_{i\geq 0}$ and $\{H_i\}_{i\geq 0}$ defined in (8) and (9) respectively. For every $i \geq 0$ we have*

$$(B_{i+1} \setminus B_i) \cap (H_{i+1} \setminus H_i) \subseteq \overline{L_{01} \cap L_{02}}. \tag{15}$$

Proof. We prove (15) by induction on $i \geq 0$.
Let us verify this property for $i = 0$. Take $x \in (B_1 \setminus B_0) \cap (H_1 \setminus H_0)$. From $x \in B_1 \setminus B_0$ we deduce that there are $a_1, b_1 \in B_0$ such that $x = \sigma(a_1, b_1)$. Similarly there are $a_2, b_2 \in H_0$ such that $x = \sigma(a_2, b_2)$ because $x \in H_1 \setminus H_0$. We can say that $x = \sigma(a_1, b_1) = \sigma(a_2, b_2)$, where $a_1, b_1 \in B_0 \cup H_0$ and $a_2, b_2 \in B_0 \cup H_0$. Thus $x \in \overline{B_0 \cup H_0}$. From the properties of the Peano algebras we deduce that $a_1 = a_2$ and $b_1 = b_2$. It follows that $a_1 \in B_0 \cap H_0$ and $b_1 \in B_0 \cap H_0$. Thus $x = \sigma(a_1, b_1) \in \overline{B_0 \cap H_0} = \overline{L_{01} \cap L_{02}}$ and (15) is true for $i = 0$.
Suppose that (15) is true for $i \in \{0, \ldots, n\}$. Take $x \in (B_{n+2} \setminus B_{n+1}) \cap (H_{n+2} \setminus H_{n+1})$.

There are $p, q \in \overline{B_0}$ such that $x = \sigma(p, q)$ because $x \in B_{n+2} \setminus B_{n+1}$. There are $\alpha, \beta \in \overline{H_0}$ such that $x = \sigma(\alpha, \beta)$ because $x \in H_{n+2} \setminus H_{n+1}$. We have $p, q \in \overline{B_0}$, therefore $p, q \in \overline{B_0 \cup H_0}$. Similarly we have $\alpha, \beta \in \overline{B_0 \cup H_0}$. It follows that $\sigma(p, q) \in \overline{B_0 \cup H_0}$ and $\sigma(\alpha, \beta) \in \overline{B_0 \cup H_0}$. Using the same property of the Peano algebras we deduce $p = \alpha$ and $q = \beta$. Thus

$$\sigma(p, q) \in (B_{n+2} \setminus B_{n+1}) \cap (H_{n+2} \setminus H_{n+1}),$$

$$p, q \in \overline{B_0} \cap \overline{H_0}.$$

From $\sigma(p, q) \in (B_{n+2} \setminus B_{n+1})$, $p, q \in \overline{B_0}$, we deduce that the following two cases are possible:

$a1)$ $p \in (B_{n+1} \setminus B_n)$ and $q \in B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)$;

$a2)$ $p \in B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)$ and $q \in (B_{n+1} \setminus B_n)$.

Similarly, from $\sigma(p, q) \in (H_{n+2} \setminus H_{n+1})$, $p, q \in \overline{H_0}$, we deduce that one of the following cases is possible:

$a3)$ $p \in (H_{n+1} \setminus H_n)$ and $q \in H_0 \cup \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)$;

$a4)$ $p \in H_0 \cup \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)$ and $q \in (H_{n+1} \setminus H_n)$.

Combining these cases we obtain:

- Suppose we have $a1)$ and $a3)$.

In this case we have

$p \in (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n)$,

$q \in [B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)] \cap [H_0 \cup \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)]$.

Applying the distributivity of the set operations we obtain

$$\begin{aligned}
&[B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)] \cap [H_0 \cup \bigcup_{j=0}^{n} (H_{j+1} \setminus B_j)] \\
&\quad = (B_0 \cap H_0) \cup (B_0 \cap \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)) \cup (H_0 \cap \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)) \\
&\qquad \cup (\bigcup_{j=0}^{n} (B_{j+1} \setminus B_j) \cap \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)) \\
&\quad = (B_0 \cap H_0) \cup \bigcup_{j=0}^{n} ((B_{j+1} \setminus B_j) \cap (H_{j+1} \setminus H_j)).
\end{aligned}$$

Thus in this case we have

$$p \in (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n), \tag{16}$$

$$q \in (B_0 \cap H_0) \cup \bigcup_{j=0}^{n} [(B_{j+1} \setminus B_j) \cap (H_{j+1} \setminus H_j)]. \tag{17}$$

By the inductive assumption we have $(B_{j+1} \setminus B_j) \cap (H_{j+1} \setminus H_j) \subseteq \overline{L_{01} \cap L_{02}}$ for every $j \in \{0, \ldots, n\}$, therefore (16) and (17) show that

$$p \in \overline{L_{01} \cap L_{02}},$$

$$q \in (L_{01} \cap L_{02}) \cup \overline{L_{01} \cap L_{02}}.$$

We know that $L_{01} \cap L_{02} \subseteq \overline{L_{01} \cap L_{02}}$, therefore in this case we have $p, q \in \overline{L_{01} \cap L_{02}}$. Thus $x = \sigma(p, q) \in \overline{L_{01} \cap L_{02}}$.

- Suppose we have $a1)$ and $a4)$.

If this is the case then

$$p \in (B_{n+1} \setminus B_n) \cap [H_0 \cup \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)], \tag{18}$$

$$q \in [B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)] \cap (H_{n+1} \setminus H_n). \tag{19}$$

But

$$(B_{n+1} \setminus B_n) \cap [H_0 \cup \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)] = [(B_{n+1} \setminus B_n) \cap H_0] \cup [(B_{n+1} \setminus B_n) \cap \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)],$$

$$(B_{n+1} \setminus B_n) \cap H_0 = \emptyset.$$

By Proposition 3.2 we have

$$(B_{n+1} \setminus B_n) \cap (H_{j+1} \setminus H_j) \neq \emptyset \Longleftrightarrow j = n.$$

On the other hand,

$$[B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)] \cap (H_{n+1} \setminus H_n) = [B_0 \cap (H_{n+1} \setminus H_n)] \cup [(H_{n+1} \setminus H_n) \cap \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)],$$

$$(H_{n+1} \setminus H_n) \cap B_0 = \emptyset,$$

and by Proposition 3.2 we have

$$(H_{n+1} \setminus H_n) \cap (B_{j+1} \setminus B_j) \neq \emptyset \Longleftrightarrow j = n.$$

Thus the relations (18) and (19) show that

$$p \in (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n),$$

$$q \in (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n).$$

Now we use the inductive assumption and deduce that $p, q \in \overline{L_{01} \cap L_{02}}$. Thus $x = \sigma(p, q) \in \overline{L_{01} \cap L_{02}}$.

- Suppose we have $a2$) and $a3$).

In this case we have

$$p \in [B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)] \cap (H_{n+1} \setminus H_n),$$

$$q \in (B_{n+1} \setminus B_n) \cap [H_0 \cup \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)].$$

Using the same relations as in the previous cases we obtain again $p \in (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n)$ and $q \in (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n)$ and finally by the inductive assumption we obtain $x = \sigma(p, q) \in \overline{L_{01} \cap L_{02}}$.

- Suppose we have $a2$) and $a4$).

In this case we have

$$p \in [B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)] \cap [H_0 \cup \bigcup_{j=0}^{n} (H_{j+1} \setminus H_j)],$$

$$q \in (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n).$$

We proceed as in the case $a1$) and $a3$). We obtain $x = \sigma(p, q) \in \overline{L_{01} \cap L_{02}}$. □

Proposition 3.4. *If $L_{01} \cap L_{02} = \emptyset$ then for every $n \geq 0$, $r \geq 0$ we have*

$$B_n \cap H_r = \emptyset. \tag{20}$$

Proof. Let's consider first the case $n = r$. We prove by induction on $k \geq 0$ the property

$$B_k \cap H_k = \emptyset. \tag{21}$$

For $k = 0$ the relation (21) is true because $L_{01} \cap L_{02} = \emptyset$. Suppose that (21) is true for $k \in \{0, \ldots, n\}$. We have

$$B_{n+1} \cap H_{n+1} = [(B_{n+1} \setminus B_n) \cup B_n] \cap [(H_{n+1} \setminus H_n) \cup H_n] =$$

$$[(B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n)] \cup [B_n \cap (H_{n+1} \setminus H_n)] \cup [(B_{n+1} \setminus B_n) \cap H_n] \cup (B_n \cap H_n).$$

But $B_n \cap H_n = \emptyset$ by the inductive assumption. We have also $B_n \cap (H_{n+1} \setminus H_n) \subseteq \Gamma_n \cap (\Gamma_{n+1} \setminus \Gamma_n) = \emptyset$ and similarly, $(B_{n+1} \setminus B_n) \cap H_n = \emptyset$. It follows that

$$B_{n+1} \cap H_{n+1} = (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n). \tag{22}$$

Suppose that $n = 0$ in (22). In other words we have

$$B_1 \cap H_1 = (B_1 \setminus B_0) \cap (H_1 \setminus H_0).$$

Suppose that $(B_1 \setminus B_0) \cap (H_1 \setminus H_0) \neq \emptyset$. Take $y \in (B_1 \setminus B_0) \cap (H_1 \setminus H_0)$. There exist $p_1, q_1 \in B_0$ such that $y = \sigma(p_1, q_1)$ and there exist $p_2, q_2 \in H_0$ such that $y = \sigma(p_2, q_2)$. We have $p_1, q_1, p_2, q_2 \in B_0 \cup H_0 \subseteq \overline{B_0 \cup H_0}$. Thus $y = \sigma(p_1, q_1) = \sigma(p_2, q_2) \in \overline{B_0 \cup H_0}$. It follows that $p_1 = p_2$ and $q_1 = q_2$. Thus $p_1 \in B_0 \cap H_0$. But this is not possible because $B_0 \cap H_0 = \emptyset$.
Suppose that $n \geq 1$ in (22). Suppose that there exists $y \in (B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n)$. Because $y \in B_{n+1} \setminus B_n$ we deduce that there are a_1, a_2 such that $y = \sigma(a_1, a_2)$ and the following two cases are possible:

- $a_1 \in B_n \setminus B_{n-1}$, $a_2 \in B_{n-1}$;
- $a_1 \in B_{n-1}$, $a_2 \in B_n \setminus B_{n-1}$.

From $y \in H_{n+1} \setminus H_n$ we deduce that there are b_1, b_2 such that
$b_1 \in H_n \setminus H_{n-1}$, $b_2 \in H_{n-1}$,
$b_1 \in H_{n-1}$, $b_2 \in H_n \setminus H_{n-1}$.
Let us observe that $a_1, a_2, b_1, b_2 \in \overline{B_0 \cup H_0}$ and $\sigma(a_1, a_2) = \sigma(b_1, b_2)$. By a basic property of a Peano algebra we deduce that $a_1 = b_1$ and $a_2 = b_2$.
It follows that we have the following four cases:

- $a_1 \in (B_n \setminus B_{n-1}) \cap (H_n \setminus H_{n-1})$, $a_2 \in B_{n-1} \cap H_{n-1}$; but $B_{n-1} \cap H_{n-1} = \emptyset$.
- $a_1 \in (B_n \setminus B_{n-1}) \cap H_{n-1}$, $a_2 \in B_{n-1} \cap (H_n \setminus H_{n-1})$; but $(B_n \setminus B_{n-1}) \cap H_{n-1} \subseteq B_n \cap H_n = \emptyset$ by the inductive assumption.
- $a_1 \in B_{n-1} \cap (H_n \setminus H_{n-1})$, $a_2 \in (B_n \setminus B_{n-1}) \cap H_{n-1}$; but $B_{n-1} \cap (H_n \setminus H_{n-1}) \subseteq B_{n-1} \cap H_n \subseteq B_n \cap H_n = \emptyset$.
- $a_1 \in B_{n-1} \cap H_{n-1}$, $a_2 \in (B_n \setminus B_{n-1}) \cap (H_n \setminus H_{n-1})$; but $B_{n-1} \cap H_{n-1} = \emptyset$ by the inductive assumption.

It follows that our assumption $(B_{n+1} \setminus B_n) \cap (H_{n+1} \setminus H_n) \neq \emptyset$ is false, therefore $B_{n+1} \cap H_{n+1} = \emptyset$.

For $n \neq r$ we will consider only the case $r < n$ as the proof is similar to the case $n < r$. We have

$$\begin{aligned} B_n \cap H_r &= [B_0 \cup \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)] \cap [H_0 \cup \bigcup_{k=0}^{r} (H_{k+1} \setminus H_k)] \\ &= (B_0 \cap H_0) \cup (B_0 \cap \bigcup_{k=0}^{r} (H_{k+1} \setminus H_k)) \cup (H_0 \cap \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j)) \\ &\quad \cup [\bigcup_{j=0}^{n} (B_{j+1} \setminus B_j) \cap \bigcup_{k=0}^{r} (H_{k+1} \setminus H_k)]. \end{aligned}$$

We obtain $B_0 \cap H_0 = L_{01} \cap L_{02} = \emptyset$, $B_0 \cap \bigcup_{k=0}^{r} (H_{k+1} \setminus H_k) = \emptyset$ and similarly $H_0 \cap \bigcup_{j=0}^{n} (B_{j+1} \setminus B_j) = \emptyset$. So we have

$$B_n \cap H_r = \bigcup_{j=0}^{n} \bigcup_{k=0}^{r} [(B_{j+1} \setminus B_j) \cap (H_{k+1} \setminus H_k)].$$

For every $j \neq k$ we have $(B_{j+1} \setminus B_j) \cap (H_{k+1} \setminus H_k) \subseteq (\Gamma_{j+1} \setminus \Gamma_j) \cap (\Gamma_{k+1} \setminus \Gamma_k) = \emptyset$. So we are left with the case $j = k$. But $r < n$ therefore we obtain

$$B_n \cap H_r = \bigcup_{k=0}^{r} [(B_{k+1} \setminus B_k) \cap (H_{k+1} \setminus H_k)].$$

And this case is now similar to the previous case when $n = r$. Thus the proposition is proved. □

The concept of morphism of universal algebra ([4], [8]) is used by the next proposition.

Proposition 3.5. *The diagram from Figure 1 is commutative. In other words, the*

$$\begin{tikzcd} \mathcal{P}(M) \times \mathcal{P}(M) \arrow[r, "\cap"] \arrow[d, "f \times f"'] & \mathcal{P}(M) \arrow[d, "f"] \\ \mathcal{P}(\overline{M}) \times \mathcal{P}(\overline{M}) \arrow[r, "\cap"'] & \mathcal{P}(\overline{M}) \end{tikzcd}$$

FIGURE 1. The operator f is a morphism.

closure operator is a morphism of universal algebras:

$$\overline{L_{01} \cap L_{02}} = \overline{L_{01}} \cap \overline{L_{02}}. \tag{23}$$

Proof. Suppose $L_{01} \cap L_{02} \neq \emptyset$. By Proposition 2.1 we have $\overline{L_{01} \cap L_{02}} \subseteq \overline{L_{01}}$ and $\overline{L_{01} \cap L_{02}} \subseteq \overline{L_{02}}$, therefore

$$\overline{L_{01} \cap L_{02}} \subseteq \overline{L_{01}} \cap \overline{L_{02}}. \tag{24}$$

In order to prove the reverse implication we observe that

$$\begin{aligned}\overline{L_{01}} \cap \overline{L_{02}} &= [B_0 \cup \bigcup_{i=0}^{n}(B_{i+1} \setminus B_i)] \cap [H_0 \cup \bigcup_{j=0}^{n}(H_{j+1} \setminus H_j)] \\ &= (B_0 \cap H_0) \cup \bigcup_{j\geq 0}[B_0 \cap (H_{j+1} \setminus H_j)] \cup \bigcup_{i\geq 0}[H_0 \cap (B_{i+1} \setminus B_i)] \\ &\quad \cup \bigcup_{i,j\geq 0}[(B_{i+1} \setminus B_i) \cap (H_{j+1} \setminus H_j)].\end{aligned}$$

But $B_0 \cap (H_{j+1} \setminus H_j) = \emptyset$ for every $j \geq 0$, $H_0 \cap (B_{i+1} \setminus B_i) = \emptyset$ for every $i \geq 0$ and $(B_{i+1} \setminus B_i) \cap (H_{j+1} \setminus H_j) \neq \emptyset$ if and only if $i = j$ by Proposition 3.2. It follows that

$$\overline{L_{01}} \cap \overline{L_{02}} = (B_0 \cap H_0) \cup \bigcup_{i\geq 0}[(B_{i+1} \setminus B_i) \cap (H_{i+1} \setminus H_i)].$$

If we use Proposition 3.3 and the inclusion $B_0 \cap H_0 \subseteq \overline{L_{01} \cap L_{02}}$ then we obtain

$$\overline{L_{01}} \cap \overline{L_{02}} \subseteq \overline{L_{01} \cap L_{02}}. \tag{25}$$

From (24) and (25) we obtain $\overline{L_{01} \cap L_{02}} = \overline{L_{01}} \cap \overline{L_{02}}$.
Now suppose that $L_{01} \cap L_{02} = \emptyset$. We have $\overline{L_{01}} = \bigcup_{n\geq 0} B_n$ and $\overline{L_{02}} = \bigcup_{k\geq 0} H_k$. We have $\overline{L_{01}} \cap \overline{L_{02}} = \bigcup_{n\geq 0,k\geq 0}(B_n \cap H_k) = \emptyset$ by Proposition 3.4. It follows that (23) is also true in this case. □

Proposition 3.6. *If* $L_{01} \cap L_{02} = \emptyset$ *then* $(H_{r+1} \setminus H_r) \cap (B_{n+1} \setminus B_n) = \emptyset$ *for every* $r \geq 0$ *and* $n \geq 0$.

Proof. Suppose that $L_{01} \cap L_{02} = \emptyset$. Let us consider $r \geq 0$ and $n \geq 0$. We have $(H_{r+1} \setminus H_r) \cap (B_{n+1} \setminus B_n) \subseteq \overline{H_0} \cap \overline{B_0} = \overline{B_0 \cap H_0} = \overline{\emptyset} = \emptyset$. It follows that $(H_{r+1} \setminus H_r) \cap (B_{n+1} \setminus B_n) = \emptyset$. □

Remark 3.1. *From Proposition 3.6 we see that Proposition 3.3 is true also if* $L_{01} \cap L_{02} = \emptyset$.

4. Conclusions and future work

In this paper we studied several properties of the closure operator in Peano algebras. The main results include the monotony under inclusion and the morphism property with respect to intersection. We intend to study the morphism property under union operation. We know that the stratified graphs can generate formal languages. We intend to apply these results to prove the closure properties of this family of languages ([9], [10]).

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