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Fundamental BCC-algebras

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Abstract. In this paper, we consider the notions of BCC-algebras and hyper BCC-algebras, give some related results, introduce the relation β on hyper BCC-algebras and let β^* be the transitive closure of β. Then by considering the concept of strongly regular equivalence relation (fundamental relation) β^* on hyper BCC-algebras, we define the notion of fundamental BCC-algebra and we prove that any countable BCC-algebra is a fundamental BCC-algebra.

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Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [12]. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups and then many researchers have been worked on this new field of modern algebra and developed it. The study of BCK-algebras was initiated by Y. Imai and K. Iseki [7] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. Since a great deal of literature has been produced on the theory of BCK-algebras. In [10] Borzooei, et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebras which is a generalization of a BCK-algebra and investigated some related properties. They introduced the notions of hyper BCK-ideals and weak hyper BCK-ideals and gave relations between theorem. Y.B. Jun et al, [9] gave a condition for a hyper BCK-algebra to be a BCK-algebra and introduced the notion of strong hyper BCK-ideal and reflexive hyper BCK-ideal. In connection with this problem Komori introduced in [11] a notion of BCC-algebra which is a generalization of a BCK-algebra and proved that the class of all BCC-algebras is not a variety. Dudek [6] followed this theory and has got a lot of related results. BCC-algebras are algebraic models of BCC-logic, implicational logic whose axiom schemes are the principal-type schemes of the combinators B, I, and K, and whose inference rules are modus ponens and modus ponens 2. So, in fact, such algebras ought to have been named BCK-algebras. In this convention, a BCK-algebra is a BCC-algebra satisfying the identity $y \longrightarrow (x \longrightarrow z) = x \longrightarrow (y \longrightarrow z)$. In [2], Borzooei, W. A. dudek and N. Koohestani, have introduced the concept of hyper BCC-algebra as common generalization of BCC-algebras and hyper BCK-algebras. In particular, they have investigated different types of hyper BCC-ideals and have

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described the relationship among them. Now, in this paper, we prove that any BCC-algebra is a fundamental BCC-algebra. But, we show that any finite BCC-algebra is not a fundamental BCC-algebra of itself.

1. Preliminaries

Definition 1.1. [7] Let X be a set with a binary operation "$*$" and a constant "0". Then, $(X, *, 0)$ is called a BCK*-algebra* if it satisfies the following conditions:
(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI-2) $(x * (x * y)) * y = 0$,
(BCI-3) $x * x = 0$,
(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(BCK-5) $0 * x = 0$.
We define a binary relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$. Then, $(X, *, 0)$ is a BCK-algebra if and only if it satisfies the following conditions:
(BCI-1′) $((x * y) * (x * z)) \leq (z * y)$,
(BCI-2′) $(x * (x * y)) \leq y$,
(BCI-3′) $x \leq x$,
(BCI-4′) $x \leq y$ and $y \leq x$ imply $x = y$,
(BCK-5′) $0 \leq x$.

Theorem 1.1. [7] *Let $(X, *, 0)$ be a BCK-algebra. Then we have the following properties:*
(a) $x \leq y$ implies $z * y \leq z * x$,
(b) $x \leq y$ implies $x * z \leq y * z$,
(c) $x \leq y$ and $y \leq z$ imply $x \leq z$,
(d) $(x * y) * z = (x * z) * y$,
(e) $x * y \leq z$ implies $x * z \leq y$,
(f) $(x * z) * (y * z) \leq x * y$,
(g) $x * y \leq x$,
(h) $x * 0 = x$.

Definition 1.2. [6] Let X be a set with a binary operation "$*$" and a constant "0". Then, $(X, *, 0)$ is called a BCC*-algebra* if it satisfies the following conditions:
(C1) $((x * y) * (z * y)) * (x * z) = 0$,
(C2) $x * 0 = x$,
(C3) $x * x = 0$,
(C4) $0 * x = 0$,
(C5) $x * y = 0$ and $y * x = 0$ imply $x = y$.
We define a binary relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$. Then, the BCK-algebra $(X, *, 0)$ satisfies in the following conditions:
(a) $0 \leq x$,
(b) $x \leq x$,
(c) $x * y \leq x$,
(d) $(x * y) * (z * y) \leq x * z$,
(e) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
A BCC-algebra is called *commutative* if for all $x, y \in X$, $x * (x * y) = y * (y * x)$.

Definition 1.3. Let $(X, *, 0)$ and $(X', *', 0')$ be two BCC-algebras. A mapping $f : X \to X'$ is called a *homomorphism* from X into X', if for any $x, y \in X$, $f(x * y) = f(x) *' f(y)$. The homomorphism f, is called an *isomorphism*, if it is onto and one to one.

Definition 1.4. [5] Let H be a nonempty set and $P^*(H)$ be the family of all nonempty subsets of H. Functions $*_{i_H} : H \times H \longrightarrow P^*(H)$, where $i \in \{1, 2, \dots, n\}$ and $n \in \mathbb{N}$, are called *binary hyperoperations*. For all x, y of H, $*_{i_H}(x, y)$ is called the *hyperproduct* of x and y. An algebraic system $(H, *_{1_H}, *_{2_H}, \dots, *_{n_H})$ is called an *n-algebraic hyperstructure* and structure $(H, *_H)$ endowed with only one hyperoperation is called a *hypergroupoid*. For any two nonempty subsets A and B of hypergropoid H and $x \in H$, we define

$$A *_H B = \bigcup_{a\in A, b\in B} a *_H b, \qquad A *_H x = \bigcup_{a\in A} a *_H x \qquad \text{and} \qquad x *_H B = \bigcup_{b\in B} x *_H b$$

Definition 1.5. [2] Let H be a non-empty set, endowed with a binary hyperoperation "$\circ$" and a constant "0". Then, $(H, \circ, 0)$ is called a *hyper BCC-algebra* if satisfies the following axioms:
(HC1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HC2) $0 \circ x = 0$,
(HC3) $x \circ 0 = x$,
(HC4) $x \ll y$ and $y \ll x$ imply $x = y$,
for all x, y, z in H, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

Nontrivial hyper BCC-algebra means that the hyperoperation "$\circ$" is not singleton and hyper BCC-algebra H is called a proper hyper BCC-algebra if H is not a hyper BCK-algebra.

Theorem 1.2. [2] *In any hyper BCC-algebra H, the following hold:*
$(a1)$ $0 \circ 0 = \{0\}$,
$(a2)$ $0 \ll x$,
$(a3)$ $x \ll x$,
$(a4)$ $x \circ y \ll x$,
$(a5)$ $A \circ 0 = A$,
$(a6)$ $0 \circ A = 0$,
for all $x, y, z \in H$ and $A \subseteq H$.

Theorem 1.3. [2] *Let H be a hyper BCC-algebra. Then H is a hyper BCK-algebra if and only if $(x \circ y) \circ z = (x \circ z) \circ y$, for all $x, y, z \in H$.*

A totally ordered set $(X, 0)$ is said to be *well-ordered* (or have a well-founded order) if every nonempty subset of X, has a least element. Every finite totally ordered set is well ordered.

Theorem 1.4. [8] (Zermelo's Well-Ordering Theorem) *Every set can be well-ordered.*

2. Some results on BCC-algebras and weak commutative hyper BCC-algebras

In this section, we get some results that we need in the next section. Specially, we construct a BCC-algebra and a hyper BCC-algebra from a nonempty set.

By Theorem 1.4, any set can be well-ordered. But in Theorem 2.2, for any countable set X, we need to well-ordered set by a new way. Then we define a special binary relation "$\leq$" and zero element $x_0 \in X$, such that $(X, \leq, x_0)$ is well-ordered set.

Theorem 2.1. *Let X and Y be two sets such that $|X| = |Y|$. If $(Y, \leq, 0)$ is a well-ordered set, then there exist a binary order relation "$\leq$" on X and $x_0 \in X$, such that $(X, \leq, x_0)$, is a well-ordered set.*

Proof. Since $|X| = |Y|$, then there exists a bijection $\psi : Y \to X$ and so for any $x, y \in X$, there exist $x', y' \in Y$ such that $x = \psi(x')$ and $y = \psi(y')$. Now, for any $x, y \in X$, we define the relation " $\leq$ " on X as follows:

$$x \leq y \Longleftrightarrow x' \leq y' \tag{2.1}$$

Then, by (2.1), for any $x, y \in X$, we have

$$\psi(x) \leq \psi(y) \Longleftrightarrow x \leq y \tag{2.2}$$

Hence $(X, \leq, x_0)$ is a well-ordered set, where $x_0 = \psi(0)$. □

Theorem 2.2. *Every countable set can be well-ordered.*

Proof. Let X be a countable set. Then, $|X| = |\mathbb{W}|$, where $\mathbb{W} = \{0, 1, 2, 3, ...\}$ or there exists $k \in \mathbb{N}$ such that $|X| = |\mathbb{N}_k|$. If X is finite then clearly there exists a binary relation " $\leq$ " on X and $x_0 \in X$, such that $(X, \leq, x_0)$ is a well-ordered set. Now, let $|X| = |\mathbb{W}|$. Since, $(\mathbb{W}, \leq, 0)$ is a well-ordered set, then by Theorem 2.1, there exist a bijection $\psi : \mathbb{W} \to X$, binary order relation " $\leq$ " on X and $x_0 \in X$ such that $x_0 = \psi(0)$ and $(X, \leq, x_0)$ is a well-ordered set. □

Lemma 2.3. *Let $(X, \leq, 0)$ be a totally ordered set. Then there exists a binary operation " $*$ " on X such that $(X, *, 0)$ is a BCC-algebra.*

Proof. For any $x, y \in X$, we define the binary operation " $*$ " on X, as follows:

$$x * y = \begin{cases} 0 & \text{, if } x \leq y \\ x & \text{, otherwise} \end{cases}$$

Then by some modification, we can prove that $(X, *, 0)$ is a BCC-algebra. □

Corollary 2.4. *Every countable set can be a BCC-algebra.*

Proof. Let X be a nonempty countable set. By Theorem 2.2, there exists a special order relation " $\leq$ " and zero element $x_0 \in X$ such that $(X, \leq, x_0)$ is a well-ordered set and by Lemma 2.3, there exists the binary relation " $*$ " on $(X, \leq, x_0)$ such that $(X, *, x_0)$ is a BCC-algebra. □

Theorem 2.5. *Let X be an infinite countable set. Then there exist $x_0 \in X$ and a binary operation "$*$" on X and $\mathbb{W}$, such that $(X, *, x_0)$ and $(\mathbb{W}, *, 0)$ are BCC-algebras and $(X, *, x_0) \cong (\mathbb{W}, *, 0)$.*

Proof. Since X is an infinite countable set, then $|X| = |\mathbb{W}|$. Now, by Theorem 2.2, there exist $x_0 \in X$, a bijection $\psi : \mathbb{W} \to X$ and an order relation " $\leq$ " on X, such that $(X, \leq, x_0)$ is a well-ordered set and by Lemma 2.3, there exists a binary operation " $*$" on X and $\mathbb{W}$ such that $(X, *, x_0)$ and $(\mathbb{W}, *, 0)$ are BCC-algebras. Now, we show that $(X, *, x_0)$ and $(\mathbb{W}, *, x_0)$ are isomorphic BCC-algebras. Let $\varphi : \mathbb{W} \to X$ be defined by $\varphi(n) = \psi(n)$, for any $n \in \mathbb{W}$. It is easy to see that φ is a bijection and $\varphi(0) = x_0$. Now, we show that φ is a homomorphism. If $m = 0$ or $n = 0$, then it is easy to check that φ is a homomorphism. Now, let $m \neq 0, n \neq 0 \in \mathbb{W}$. If $m > n$, then by (3.2), $\psi(m) > \psi(n)$ and so

$$\varphi(m * n) = \varphi(m) = \psi(m) = \psi(m) * \psi(n) = \varphi(m) * \varphi(n)$$

Now, let $m \leq n$. Then by (3.2), $\psi(m) \leq \psi(n)$ and so

$$\varphi(m * n) = \varphi(0) = \psi(0) = x_0 = \psi(m) * \psi(n) = \varphi(m) * \varphi(n)$$

Therefore, φ is a homomorphism and so it is an isomorphism. □

Theorem 2.6. *Every set can be a commutative BCC-algebra.*

Proof. Let X be a nonempty set and $x_0 \in X$. For any $x, y \in X$, we define the binary operation " $*$ " on X as follows:

$$x * y = \begin{cases} x_0 & \text{, if } x = y \\ x & \text{, otherwise} \end{cases}$$

Then by some modification, it is easy to see that $(X, *, x_0)$ is a BCC-algebra. □

Definition 2.1. Let $(X, \circ, 0)$ be a hyper BCC-algebra. Then, $(X, \circ, 0)$ is called a weak commutative hyper BCC-algebra, if for any $x, y \in X$, $(x\circ(x\circ y)) \bigcap (y\circ(y\circ x)) \neq \emptyset$.

Example 2.1. Let $X = \{0, 1, 2\}$. Then, $(X, \circ)$ is a hyper BCC-algebra which is defined as follows:

∘	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	c	{0, c}

We have $a \circ (a \circ c) = a$, while $c \circ (c \circ a) = \{0, c\}$. Then, $a \circ (a \circ c) \neq c \circ (c \circ a)$ and $a \circ (a \circ c) \bigcap c \circ (c \circ a) = \emptyset$. Therefore, if $(X, \circ)$ is a hyper BCC-algebra, it necessarily dose not hold that $x \circ (x \circ y) \bigcap y \circ (y \circ x) \neq \emptyset$.

Theorem 2.7. *Every nonempty set can be a weak commutative hyper BCC-algebra.*

Proof. Let X be a nonempty set and $x_0 \in X$. For any $x, y \in X$, we define the binary hyperoperation " $\circ$ " on X as follows:

$$x \circ y = \begin{cases} \{x_0, x\} & \text{, if } x = y \\ x & \text{, otherwise} \end{cases}$$

Moreover, for any $x, y \in X$, we define $x \ll y$ by $x_0 \in x \circ y$. Clearly for any $x \in X$, $x_0 \in x \circ x$, and $x_0 \in x_0 \circ x$. Then, $x \ll x$ and $x_0 \ll x$. Now, we show that $(X, \circ, x_0)$ is a hyper BCC-algebra.

(HC1): Let $x, y, z \in X$. We consider the following cases:
Case 1: $x = y \neq z$. Then, $(x \circ z) \circ (y \circ z) = x \circ y = x \circ x = \{x_0, x\} \ll \{x_0, x\} = x \circ y$.
Case 2: $x = z \neq y$. Then, $(x \circ z) \circ (y \circ z) = \{x_0, x\} \circ y = \{x_0, x\} \ll x = x \circ y$.
Case 3: $y = z \neq x$. Then, $(x \circ z) \circ (y \circ z) = x \circ \{x_0, x\} = \{x_0, x\} \ll x = x \circ y$.
Case 4: $x = y = z$. Then, $(x \circ z) \circ (y \circ z) = \{x_0, x\} \ll \{x_0, x\} = x \circ y$.

(HC2): Let $x \in X$. Then $x_0 \circ x = \{x_0\}$.

(HC3): Let $x \in X$. Then $x \circ x_0 = \{x\}$.

(HC4): Let $x, y \in X$. If $x \ll y$ and $y \ll x$, then $x_0 \in x \circ y$ and $x_0 \in y \circ x$ and so $x = y$. Now, we show that it is weak commutative. For $x = y$, the proof is clear. Let $x \neq y$. Then $x \circ (x \circ y) = x \circ x = \{x_0, x\} = y \circ y = y \circ (y \circ x)$.
Therefore, $(X, \circ, x_0)$ is a weak commutative hyper BCC-algebra. □

Theorem 2.8. *Let X and Y be two nonempty sets and $|X| = |Y|$. Then for $x_0 \in X$ and $y_0 \in Y$, there exists a binary hyperoperation " $\circ$" on X and Y, such that $(X, \circ, x_0)$ and $(Y, \circ, y_0)$ are two isomorphic weak commutative hyper BCC-algebras.*

Proof. Let $x_0 \in X$ and $y_0 \in Y$. By Theorem 2.7, there exists a binary hyperoperation " $\circ$ " on X and Y, such that $(X, \circ, x_0)$ and $(Y, \circ, y_0)$ are weak commutative hyper BCC-algebras. Now, since $|X| = |Y|$, then there exists a bijection $\psi : X \to Y$. Let

$\varphi : X \to Y$ is defined by $\varphi(x) = \psi(x)$, for any $x \in X$ and $\varphi(x_0) = y_0$. It is easy to see that φ is a bijection. Now, we show that φ is a homomorphism. Let $x, y \in X$. If $x \neq y$, then $\psi(x) \neq \psi(y)$ and so $\psi(x) \circ \psi(y) = \psi(x)$. Now, if $x = y$, then $\psi(x) = \psi(y)$ and $\psi(x) \circ \psi(y) = \{y_0, \psi(x)\}$. Hence,

$$\begin{aligned}\varphi(x \circ y) = \varphi(x \circ x) &= \{\varphi(x_0), \varphi(x)\} = \{y_0, \varphi(x)\} \\ &= \{y_0, \psi(x)\} = \psi(x) \circ \psi(x) \\ &= \psi(x) \circ \psi(y) = \varphi(x) \circ \varphi(y)\end{aligned}$$

Therefore, φ is a homomorphism and so it is an isomorphism. □

Theorem 2.9. *Let $(A, *_A, 0_A)$ and $(B, *_B, 0_B)$ be two BCC-algebras. Then there exists a hyperoperation " $\circ$ " on $A \times B$, such that $(A \times B, \circ, (0_A, 0_B))$ is a hyper BCC-algebra.*

Proof. Let $(A, *_A, 0_A)$ and $(B, *_B, 0_B)$ be two BCC-algebras. For any $(x_1, y_1), (x_2, y_2) \in A \times B$, we define the binary hyperoperation " $\circ$ " on $A \times B$ as follows:

$$(x_1, y_1) \circ (x_2, y_2) = \{(x_1 *_A x_2, y_1), (x_1 *_A x_2, y_1 *_B y_2)\}$$

and for $(x, y), (z, w) \in A \times B$, we define

$$(x, y) \ll (z, w) \Longleftrightarrow (0_A, 0_B) \in (x, y) \circ (z, w)$$

First, we show that the hyperoperation " $\circ$ " is well defined. Let $(x_1, y_1) = (x'_1, y'_1)$ and $(x_2, y_2) = (x'_2, y'_2)$. Then,

$$\begin{aligned}(x_1, y_1) \circ (x_2, y_2) &= \{(x_1 *_A x_2, y_1), (x_1 *_A x_2, y_1 *_B y_2)\} \\ &= \{(x'_1 *_A x'_2, y'_1), (x'_1 *_A x'_2, y'_1 *_B y'_2)\} \\ &= (x'_1, y'_1) \circ (x'_2, y'_2) \qquad (2.3)\end{aligned}$$

Moreover, we show that for any $(x, y), (z, w) \in A \times B$,

$$(x, y) \ll (z, w) \text{ if and only if } x \leq z \text{ and } y \leq w \qquad (2.4)$$

For this, let $(x, y) \ll (z, w)$. Then by the hypotheses, $(0_A, 0_B) \in (x, y) \circ (z, w) = \{(x *_A z, y), (x *_A z, y *_B w)\}$, and so $(0_A, 0_B) = (x *_A z, y)$ or $(0_A, 0_B) = (x *_A z, y *_B w)$. If $(0_A, 0_B) = (x *_A z, y)$, then $x \leq z$ and $y = 0_B \leq w$. If $(0_A, 0_B) = (x *_A z, y *_B w)$, then $x \leq z$ and $y \leq w = 0$. Therefore, for any cases, we have, $x \leq z$ and $y \leq w$. Conversely, let $x \leq z$ and $y \leq w$. Then $x *_A z = 0_A$ and $y *_B w = 0_B$. Hence $(0_A, 0_B) \in (x, y) \circ (z, w)$. Therefore, $(x, y) \ll (z, w)$.
Now, we show that $(A \times B, \circ, (0_A, 0_B))$ is a hyper BCC-algebra.

<u>(HC1):</u> Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A \times B$. Since, $(A, *_A)$ and $(B, *_B)$ are BCC-algebras, then by (2.4),

$$\begin{aligned}&((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_2, y_2)) \\ &\quad = \{(x_1 *_A x_2, y_1), (x_1 *_A x_2, y_1 *_B y_2)\} \circ \{(x_3 *_A x_2, y_3), (x_3 *_A x_2, y_3 *_B y_2)\} \\ &\quad = \{((x_1 *_A x_2) *_A (x_3 *_A x_2), y_1), ((x_1 *_A x_2) *_A (x_3 *_A x_2), y_1 *_B y_3), \\ &\qquad ((x_1 *_A x_2) *_A (x_3 *_A x_2), y_1 *_B (y_3 *_B y_2)), \\ &((x_1 *_A x_2) *_A (x_3 *_A x_2), y_1 *_B y_2), ((x_1 *_A x_2) *_A (x_3 *_A x_2), (y_1 *_B y_2) *_B (y_3 *_B y_2))\} \\ &\quad \ll \{(x_1 *_A x_3, y_1), (x_1 *_A x_3, y_1 *_B y_3)\} \\ &\quad = (x_1, y_1) \circ (x_3, y_3)\end{aligned}$$

<u>(HC2):</u> Let $(x, y) \in A \times B$. Then,

$$(0, 0) \circ (x, y) = \{(0 * 0, 0), (0 * 0, 0 * y)\} = \{(0, 0)\}$$

(HC3): Let $(x,y) \in A \times B$. Then,

$$(x,y) \circ (0,0) = \{(x*0,y),(x*0,y*0)\} = \{(x,y)\}$$

(HC4): Let $(x,y),(z,w) \in A \times B$. If $(x,y) \ll (z,w)$ and $(z,w) \ll (x,y)$, then by (2.4), we have $x \leq z$, $y \leq w$ and $z \leq x, w \leq y$. Then $x = z$ and $w = y$ and so $(x,y) = (z,w)$.

Therefore, $(A \times B, \circ, (0_A, 0_B))$ is a hyper BCC-algebra. □

Theorem 2.10. *Let $(A, *_A, 0_A)$ and $(B, *_B, 0_B)$ be two BCC-algebras such that $(A, *_A, 0_A)$ is commutative. Then there exists a hyperoperation " $\circ$ " on $A \times B$, such that $(A \times B, \circ, (0_A, 0_B))$ is a weak commutative hyper BCC-algebra.*

Proof. Since, $(A, *_A, 0_A)$ and $(B, *_B, 0_B)$ are two BCC-algebras and $(A, *_A, 0_A)$ is commutative, then for any $a, a' \in A$ we have $a*(a*a') = a'*(a'*a)$. By Theorem 2.9, $(A \times B, \circ, (0_A, 0_B))$ is a hyper BCC-algebra. Now, we show that $(A \times B, \circ, (0_A, 0_B))$ is a weak commutative hyper BCC-algebra. For any $(x_1, y_1), (x_2, y_2) \in A \times B$, since $(A, *_A, 0_A)$ is commutative, then

$$\begin{aligned}(x_1,y_1) \circ ((x_1,y_1) \circ (x_2,y_2)) &= (x_1,y_1) \circ \{(x_1 *_A x_2, y_1), (x_1 *_A x_2, y_1 *_B y_2)\} \\ &= \{(x_1 *_A (x_1 *_A x_2), y_1)), (x_1 *_A (x_1 *_A x_2), 0_B) \\ &, \quad (x_1 *_A (x_1 *_A x_2), y_1 *_A (y_1 *_A y_2))\}.\end{aligned}$$

Moreover,

$$\begin{aligned}(x_2,y_2) \circ ((x_2,y_2) \circ (x_1,y_1)) &= (x_2,y_2) \circ \{(x_2 *_A x_1, y_2), (x_2 *_A x_1, y_2 *_B y_1)\} \\ &= \{(x_2 *_A (x_2 *_A x_1), y_2)), (x_2 *_A (x_2 *_A x_1), 0_B) \\ &, \quad (x_2 *_A (x_2 *_A x_1), y_2 *_B (y_2 *_B y_1))\}.\end{aligned}$$

Then,

$$(x_1,y_1) \circ ((x_1,y_1) \circ (x_2,y_2)) \cap (x_2,y_2) \circ ((x_2,y_2) \circ (x_1,y_1)) \neq \emptyset.$$

Therefore, $(A \times B, \circ, (0_A, 0_B))$ is a weak commutative hyper BCC-algebra. □

3. Fundamental BCC-algebras

In this section, we define the notion of fundamental relation (strongly regular equivalence relation) on (weak commutative) hyper BCC-algebras, we define the concept of fundamental BCC-algebra and we prove that any countable BCC-algebra is a fundamental BCC-algebra.

Definition 3.1. Let $(X, \circ)$ be a hyper BCC-algebra and R be an equivalence relation on X. If A and B are nonempty subsets of X, then
(i) $A\overline{R}B$ means that for all $a \in A$, there exists $b \in B$ such that aRb and for all $b' \in B$, there exists $a' \in A$ such that $b'Ra'$.
(ii) $A\overline{\overline{R}}B$ means that for all $a \in A$, and $b \in B$, we have aRb.
(iii) R is called regular on the right (on the left) if for all x of X, from aRb, it follows that $(a \circ x)\overline{R}(b \circ x)$ $((x \circ a)\overline{R}(x \circ b)$ respectively).
(iv) R is called strongly regular on the right (on the left) if for all x of X, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively).
(v) R is called regular (strongly regular) if it is regular (strongly regular) on the right and on the left. (vi) R is called good, if $(a \circ b)R0$ and $(b \circ a)R0$ imply aRb, for all $a, b \in X$.

Let $(X, \circ)$ be a hyper BCC-algebra and A a subset of X. Then we let $\mathcal{L}(A)$, denote the set of all finite combinations of elements A with $\circ$. Now, in the following, the well-known idea of β^* relation on hyperstructure [5, 15, 16] is transferred and applied to hyper BCC-algebras.

Definition 3.2. Let $(X, \circ)$ be a hyper BCC-algebra. Then we set:

$$\beta_1 = \{(x, x) \mid x \in X\}$$

and, for every integer $n \geq 1$, β_n is the relation defined as follows:

$$x\beta_n y \Longleftrightarrow \exists (a_1, a_2, \ldots, a_n) \in X^n,\ \exists u \in \mathcal{L}(a_1, a_2, \ldots, a_n) \text{ such that } \{x, y\} \subseteq u.$$

Obviously, for every $n \geq 1$, the relations β_n are symmetric and the relation $\beta = \bigcup\limits_{n\geq 1} \beta_n$ is reflexive and symmetric. Let β^* be the *transitive closure* of β. Then in the following theorem we show that β^* is a strongly regular relation.

Theorem 3.1. *Let $(X, \circ)$ be a hyper BCC-algebra. Then β^* is a strongly regular relation on X.*

Proof. Let $x, y \in X$ and $x\beta^* y$. Then there exist $a_0, a_1, \ldots, a_n \in X$ such that $a_0 = x, a_n = y$ and there exist $\beta_{q_1}, \beta_{q_2}, \ldots, \beta_{q_n} \in \mathbb{N}$ such that $x = a_0\beta_{q_1}a_1\beta_{q_2}a_2 \ldots a_{n-2}\beta_{q_{n-1}}$ $a_{n-1}\beta_{q_n}a_n = y$, where $n \in \mathbb{N}$. Since for any $1 \leq i \leq n$, $a_{i-1}\beta_{q_i}a_i$, then there exist $z_t^j \in X$ such that $\{a_i, a_{i+1}\} \subseteq \prod\limits_{t=1}^{q_{i+1}} z_t^{i+1}$, where for $1 \leq m \leq n-1$, we have $1 \leq t \leq q_m$ and $1 \leq j \leq n-1$. Now, let $s \in X$. Then for all $0 \leq i \leq n-1$, $a_i \circ s \subseteq \prod\limits_{t=1}^{q_{i+1}} z_t^{i+1} \circ s = z_1^{i+1} \circ z_2^{i+1} \circ \ldots \circ z_{q_{i+1}}^{i+1} \circ s$ and similarly $a_{i+1} \circ s \subseteq \prod\limits_{t=1}^{q_{i+1}} z_t^{i+1} \circ s = z_1^{i+1} \circ z_2^{i+1} \circ \ldots \circ z_{q_{i+1}}^{i+1} \circ s$. Then for all $0 \leq i \leq n$ and for all $u \in a_i \circ s$, $v \in a_{i+1} \circ s$ we have $u\beta_{q_{i+1}}v$, and so for all $z \in a_0 \circ s = x \circ s$, $w \in a_n \circ s = y \circ s$ we have $z\beta^* w$. Then β^* is a right strongly regular and similarly is a left strongly regular relation. Therefore, β^* is a strongly regular relation. $\square$

Theorem 3.2. *Let $(X, \circ)$ be a weak commutative hyper BCC-algebra. Then, β^* is a good strongly regular relation on X.*

Proof. Since, $(X, \circ)$ is a hyper BCC-algebra, then by Theorem 3.1, β^* is a strongly regular relation on X. Now, we show that β^* is good. Let $0\beta^*(a \circ b)$ and $0\beta^*(b \circ a)$. Since, β^* is strongly regular relation, then $(a \circ 0)\overline{\overline{\beta^*}}(a \circ (a \circ b))$ and similarly $(b \circ 0)\overline{\overline{\beta^*}}(b \circ (b \circ a))$. Since $(X, \circ)$ is a weak commutative hyper BCC-algebra, then there exists $t \in (a \circ (a \circ b)) \bigcap (b \circ (b \circ a))$. Now, by Theorem 1.2, we have, $a \in (a \circ 0)\overline{\overline{\beta^*}}(a \circ (a \circ b)), b \in (b \circ 0)\overline{\overline{\beta^*}}(b \circ (a \circ b))$ and there exists, $t \in (a \circ (a \circ b)) \bigcap (b \circ (b \circ a))$. Now, since β^* is strongly regular, then by Definition 3.1, $a\beta^* t\beta^* b$ and since β^* is transitive, then $a\beta^* b$. Therefore, β^* is a good strongly regular relation on X. $\square$

Now, we define the product "$\overline{\circ}$" on $\frac{X}{\beta^*}$ in the usual manner:

$$\beta^*(x)\overline{\circ}\beta^*(y) = \{\beta^*(z) \mid z \in x\circ y\}$$

for all $x, y \in X$.

Theorem 3.3. *Let $(X, \circ)$ be a weak commutative hyper BCC-algebra. Then, $(\frac{X}{\beta^*}, \overline{\circ})$ is a BCC-algebra.*

Proof. By Theorem 3.1, β^* is a strongly regular equivalence relation. Then for any $x, y \in X, \beta^*(x)\bar{\circ}\beta^*(y)$ is singleton and so,

$$\beta^*(x)\bar{\circ}\beta^*(y) = \beta^*(z)$$

for all $z \in \beta^*(x)\circ\beta^*(y)$. First, we show that $(\frac{X}{\beta^*}, \bar{\circ})$ satisfies in the conditions (C-1), (C-2), (C-3) and (C-4).

(C-1): Let $x, y, z \in X$. Then $((\beta^*(x)\bar{\circ}\beta^*(y)))\bar{\circ}((\beta^*(x)\bar{\circ}\beta^*(z))) = \beta^*(r)\bar{\circ}\beta^*(s) = \beta^*(t)$, for all $t \in r \circ s$, $r \in x \circ y$ and $s \in x \circ z$. Then for all $t \in r \circ s \subseteq ((x \circ y) \circ (x \circ z))$, we have $((\beta^*(x)\bar{\circ}\beta^*(y)))\bar{\circ}((\beta^*(x)\bar{\circ}\beta^*(z))) = \beta^*(t)$. Since $(X, \circ)$ is a hyper BCC-algebra, then by (H_1), $(x \circ y) \circ (x \circ z) \ll z \circ y$, and so for all $t \in (x \circ y) \circ (x \circ z)$, there exist $m \in z \circ y$ such that $t \ll m$, and so $0 \in t \circ m$. Now, since $t \in r \circ s$, $r \in x \circ y$, $s \in x \circ z$, $m \in z \circ y$ and $0 \in t \circ m$, then by the definition, we have,

$$\beta^*(0) = \beta^*(t)\bar{\circ}\beta^*(m) = (((\beta^*(x)\bar{\circ}\beta^*(y)))\bar{\circ}((\beta^*(x)\bar{\circ}\beta^*(z)))\bar{\circ}(\beta^*(z)\bar{\circ}\beta^*(y)).$$

Therefore,

$$(((\beta^*(x)\bar{\circ}\beta^*(y)))\bar{\circ}((\beta^*(x)\bar{\circ}\beta^*(z)))\bar{\circ}(\beta^*(z)\bar{\circ}\beta^*(y)) = \beta^*(0).$$

(C-2): Let $x \in X$. Then $\beta^*(x)\bar{\circ}\beta^*(0) = \beta^*(t)$, for all $t \in x \circ 0$. By Definition 1.5, $x \circ 0 = \{x\}$, then $\beta^*(x)\bar{\circ}\beta^*(0) = \beta^*(x)$.

(C-3): Let $x \in X$. Then, $\beta^*(x)\bar{\circ}\beta^*(x) = \beta^*(t)$, for all $t \in x \circ x$. By Theorem 1.2, $0 \in x \circ x$, then $\beta^*(t) = \beta^*(0)$ and so $\beta^*(x)\bar{\circ}\beta^*(x) = \beta^*(0)$.

(C-4): Let $x \in X$. Then, $\beta^*(0)\bar{\circ}\beta^*(x) = \beta^*(t)$, for all $t \in 0 \circ x$. By Theorem 1.2, $0 \in 0 \circ x$, then $\beta^*(t) = \beta^*(0)$ and so $\beta^*(0)\bar{\circ}\beta^*(x) = \beta^*(0)$.

Secondly, we use the concept of weak commutative hyper BCC-algebra and prove the property (C-5).

(C-5): Let $x, y \in X$. If $\beta^*(x)\bar{\circ}\beta^*(y) = \beta^*(0)$ and $\beta^*(y)\bar{\circ}\beta^*(x) = \beta^*(0)$, then respectively we have $(x\circ y)\beta^*0$ and $(y\circ x)\beta^*0$. Now, since $(X, \circ)$ is a weak commutative hyper BCC-algebra, by Theorem 3.2, β^* is good, then $\beta^*(x) = \beta^*(y)$. Therefore, $(\frac{X}{\beta^*}, \bar{\circ})$ is a BCC-algebra. □

Remark 3.1. We consider the hyper BCC-algebra $(X, \circ)$ that is defined in Example 2.1. Clearly $\beta^*(0) = \{0, c\}$, $\beta^*(a) = \{a\}$, $\beta^*(b) = \{b\}$ and for any $x \in X, \beta^*(0)\bar{\circ}\beta^*(x) = \beta^*(0)$. Now, $\beta^*(a)\bar{\circ}\beta^*(c) = \beta^*(0)$ and $\beta^*(c)\bar{\circ}\beta^*(a) = \beta^*(0)$, but $\beta^*(a) \neq \beta^*(c)$.

If $(X, \circ, 0)$ is a hyper BCC-algebra, then for the strongly regular β^*, structure $\frac{X}{\beta^*}$ necessarily does not satisfy in the condition (C-5). For this, we need if $\beta^*(x)\bar{\circ}\beta^*(y) = \beta^*(0)$ and $\beta^*(y)\bar{\circ}\beta^*(x) = \beta^*(0)$, then $0\beta^*(x \circ y)$ and $0\beta^*(y \circ x)$, and so $\beta^*(x) = \beta^*(y)$. Therefore, must be for any $x, y \in X, (x \circ (x \circ y)) \bigcap (y \circ (y \circ x)) \neq \emptyset$.

Theorem 3.4. *Let $(X, \circ)$ be a weak commutative hyper BCC-algebra. Then β^* is the smallest strongly regular equivalence relation on X, such that $\frac{X}{\beta^*}$ is a BCC-algebra.*

Proof. By Theorem 3.1, β^* is a strongly regular equivalence relation on X and so $\frac{X}{\beta^*}$ is a BCC-algebra. Now, we show that it is the smallest. Let R be a strongly regular equivalence relation on X, such that $\frac{X}{R}$ is a BCC-algebra. By induction, we prove that for all $n \in \mathbb{N}, \beta_n \subseteq R$. Since R is an equivalence relation, then $\beta_1 \subseteq R$. Let for all $k < n, \beta_k \subseteq R$, then we show that $\beta_n \subseteq R$. For $x, y \in X$, if $x\beta_n y$, then there exist $a_1, a_2, \ldots, a_n \in X$, such that $\{x, y\} \subseteq \prod_{i=1}^{n} a_i = \prod_{i=1}^{k-1} a_i \circ \prod_{i=k+1}^{n} a_i$. Then there exist

$\{u, u'\} \subseteq \prod_{i=1}^{k-1} a_i$ and $\{v, v'\} \subseteq \prod_{i=k+1}^{n} a_i$, such that $x \in u \circ v$ and $y \in u' \circ v'$. Moreover, $u\beta_k u'$ and $v\beta_{k+1}v'$. Now, since $k < n$ and $n - k < n$, then by induction hypotheses, uRu' and vRv'. But R is strongly regular then $u \circ v\overline{\overline{R}}u' \circ v'$, and then xRy. Hence, for all $n \geq 1$, $\beta_n \subseteq R$ and so $\beta = \bigcup_{n\geq 1} \beta_n \subseteq R$. By Theorem 3.1, $\beta^* = \overline{\beta}$ and since R is transitive, then $\beta^* = \overline{\beta} \subseteq \overline{R} = R$. Therefore, β^* is the smallest strongly regular equivalence relation on X, such that $\frac{X}{\beta^*}$ is a BCC-algebra. □

Lemma 3.5. *Let $(A, \circ_A)$ and $(B, \circ_B)$ be two hyper BCC-algebras. Then for any $(a, c) \in A^2$, $(b, d) \in B^2$, $(a, b)\beta^*_{A\times B}(c, d)$ if and only if $a\beta^*_A c$ and $b\beta^*_B d$.*

Proof. We know that,

$$\mathcal{L}(A) = \{u \mid u = x_1 \circ_A x_2 \circ_A \ldots \circ_A x_n, (x_1, x_2, \ldots, x_n) \in A^n, n \in \mathbb{N}\} \tag{3.1}$$

Now, let $(a, c) \in A^2$ and $(b, d) \in B^2$. Then $a\beta^*_A c$ and $b\beta^*_B d$ if and only if there exist $u \in \mathcal{L}(A)$ and $v \in \mathcal{L}(B)$ such that $\{a, c\} \subseteq u$ and $\{b, d\} \subseteq v$, if and only if $\{(a, b), (c, d)\} \subseteq u \times v$ if and only if $(a, c)\beta^*_{A\times B}(b, d)$. Then the proof is complete. □

Lemma 3.6. *Let $(A, \circ_A, 0_A, \ll_A)$ and $(B, \circ_B, 0_B, \ll_A)$ be two hyper BCC-algebras. Then there exists a binary hyperoperation "$\circ$" on $A\times B$ such that $(A\times B, \circ_{A\times B}, (0_A, 0_B), \ll)$ is a hyper BCC-algebra.*

Proof. For two hyper BCC-algebras $(A, \circ_A, 0_A, \ll_A)$ and $(B, \circ_B, 0_B, \ll_A)$, we define the hyperoperation " $\circ$ " as follows:

$$(a_1, b_1) \circ (a'_1, b'_1) = \{(a, b) \mid a \in a_1 \circ_A a'_1, b \in b_1 \circ_B b'_1\}$$

for any $(a_1, a'_1) \in A^2$ and $(b_1, b'_1) \in B^2$. First we show that is well-defined. For any $((a_1, b_1), (a_2, b_2)) \in A \times B$, have $((a_1, b_1), (a_2, b_2)) = ((a'_1, b'_1), (a'_2, b'_2))$ if and only if $a_1 = a'_1, a_2 = a'_2$ and $b_1 = b'_1, b_2 = b'_2$, if and only if $\{(a, b) \mid a \in a_1 \circ_A a'_1,\ b \in b_1 \circ_B b'_1\} = \{(a, b) \mid a \in a_2 \circ_A a'_2,\ b \in b_2 \circ_B b'_2\}$ if and only if $(a_1, b_1) \circ (a'_1, b'_1) = (a_2, b_2) \circ (a'_2, b'_2)$.

Now, we define, $(a, b) \ll (c, d)$ if and only if $(0_A, 0_B) \in (a, b) \circ_{A\times B} (c, d)$. Clearly $(a, b) \ll (c, d)$ if and only if $0_A \in a \circ_A c$ and $0_B \in b \circ_B d$ if and only if $a \ll_A c$ and $b \ll_B d$, for all $(a, b), (c, d) \in A \times B$. Then,

$$(a, b) \ll (c, d) \Longleftrightarrow a \ll_A c \text{ and } b \ll_B d \tag{3.2}$$

Now, since $(A, \circ_A)$ and $(B, \circ_B)$ are two hyper BCC-algebras, then by (3.2), clearly $(A \times B, \circ_{A\times B}, (0_A, 0_B), \ll)$ is a hyper BCC-algebra. □

Theorem 3.7. *Let $(A, \circ_A)$ and $(B, \circ_B)$ be two hyper BCC-algebras. Then,*

$$\frac{(A \times B, \circ_{A\times B})}{\beta^*_{A\times B}} \cong \frac{(A, \circ_A)}{\beta^*_A} \times \frac{(B, \circ_B)}{\beta^*_B}$$

.

Proof. Let

$$\varphi : (\frac{A \times B}{\beta^*_{A\times B}}, \overline{\circ}) \longrightarrow (\frac{A}{\beta^*_A} \times \frac{B}{\beta^*_B}, \overline{\circ})$$

be defined by $\varphi(\beta^*_{A\times B}(a, b)) = (\beta^*_A(a), \beta^*_B(b))$. First, we show that φ is well defined and one to one. By Lemma 3.5, $\beta^*_{A\times B}(a_1, b_1) = \beta^*_{A\times B}(a_2, b_2)$ if and only if $\beta^*_A(a_1) =$

$\beta^*_A(a_2)$ and $\beta^*_B(b_1) = \beta^*_B(b_2)$ if and only if $\varphi(\beta^*_{A\times B}(a_1, b_1)) = \varphi(\beta^*_{A\times B}(a_2, b_2))$. Now, we show that φ is a homomorphism. By Lemma 3.6,

$$\begin{aligned}
\varphi(\beta^*_{A\times B}(a_1, b_1)\overline{\circ}(\beta^*_{A\times B}(a_2, b_2)) &= \varphi(\beta^*_{A\times B}(c, d)) \\
&= (\beta^*_A(c), \beta^*_B(d)) \\
&= (\beta^*_A(a_1), \beta^*_B(b_1))\overline{\circ}(\beta^*_A(a_2), \beta^*_B(b_2)) \\
&= \varphi(\beta^*_{A\times B}(a_1, b_1))\overline{\circ}\varphi(\beta^*_{A\times B}(a_2, b_2)) \quad (3.3)
\end{aligned}$$

for any $c \in \beta^*_A(a_1)\overline{\circ}\beta^*_A(a_2)$ and $d \in \beta^*_B(b_1)\overline{\circ}\beta^*_B(b_2)$. Clearly φ is a bijection. Therefore, φ is an isomorphism. □

Corollary 3.8. *Let $(X_i, \circ_i)$ be a hyper BCC-algebra and β^*_i be a fundamental relation on X_i, for any $i = 1, 2, \dots, n$. Then,*

$$\frac{X_1 \times X_2 \times \ldots \times X_n}{\beta^*_{X_1\times X_2\times\ldots\times X_n}} \cong \frac{X_1}{\beta^*_1} \times \frac{X_2}{\beta^*_2} \times \ldots \times \frac{X_n}{\beta^*_n}.$$

Proof. By Theorem 3.7, the proof is clear. □

Theorem 3.9. *Let X and Y be two nonempty sets and $|X| = |Y|$. Then there exists a binary hyperoperation " $\circ$ " on X and Y, such that $(\frac{(X,\circ)}{\beta^*}, \overline{\circ}) \cong (\frac{(Y,\circ)}{\beta^*}, \overline{\circ})$.*

Proof. By Theorem 2.8, for $x_0 \in X$ and $y_0 \in Y$, there exist a binary hyperoperation " $\circ$ " and an isomorphism $f : (X, \circ, x_0) \longrightarrow (Y, \circ, y_0)$, such that $f(x_0) = y_0$. Now, we show that $\beta^*(x_0) = X$ and for any $x_0 \neq x \in X$, $\beta^*(x) = \beta^*(x_0)$. Let $u = \prod_{i=1}^{n} a_i \in \mathcal{L}(X)$. If $a_1 = a_2$, then $u = \{x_0, a_1\} \bigcup \prod_{i=3}^{n} a_i$. In this case, since for any $x \in X$, $\{x_0, x\} \circ x = \{x_0, x\}$, then $u = \{x_0, a_1\}$. Since a_1 is arbitrary then for any $x \in X, \beta^*(x) = \beta^*(x_0)$. Now, if $a_1 \neq a_2$, then $u = \prod_{i=1}^{n} a_i = a_1$ and so $|u| = 1$. Hence in X, $\beta^*(x_0) = X$ and for any $x, y \in X$, we have $\beta^*(x) = \beta^*(y)$. By the similar way, for weak commutative hyper BCC-algebra $(Y, \circ, y_0)$, we have $\beta^*(y_0) = Y$ and for any $x, y \in Y$, we have $\beta^*(x) = \beta^*(y)$. Now, we define $\varphi : (\frac{(X,\circ,x_0)}{\beta^*}, \overline{\circ}) \to (\frac{(Y,\circ,y_0)}{\beta^*}, \overline{\circ})$ by $\varphi(\beta^*(x)) = \beta^*(f(x))$. First, we show that φ is well-defined and one to one. Let $x, y \in X$. Since for any $x, y, \in X, \beta^*(x) = \beta^*(y)$, and for any $x', y' \in Y, \beta^*(x') = \beta^*(y')$, then $\varphi(\beta^*(x)) = \varphi(\beta^*(y))$ if and only if $\beta^*(f(x)) = \beta^*(f(y))$. Therefore, φ is well-defined and one to one. Since f is onto then φ is onto. Now, we show that φ is a homomorphism. Let $x, y \in X$. Since for any $x, y, \in X, \beta^*(x) = \beta^*(y)$ and for any $x', y' \in Y, \beta^*(x') = \beta^*(y')$, then

$$\begin{aligned}
\varphi(\beta^*(x)\overline{\circ}\beta^*(y)) &= \varphi(\beta^*(x)\overline{\circ}\beta^*(x)) \\
&= \varphi(\beta^*(x_0)) = \beta^*(f(x_0)) = \beta^*(y_0) \\
&= \beta^*(f(x)\circ f(x)) = \beta^*(f(x))\overline{\circ}\beta^*(f(x)) \\
&= \beta^*(f(x))\overline{\circ}\beta^*(f(y)) = \varphi(\beta^*(x))\overline{\circ}\varphi(\beta^*(y)) \quad (3.4)
\end{aligned}$$

Hence, φ is a homomorphism and so it is an isomorphism. Therefore, $(\frac{(X,\circ)}{\beta^*}, \overline{\circ}) \cong (\frac{(Y,\circ)}{\beta^*}, \overline{\circ})$. □

Definition 3.3. A BCC-algebra $(X, *)$, is called a *fundamental BCC-algebra*, if there exists a nontrivial weak commutative hyper BCC-algebra $(H, \circ)$, such that $(\frac{(H,\circ)}{\beta^*}, \overline{\circ}) \cong (X, *)$.

Theorem 3.10. *Every commutative BCC-algebra is a fundamental BCC-algebra.*

Proof. Let $(A, *_A, 0_A)$ be a commutative BCC-algebra. Then by Theorem 2.10, for any BCC-algebra $(B, *_B, 0_B)$, $(A \times B, \circ, (0_A, 0_B), \ll)$ is a weak commutative hyper BCC-algebra. First, we show that for any $(a, b) \in A \times B$, $\beta^*(a, b) = \{(a, x) \mid x \in B\}$. For this, let $u = \prod_{i=1}^{n}(a_i, b_i) \in \mathcal{L}(A \times B)$, where $(a_i, b_i) \in A \times B$. We have

$$u = ((\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i)) \bigcup (\prod_{i=1}^{n} a_i, b_i) = \{(\prod_{i=1}^{n} a_i, t) \mid (a_i, t) \in A \times B\}$$

Then for any product u in $\mathcal{L}(A \times B), u = \{(a, b_i) \mid a \in A$ is fixed and $b_i \in B\}$. Hence, for any $(a, b), (c, d) \in A \times B, (a, b)\beta^*(c, d)$ if and only if $a = c$. Now, we define the map

$$\varphi : (\frac{(A \times B, \circ, (0_A, 0_B))}{\beta^*}, \overline{\circ}) \longrightarrow (A, *_A, 0_A)$$

by $\varphi(\beta^*(a, b)) = a$. It is clear that $\beta^*((a, b)) = \beta^*((a', b'))$ if and only if $a = a'$ if and only if $\varphi(\beta^*(a, b)) = \varphi(\beta^*(a', b'))$. Then, φ is well defined and one to one. Now, we show that φ is a homomorphism. For this we have,

$$\varphi(\beta^*(a, b)\overline{\circ}(\beta^*(a', b'))) = \varphi(\beta^*(a *_A a', b)) = a *_A a' = \varphi(\beta^*(a, b)) *_A \varphi(\beta^*(a', b'))$$

Clearly φ is onto. Therefore, φ is a homomorphism and so it is an isomorphism. $\square$

Corollary 3.11. *Every nonempty set can be a fundamental BCC-algebra.*

Proof. By Theorem 2.6, there exists a binary operation " $*$ ", such that $(X, *, 0)$ is a commutative BCC-algebra and so by Theorem 3.10, $(X, *, 0)$ is a fundamental BCC-algebra. $\square$

Theorem 3.12. *Let $(X, *, 0)$ be any finite BCC-algebra. Then for any hyperoperation " $\circ$ " on X, such that $(X, \circ, 0)$ is a weak commutative hyper BCC-algebra, there is not any isomorphic between $(X, *)$ and $(\frac{(X,\circ)}{\beta^*}, \overline{\circ})$, that is $(X, \circ, 0) \ncong (\frac{(X,\circ,0)}{\beta^*}, \overline{\circ})$.*

Proof. Let $(X, *, 0)$ be a finite BCC-algebra, $|X| = n$ and " $\circ$ " be a hyperoperation on X, such that $(X, \circ, 0)$ be a weak commutative hyper BCC-algebra. Then there exist $x, y \in X$ such that $|x \circ y| \geq 2$. Hence, there are $m, n \in x \circ y$ such that $\beta^*(m) = \beta^*(n)$. Since $\frac{X}{\beta^*} = \{\beta^*(x) \mid x \in X\}$, then, $|\frac{X}{\beta^*}| < n = |X|$. Therefore, $(\frac{(X,\circ,0)}{\beta^*}, \overline{\circ}) \ncong (X, *, 0)$. $\square$

Now, in what follows we try to show that for any infinite countable set X, there exist an operation " $*$ " and a hyperoperation " $\circ$ " on X, such that $(X, *)$ is a BCC-algebra and $(X, \circ)$ is a weak commutative hyper BCC-algebra and $\frac{(X,\circ)}{\beta^*} \cong (X, *)$.

Theorem 3.13. *Let X be an infinite countable set. Then there exist an operation " $*$ " and a hyperoperation " $\circ$ " on X, such that $(\frac{(X,\circ,0)}{\beta^*}, \overline{\circ}) \cong (X, *, 0)$. That is X is a fundamental BCC-algebra of itself.*

Proof. By Theorem 2.2, there exist a binary relation " $\leq$ " on X and $x_0 \in X$ such that $(X, \leq, x_0)$ is well-ordered. Moreover, by Theorem 2.5, there exists a binary operation " $*$ " on X and $\mathbb{W}$, such that $(X, *, x_0)$ and $(\mathbb{W}, *, 0)$ are BCC-algebras and $(X, *, x_0) \cong (\mathbb{W}, *, 0)$. Let $\psi : (\mathbb{W}, *, 0) \to (X, *, x_0)$ be an isomorphism. Since "ψ" is a monotone and for any $0 \neq x \in \mathbb{W}, 1 \leq x$, then for any $x \in X$, $\psi(1) \leq \psi(x)$.

Let $a = \psi(1)$, then for any $x_0 \neq x, a \leq x$. Now, for any $x, y \in X$, define a binary hyperoperation " $\circ$ " on X, as follows,

$$x \circ y = \begin{cases} x_0 & , \text{if } x = x_0 \\ \{x_0, a\} & , \text{if } x = y \text{ and } x \neq x_0 \\ x & , \text{otherwise} \end{cases}$$

Clearly " $\circ$ " is well-defined. Now, we define, $x \ll y$ if and only if $x_0 \in x \circ y$. Since, $x_0 \in x \circ y$ if and only if $x = y$ or $x = x_0$, if and only if $x \leq y$, Then for any $x, y \in X$,

$$x \ll y \text{ if and only if } x \leq y \tag{3.5}$$

Hence, for any $x \in X$, $x_0 \ll x$, $x \ll x$. In the follow, we show that $(X, \circ, x_0)$ is a hyper BCC-algebra.

(HC1): Let $x, y, z \in X$. We consider the following cases:
Case 1: $x = x_0$. Then for any $y, z \in X$, $(x \circ z) \circ (y \circ z) = x_0 \circ (y \circ z) = x_0 \ll x_0 = x \circ y$.
Case 2: $x = y \neq z$. Then, $(x \circ z) \circ (y \circ z) = x \circ y = x_0 \ll x_0 = x \circ y$.
Case 3: $x = z \neq y$. Then, $(x \circ z) \circ (y \circ z) = x_0 \circ y = x_0 \ll x = x \circ y$.
Case 4: $y = z \neq x$. Then, $(x \circ z) \circ (y \circ z) = x \circ x_0 = x \ll x = x \circ y$.
Case 5: $x = y = z$. Then, $(x \circ z) \circ (y \circ z) = x_0 \circ x_0 = x_0 \ll x_0 = x \circ y$.

(HC2): Let $x \in X$. Then by hypotheses $x_0 \circ x = \{x_0\}$.

(HC3): Let $x \in X$. Then by hypotheses $x \circ x_0 = \{x\}$.

(HC4): Let $x, y \in X$. If $x \ll y$ and $y \ll x$, then $x \leq y$ and $y \leq x$ and so by, (3.5) $x = y$. Therefore, $(X, \circ, x_0)$ is a hyper BCC-algebra.

Now, we show that $(X, \circ, x_0)$ is a weak commutative hyper BCC-algebra. Let $x, y \in X$. If $x = y$, then clearly $x \circ (x \circ y) = y \circ (y \circ x)$. If $x = x_0$ and $y \neq x$, then $x \circ (x \circ y) = x_0 \circ x_0 = x_0$. Moreover, $y \circ (y \circ x) = y \circ y = x_0$. Then, $(x \circ (x \circ y)) \bigcap (y \circ (y \circ x)) \neq \emptyset$. If $x \neq x_0$, and $y = x$, then $x \circ (x \circ y) = x \circ \{x_0, a\} = x$. Moreover, $y \circ (y \circ x) = y \circ \{x_0, a\} = y$. Then, $(x \circ (x \circ y)) \bigcap (y \circ (y \circ x)) \neq \emptyset$. If $x \neq x_0$, and $y \neq x$, then $x \circ (x \circ y) = x \circ x = \{x_0, a\}$. Moreover, $y \circ (y \circ x) = y \circ y = \{x_0, a\}$. Then, $(x \circ (x \circ y)) \bigcap (y \circ (y \circ x)) \neq \emptyset$. Therefore, $(X, \circ, x_0)$ is a weak commutative hyper BCC-algebra. Clearly, $\beta^*(x_0) = \{x_0, a\}$ and for any $x \notin \beta^*(x_0)$, $\beta^*(x) = \{x\}$. Hence, we define the map $\theta : (\frac{(X, \circ, x_0)}{\beta^*}, \overline{\circ}) \longrightarrow (X \setminus \{a\}, *, x_0)$ by, $\theta(\beta^*(x_0)) = \theta(\beta^*(a)) = x_0$ and for any $x \notin \beta^*(x_0)$, $\theta(\beta^*(x)) = x$. Since, $\theta(\beta^*(x_0)) = \theta(\beta^*(a)) = x_0$ and for any $x \notin \beta^*(x_0), |\beta^*(x_0)| = 1$, then, $\theta(\beta^*(x)) = \theta(\beta^*(y))$ if and only if $x = y$. Hence, θ, is well defined and one to one. Moreover, $\theta(\beta^*(a)) = x_0$ and for any $x \in X \setminus \{a\}$, since $x \neq a$ and $\theta(\beta^*(x)) = x$, then θ is onto. Now, we show that θ is a homomorphism. If $x \in \beta^*(x_0)$, then for any $y \in X$

$$\theta(\beta^*(x) \overline{\circ} \beta^*(y)) = \theta(\beta^*(x_0)) = x_0 = x_0 * y = \theta(\beta^*(x)) * \theta(\beta^*(y))$$

If $x \notin \beta^*(x_0)$ and $x = y$, then

$$\theta(\beta^*(x) \overline{\circ} \beta^*(y)) = \theta(\beta^*(x) \overline{\circ} \beta^*(x)) = \theta(\beta^*(x_0)) = x_0 = x * x = \theta(\beta^*(x)) * \theta(\beta^*(y))$$

If $x \neq y$, then

$$\theta(\beta^*(x) \overline{\circ} \beta^*(y)) = \theta(\beta^*(x)) = x = x * y = \theta(\beta^*(x)) * \theta(\beta^*(y))$$

Hence, θ is a homomorphism and so it is an isomorphism. Moreover, since X is an infinite countable set and $|X \setminus \{a\}| = |X|$, for $a \in X$. Then by Theorem 2.5, $(X \setminus \{a\}, *, 0) \cong (\mathbb{W}, *, 0) \cong (X, *, 0)$. Therefore, there exist a binary operation " $*$ " and hyperoperation " $\circ$ " on X, such that $(X, *, x_0) \cong (\frac{(X, \circ, x_0)}{\beta^*}, \overline{\circ})$. $\square$

4. Conclusions and further research

In this paper we proved that for any nonempty set X we can construct a BCC-algebra and a hyper BCC-algebra on X. Moreover, we shown that for any infinite countable BCC-algebra X, then X is as a fundamental BCC-algebra of itself. Now, in the further research, we should investigate that if X is an infinite non-countable BCC-algebra, then is it X as a fundamental BCC-algebra of itself?

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