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Statistical convergence of triple sequences in topological groups

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Abstract. The idea of triple statistical convergence was introduced by Şahiner et.al [3] while the idea of double statistical sequences was introduced by Mursaleen and Edely [4] . In this paper, we give an extension of statistical convergence of triple sequences in topological groups and give some theorems which generalize Cakallı and Savas's theorems given by in [1] earlier.

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1. Introduction

The idea of statistical convergence for single sequences was introduced by Fast [9] and then studied by various authors, e.g., Salat [10], Fridy [6], Connor [2], Esi [1] and many others. This notion was used by Kolk [7] to extend statistical convergence to normed spaces and Maddox [8] extended to locally convex spaces. Recently, Çakallı [4] extended lacunary statistical convergence to topological groups and Çakallı and Savaş [3] extended double statistical convergence to topological groups.

Let $K \subset \mathbb{N} \times \mathbb{N}$ be two-dimensional set of positive integers and let $K(n, m)$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of a set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = P - \liminf_{n,m} \frac{K(n,m)}{nm}.$$

In this case $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\delta_2(K) = P - \lim_{n,m} \frac{K(n,m)}{nm}.$$

For example, let $K = \left\{\left(i^2, j^2\right) : \ i, j \in \mathbb{N}\right\}$. Then

$$\delta_2(K) = P - \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

i.e., the set K has double natural density zero, while the set $L = \{(i, 2j) : \ i, j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$.

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A real double sequence $x=(x_{jk})$ is said to be statistically convergent to a number L provided that, for each $\varepsilon>0$, the set

$$\{(j,k): \ |x_{jk}-L|\geq\varepsilon\}$$

has double natural density zero. In this case, one writes $st_2-\lim x=L$, [9].

A real double sequence $x=(x_{jk})$ is said to be statistically Cauchy provided that, for every $\varepsilon>0$ there exist $M=M(\epsilon)$ and $T=T(\varepsilon)$ such that for all $j,p\geq M$, $k,q\geq T$, the set

$$\{(j,k)\in\mathbb{N}\times\mathbb{N}: \ |x_{jk}-x_{pq}|\geq\varepsilon\}$$

has double natural density zero, [9].

A subset K of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to be natural density $\delta_3(K)$ if

$$\delta_3(K)=\lim_{p,q,r\to\infty}\frac{K(p,q,r)}{pqr} \text{ exists,}$$

where $K(p,q,r)$ denotes the number of (j,k,l) in K such that $j\leq p, k\leq q$ and $l\leq r$, [12]. For example, let $K=\left\{\left(j^3,k^3,l^3\right): \ j,k,l\in\mathbb{N}\right\}$, then

$$\delta_3(K)=\lim_{p,q,r\to\infty}\frac{K(p,q,r)}{pqr}\leq\lim_{p,q,r\to\infty}\frac{\sqrt[3]{p}\sqrt[3]{q}\sqrt[3]{r}}{pqr}=0$$

i.e., the set K has triple natural density zero, while the set $L=\{(j,3k,5l):j,k,l\in\mathbb{N}\}$ has triple natural density $\frac{1}{15}$.

In a topological group, double sequence $x=(x_{jk})$ is called statistically convergent to a point L of X if for each neighbourhood U of 0 the set

$$\{(j,k):j\leq n,k\leq m; \ x_{jk}-L\in U\}$$

has double natural density zero. In this case we write $st_2(X)-\lim_{j,k}x_{jk}=L$ and we will denote the set of all statistically convergent double sequences by $st_2(X)$.

A triple sequence $x=(x_{jkl})$ is said to be convergent if for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{jkl}-L|<\varepsilon$ whenever $j,k,l\geq N$. A triple sequence $x=(x_{jkl})$ is said to be Cauchy sequence if for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{pqr}-x_{jkl}|<\varepsilon$ for all $p\geq j\geq N$, $q\geq k\geq N$, $r\geq l\geq N$.

A real triple sequence $x=(x_{jkl})$ is said to be statistically convergent to the number L if for each $\varepsilon>0$

$$\delta_3(\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ |x_{jkl}-L|\geq\varepsilon\})=0.$$

In this case, one writes $st_3-\lim x=L$, [12].

The concept of statistical convergence for triple sequences was first introduced by Sahiner et. al. [3] who have given main definition of statistical convergence and statistical Cauchy for triple sequences $x=(x_{jkl})$ as follows: a real triple sequence $x=(x_{jkl})$ is said to be statistically convergent to the number L if for each $\varepsilon>0$

$$\delta_3(\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ |x_{jkl}-L|\geq\varepsilon\})=0.$$

In this case, we write $st_3-\lim x=L$.

A triple sequence $x=(x_{jkl})$ is said to be statistically Cauchy sequence if for every $\varepsilon>0$, there exists $N=N(\varepsilon), M=M(\varepsilon)$ and $Z=Z(\varepsilon)\in\mathbb{N}$ such that

$$\delta_3(\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ |x_{jkl}-x_{NMZ}|\geq\varepsilon\})=0.$$

The purpose of this paper is to study statistical convergence of triple sequences in topological groups and to give some important theorems.

2. Definitions and Notations

By X, we will denote an abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability.

A triple sequence $x = (x_{jkl})$ is said to be convergent in a topological group X if for every neighbourhood U of 0 there exists $N \in \mathbb{N}$ such that $x_{jkl} - L \in U$ whenever $j, k, l \geq N$. L is called a triple limit of $x = (x_{jkl})$. For a subset A of X, $S_3(A)$ will denote the set of all triple sequences $x = (x_{jkl})$ of points in A and $C_3(X)$ will denote the set of all convergent triple sequences of points in X.

A triple sequence $x = (x_{jkl})$ is said to be a Cauchy sequence if for every neighbourhood U of 0 there exists $N \in \mathbb{N}$ such that $x_{pqr} - x_{jkl} \in U$ whenever $p \geq j \geq N$, $q \geq k \geq N, r \geq l \geq N$.

In a topological group, triple sequence $x = (x_{jkl})$ is called statistically convergent to a point L of X if for each neighbourhood U of 0 the set

$$\{(j,k,l) : \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - L \notin U\}$$

has triple natural density zero. In this case we write $st_3(X) - \lim_{j,k,l} x_{jkl} = L$ and we will denote the set of all statistically convergent triple sequences by $st_3(X)$. If $x = (x_{jkl})$ is statistically convergent, then $x = (x_{jkl})$ need not be convergent. For instance, let

$$x_{jkl} = \begin{cases} jkl & ; \quad j,k,l \text{ are cubes} \\ z & ; \quad \text{otherwise} \end{cases}$$

where z is a fixed non-zero element of X. It is easy to see that $st_3(X) - \lim_{j,k,l} x_{jkl} = z$, since the cardinality of the set

$$|\{(j,k,l) : \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - z \notin U\}| \leq \sqrt[3]{j}\sqrt[3]{k}\sqrt[3]{l}$$

for every neighbourhood U of 0. But $x = (x_{jkl})$ is neither convergent nor bounded.

In a topological group, triple sequence $x = (x_{jkl})$ is called statistically Cauchy if for each neighbourhood U of 0 there exist $N = N(U)$, $M = M(U)$, $Z = Z(U)$ such that for all $j, p \geq N, k, q \geq M, l, r \geq Z$ the set

$$\{(j,k,l) : \ j,p \geq N, k,q \geq M, l,r \geq Z; \ x_{jkl} - x_{pqr} \notin U\}$$

has triple natural density zero. In this case we denote the set of all statistically Cauchy triple sequences by $st_3C(X)$.

3. Main Results

Theorem 3.1. *A triple sequence $x = (x_{jkl})$ of points in X is statistically convergent to L if and only if there exists a subset $K \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3(K) = 1$ and $\lim_{j,k,l} x_{jkl} = L$ where the limit is being taken over the set K, i.e., $(j,k,l) \in K$.*

Proof. Let $x = (x_{jkl})$ be statistically convergent to L and (U_q) be a base of nested closed neighbourhood of 0. We write for $q = 1, 2, ...$

$$K_q = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \ x_{jkl} - L \notin U_q\}$$

and

$$M_q = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \ x_{jkl} - L \in U_q\}.$$

Then $\delta_3(K_q) = 0$ and

$$M_1 \supset M_2 \supset ... \supset M_i \supset M_{i+1} \supset ... \tag{3.1}$$

and

$$\delta_3(M_q) = 1, \ q = 1, 2, ... \ . \tag{3.2}$$

Now we will show that for $(j,k,l) \in M_q$, $x = (x_{jkl})$ is convergent to L. Suppose that $x = (x_{jkl})$ is not convergent to L so that there is a neighbourhood U of 0 such that $x_{jkl} - L \notin U$, for infinitely many terms. Let $U_q \subset U$ for $q = 1, 2, ...$ and

$$M_U = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \ x_{jkl} - L \in U\}.$$

Then $\delta_3(M_U) = 0$ by (3.1), $M_q \subset M_U$. Hence $\delta_3(M_q) = 0$ which is contradiction to (3.2). Thus $x = (x_{jkl})$ is convergent to L.

Conversely, suppose that there exists a subset $K \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_3 (K) = 1$ and $\lim_{j,k,l} x_{jkl} = L$; i.e., there exists $N \in \mathbb{N}$ such that for each neighbourhood U of 0, $x_{jkl} - L \in U$ for $j,k,l \geq N$. Now

$$\begin{aligned} K_U &= \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \ x_{jkl} - L \notin U\} \\ &\subset \mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus \{(j_{N+1}, k_{N+1}, l_{N+1}), (j_{N+2}, k_{N+2}, l_{N+2}), ...\}. \end{aligned}$$

Hence $\delta_3(K_U) \leq 1 - 1 = 0$. It follows that $x = (x_{jkl})$ is statistically convergent to L. □

Corollary 3.1. *If a triple sequence $x = (x_{jkl})$ is statistically convergent to a point L, then there exists a sequence $y = (y_{jkl})$ such that $\lim_{j,k,l} y_{jkl} = L$ and*

$$\delta_3 (\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \ x_{jkl} = y_{jkl}\}) = 1$$

i.e., $x_{jkl} = y_{jkl}$ for almost all $j,k,l \in \mathbb{N}$.

Theorem 3.2. *Let X be complete. A triple sequence $x = (x_{jkl})$ of points in X is statistically convergent if and only if $x = (x_{jkl})$ is statistically Cauchy.*

Proof. Suppose that the triple sequence $x = (x_{jkl})$ is statistically convergent to L. Let U be any neighbourhood of 0. Then we may choose a symmetric neighbourhood W of 0 such that $W + W \subset U$. Then for this neighbourhood W of 1, the set

$$\{(j,k,l) : \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - L \in W\}$$

has triple natural density 0. For each neighbourhood U of 0, the set

$$\{(j,k,l) : \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - L \notin U\}$$

has triple natural density zero. Then we may choose numbers N, M and Z such that $x_{NMZ} - L \notin U$. Now we write

$$A_U = \{(j,k,l) : \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - x_{NMZ} \notin U\},$$

$$B_W = \{(j,k,l) : \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - L \notin W\}$$

and

$$C_W = \{(j,k,l) : \ j = N \leq n, k = M \leq m, l = Z \leq r; \ x_{NMZ} - L \notin W\}.$$

Then $A_U \subset B_W \subset C_W$ and hence $\delta_3(A_U) \leq \delta_3(B_W) \leq \delta_3(C_W) = 0$. Therefore we get that $x = (x_{jkl})$ is statistically Cauchy.

Conversely, suppose that there is a statistically Cauchy sequence $x = (x_{jkl})$ but it is not statistically convergent. Then we may find natural numbers N, M and Z such that the set

$$A_U = \{(j,k,l) : \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - x_{NMZ} \notin U\}$$

has triple natural density zero. It follows from this the set

$$E_U = \{(j,k,l) : \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - x_{NMZ} \in U\}$$

has triple natural density one. Now, we may choose a neighbourhood W of 0 such that $W+W \subset U$. Now take any fixed non-zero element L of X. Write $x_{jkl}-x_{NMZ} = x_{jkl}-L+L-x_{NMZ}$. It follows from this equality that $x_{jkl}-x_{NMZ} \in U$ if $x_{jkl}-L \in W$. Since $x=(x_{jkl})$ is not statistically convergent to L, the set

$$B_W = \{(j,k,l): \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - L \notin W\}$$

has triple natural density one. Hence the set

$$E_U = \{(j,k,l): \ j \leq n, k \leq m, l \leq r; \ x_{jkl} - x_{NMZ} \in U\}$$

has triple natural density zero, i.e., the set A_U has triple natural density one, which is contradiction. This completes the proof. □

Now from Theorem 3.1 and Theorem 3.2 we can state the following theorem and since the proof is easy, then we omit it.

Theorem 3.3. *If X is complete, the the following conditions are equivalent:*
a) $x=(x_{jkl})$ is statistically convergent to L,
b) $x=(x_{jkl})$ is statistically Cauchy,
c) There exists a subsequence $y=(y_{jkl})$ of $x=(x_{jkl})$ such that $\lim_{j,k,l} y_{jkl} = L$.

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