Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 40(2), 2013, Pages 167–173
ISSN: 1245-4563

# The Hadamard's inequality for quasi-convex functions via fractional integrals

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Abstract. In this paper, we give the Riemann-Liouville fractional integrals definitions. We use these Riemann-Liouville fractional integrals to establish some new integral inequalities for quasi-convex functions. Also, some applications for special means of real numbers are provided.

*2010 Mathematics Subject Classification.* 26D07, 26D10, 26D15, 26A33.
*Key words and phrases.* Quasi convex functions, Hadamard's inequality, Riemann-Liouville fractional integral, Power-mean inequality.

## 1. Introduction

Let real function $f$ be defined on some nonempty interval $I$ of real line $\mathbb{R}$. The function $f$ is said to be *quasi-convex* on $I$ if inequality

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \qquad (QC)$$

holds for all $x, y \in I$ and $t \in [0,1]$ (see [15]).

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of $I$ of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well-known in the literature as Hadamard's inequality. For several recent results concerning the inequality (1) we refer the interested reader to ([1], [2], [3], [9], [12]-[15]). Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex. For example, consider the following:

Let $f : \mathbb{R}^+ \to \mathbb{R}$,

$$f(x) = \ln x, \quad x \in \mathbb{R}^+.$$

This function is quasi-convex. However $f$ is not convex functions.

**Definition 1.1.** [15] The mapping $f : I \to \mathbb{R}$ is *Jensen-* or *J-quasi-convex* if

$$f\left(\frac{x+y}{2}\right) \leq \max\{f(x), f(y)\} \qquad (JQC)$$

for all $x, y \in I$.

**Definition 1.2.** [15] For $I \subset \mathbb{R}$, the mapping $f : I \to \mathbb{R}$ is *Wright-quasi-convex* if, for all $x, y \in I$ and $t \in [0,1]$, one has the inequality

$$\frac{1}{2}\left[f(tx + (1-t)y) + f((1-t)x + ty)\right] \leq \max\{f(x), f(y)\} \qquad (WQC)$$

Received March 5, 2013. Accepted September 4, 2013.

or equivalently

$$\frac{1}{2}\left[f(y)+f(x+\delta)\right]\leq \max\left\{f(x),f(y+\delta)\right\}$$

for every $x,y+\delta\in I$ with $x<y$ and $\delta>0$.

In [15], Dragomir and Pearce proved the following results connected with the inequality (1):

**Theorem 1.1.** [15] *Let $f:I\to\mathbb{R}$ be a Wright-quasi-convex map on $I$ and suppose $a,b\in I\subseteq\mathbb{R}$ with $a<b$ and $f\in L_1[a,b]$. Then we have the inequality*

$$\frac{1}{b-a}\int_a^b f(t)dt\leq \max\left\{f(a),f(b)\right\}. \tag{2}$$

**Theorem 1.2.** [15] *Let $WQC(I)$ denote the class of Wright-quasi-convex functions on $I\subseteq\mathbb{R}$. Then*

$$QC(I)\subset WQC(I)\subset JQC(I).$$

In [12], Ion proved the following results connected with quasi-convex function:

**Theorem 1.3.** [12] *Assume $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ is a differentiable function on $(a,b)$. If $|f'|$ is quasi-convex on $[a,b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}. \tag{3}$$

**Theorem 1.4.** [12] *Assume $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ is a differentiable function on $(a,b)$. Assume $p\in\mathbb{R}$ with $p>1$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a,b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{2(p+1)^{1/p}}\left[\max\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right]^{\frac{p-1}{p}}. \tag{4}$$

In [2], Alomari *et al.* proved the following theorem for quasi-convex function:

**Theorem 1.5.** [2] *Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^{\circ}$ , $a,b\in I^{\circ}$ with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q\geq1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}. \tag{5}$$

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 1.3.** [11] Let $f\in L_1[a,b]$. The *Riemann-Liouville integrals* $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ *of order* $\alpha>0$ with $a\geq0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\quad x<b$$

respectively where $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du$. Here is $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see ([4]-[8]).

In [13], Sarıkaya *et al.* proved the following Lemma and established some inequalities for fractional integrals

**Lemma 1.6.** [13] *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on $(a,b)$ with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] = \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right] f'(ta+(1-t)b)dt.$$

The aim of this paper is to establish Hadamard type inequalities for quasi-convex functions via Riemann-Liouville fractional integral.

## 2. MAIN RESULTS

**Theorem 2.1.** *Let $f : [a,b] \to \mathbb{R}$, be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If $f$ is a quasi-convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \le \max\{f(a),f(b)\}$$

*with $\alpha > 0$.*

*Proof.* Since $f$ is quasi-convex function on $[a,b]$, we have

$$f(ta+(1-t)b) \le \max\{f(a),f(b)\}$$

and

$$f((1-t)a+tb) \le \max\{f(a),f(b)\}.$$

By adding these inequalities we get

$$\frac{1}{2}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right] \le \max\{f(a),f(b)\} \tag{6}$$

Then multiplying both sides of (6) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned} \int_0^1 t^{\alpha-1}f(ta+(1-t)b)\,dt &+ \int_0^1 t^{\alpha-1}f((1-t)a+tb)\,dt \\ &= \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{du}{a-b} + \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\frac{dv}{b-a} \\ &\le \frac{2}{\alpha}\max\{f(a),f(b)\}, \end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \le \max\{f(a),f(b)\}.$$

The proof is complete. □

**Remark 2.1.** If we choose $\alpha = 1$ in Theorem 2.1, we have the inequality (2).

**Theorem 2.2.** *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on $(a,b)$ with $a < b$. If $|f'|$ is quasi-convex on $[a,b]$ and $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]\right| \leq \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{|f'(a)|, |f'(b)|\right\}. \tag{7}$$

*Proof.* Using Lemma 1.6 and the quasi-convex of $|f'|$ with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]\right| \\
&\leq \frac{b-a}{2}\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\,|f'(ta+(1-t)b)|\,dt \\
&\leq \frac{b-a}{2}\int_0^1 |(1-t)^{\alpha} - t^{\alpha}| \max\left\{|f'(a)|, |f'(b)|\right\} dt \\
&= \frac{b-a}{2}\max\left\{|f'(a)|, |f'(b)|\right\}\left\{\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha} - t^{\alpha}\right]dt + \int_{\frac{1}{2}}^1\left[t^{\alpha} - (1-t)^{\alpha}\right]dt\right\} \\
&= \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{|f'(a)|, |f'(b)|\right\}.
\end{aligned}$$

where we use the fact that

$$\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\,dt = \int_0^{\frac{1}{2}}\left[(1-t)^{\alpha} - t^{\alpha}\right]dt + \int_{\frac{1}{2}}^1\left[t^{\alpha} - (1-t)^{\alpha}\right]dt = \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)$$

which completes the proof. □

**Remark 2.2.** If we choose $\alpha = 1$ in (7), then the inequality (7) reduces to the inequality (3) of Theorem 1.3.

**Theorem 2.3.** *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on $(a,b)$ with $a < b$ such that $f' \in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$, and $p > 1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]\right| \leq \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^{q}, |f'(b)|^{q}\right\}\right)^{\frac{1}{q}} \tag{8}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.*

*Proof.* From Lemma 1.6 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(ta+(1-t)b)\right|dt \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}.
\end{aligned}$$

We know that for $\alpha\in[0,1]$ and $\forall t_1,t_2\in[0,1]$,

$$\left|t_1^{\alpha}-t_2^{\alpha}\right|\leq\left|t_1-t_2\right|^{\alpha},$$

hence

$$\begin{aligned}
\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|^{p}dt &\leq \int_0^1 \left|1-2t\right|^{\alpha p}dt \\
&= \int_0^{\frac{1}{2}} \left[1-2t\right]^{\alpha p}dt+\int_{\frac{1}{2}}^1 \left[2t-1\right]^{\alpha p}dt \\
&= \frac{1}{\alpha p+1}.
\end{aligned}$$

Since $|f'|^q$ is quasi-convex on $[a,b]$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \\
&\qquad\leq \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. □

**Remark 2.3.** If in Theorem 2.3, we choose $\alpha=1$, then the inequality (8) becomes the inequality (4) of Theorem 1.4.

**Theorem 2.4.** *Let $f:[a,b]\to\mathbb{R}$, be a differentiable mapping on $(a,b)$ with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q\geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \\
&\qquad\leq \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}
\end{aligned} \tag{9}$$

*with $\alpha>0$.*

*Proof.* From Lemma 1.6 and using power-mean inequality with properties of modulus, we can write

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \\ \leq\ & \frac{b-a}{2}\int_{0}^{1}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(ta+(1-t)b)\right|dt \\ \leq\ & \frac{b-a}{2}\left(\int_{0}^{1}\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^{q}$ is quasi-convex on $[a,b]$, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \\ \leq\ & \frac{b-a}{2}\left(\max\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}\left(\int_{0}^{1}\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right) \\ =\ & \frac{b-a}{2}\left(\max\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}\left\{\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]dt\right\} \\ =\ & \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. □

**Remark 2.4.** We note that the obtained inequality (9) is better than the inequality (8) meaning that the approach via the power mean inequality is a better approach than that through Hölder's inequality.

**Remark 2.5.** If in Theorem 2.4, we choose $\alpha = 1$, then the inequality (9) becomes the inequality (5) of Theorem 1.5.

## 3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers $\alpha, \beta$ $(\alpha \neq \beta)$. We take

(1) *Arithmetic mean* :

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},\quad \alpha,\beta\in\mathbb{R}^{+}.$$

(2) *Logarithmic mean*:

$$L(\alpha,\beta)=\frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|},\qquad |\alpha|\neq|\beta|\,,\ \alpha,\beta\neq 0,\ \alpha,\beta\in\mathbb{R}^{+}.$$

(3) *Generalized log − mean*:

$$L_{n}(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}},\qquad n\in\mathbb{Z}\backslash\{-1,0\},\ \alpha,\beta\in^{+}.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

**Proposition 3.1.** *Let $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{Z}$. Then, we have*

$$|A(a^n, b^n) - L_n^n(a, b)| \leq \frac{b-a}{4} \max\{|a|^n, |b|^n\}.$$

*Proof.* The assertion follows from Theorem 2.2 applied to the quasi-convex mapping $f(x) = x^n$, $x \in \mathbb{R}$ and $\alpha = 1$. □

**Proposition 3.2.** *Let $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{Z}$. Then, for all $q \geq 1$, we have*

$$|A(a^n, b^n) - L_n^n(a, b)| \leq \frac{b-a}{4} \left(\max\left\{(|a|^n)^q, (|b|^n)^q\right\}\right)^{\frac{1}{q}}.$$

*Proof.* The assertion follows from Theorem 2.4 applied to the m-convex mapping $f(x) = x^n$, $x \in \mathbb{R}$ and $\alpha = 1$. □

## References

[1] M. Alomari and M. Darus, On the Hadamard's inequality for log-convex functions on the coordinates, *Journal of Inequalities and Applications* (2009), Article ID 283147, 13 pages.

[2] M. Alomari, M. Darus and S.S. Dragomir, Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are quasi-convex, *RGMIA Res. Rep. Coll.* **12**, Supplement, Article 14.

[3] M.K. Bakula, M.E. Özdemir, J. Pečarić, Hadamard type inequalities for $m-convex$ and $(\alpha - m) - convex$ functions, *J. Ineq. Pure and Appl. Math.* **9** (2008), no. 4, Article 96.

[4] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, *J. Ineq. Pure and Appl. Math.* **10** (2009), no. 3, Article 86.

[5] Z. Dahmani, New inequalities in fractional integrals, *International Journal of Nonlinear Science* **9** (2010), no. 4, 493–497.

[6] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, *Ann. Funct. Anal.* **1** (2010), no. 1, 51–58.

[7] Z. Dahmani, L. Tabharit, S. Taf, Some fractional integral inequalities, *Nonl. Sci. Lett. A.* **1** (2010), no. 2, 155–160.

[8] Z. Dahmani, L. Tabharit, S. Taf, New generalizations of Gruss inequality using Riemann-Liouville fractional integrals, *Bull. Math. Anal. Appl.* **2** (2010), no. 3, 93–99.

[9] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[10] S.S. Dragomir and C.E.M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377–385.

[11] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien, 1997, 223–276.

[12] D.A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, *Annals of University of Craiova, Math. Comp. Sci. Ser.* **34** (2007), 82–87.

[13] M.Z. Sarıkaya, E. Set, H. Yaldız and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, , **57** (2013), no. 9-10, 2403–2407.

[14] E. Set, M.E. Özdemir and S.S. Dragomir, On the Hermite-Hadamard inequality and other integral inequalities involving two functions, *Journal of Inequalities and Applications* (2010), Article ID 148102, 9 pages.

[15] M.E. Özdemir, S.S. Dragomir and Ç. YILDIZ, The Hadamard Inequality for convex function via fractional integrals, *Acta Math. Sci.* **33B** (2013), no. 5, 1293–1299.

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