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Quasi-invariant convergence in a normed space

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Abstract. In this study, notions of quasi-invariant convergence and quasi-invariant statistical convergence, which are related to invariant limits, are defined and discussed.

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1. Introduction

Let σ be a one-to-one mapping of the set of positive integer into itself such that $\sigma^m(n) \neq n$ for all positive integers m and n, where $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$; $m = 1, 2, 3, ...$

Let X be a real normed space. A continuous linear functional φ on the space of bounded sequences is an *invariant mean* or *σ-limit* if
(1) $\varphi(x) \geq 0$ when the sequence $x = (x_n) \in X$ has $x_n \geq 0$ for all n,
(2) $\varphi(1, 1, 1, ...) = 1$ and
(3) $\varphi(x_{\sigma(n)}) = \varphi(x)$
for all bounded sequences x. We denote by V_σ the set of bounded sequences all of whose invariant means are equal. In case $\sigma(n) = n + 1$, a σ-limit is often called a *Banach limit* and V_σ is the set of almost convergent sequences. It is known that a bounded sequence $x = (x_n) \in X$ is invariant convergent to $s \in X$ if and only if

$$\lim_{p\to\infty} \|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(k)} - s\| = 0 \tag{1}$$

uniformly in $k(= 1, 2, 3, ...)$. It is known that $c \subset V_\sigma \subset l_\infty$ where c is the space of all convergent sequences and l_∞ is the space of all bounded sequences in a real normed space X. Over the years invariant convergence has been examined in summability theory.

A sequence $(x_i) \in X$ is said to be *statistically convergent* to $s \in X$ if for each $\epsilon > 0$

$$\lim_{p\to\infty} \frac{1}{p}|\{i \leq p : \|x_i - s\| \geq \epsilon\}| = 0$$

where $|A|$ denotes the number of members of a set A. The concept of statistical convergence was first introduced by Fast [2] and also independently by Buck [1] and Schoenberg [8] for real and complex sequences. Further this concept was studied by Salat [6], Fridy [3] and many others. Recently Savas and Nuray [5] introduced σ-statistical convergence for real and complex sequences as follows: A sequence (x_i) is

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said to be *invariant* or σ- *statistically convergent* to real or complex number s if for each $\epsilon > 0$

$$\lim_{p\to\infty} \frac{1}{p} |\{i \leq p : |x_{\sigma^i(k)} - s| \geq \epsilon\}| = 0$$

uniformly in k. We can generalize this definition to the sequences in a real normed space X as follows: A sequence $(x_i) \in X$ is said to be invariant or σ- statistically convergent to $s \in X$ if for each $\epsilon > 0$

$$\lim_{p\to\infty} \frac{1}{p} |\{i \leq p : \|x_{\sigma^i(k)} - s\| \geq \epsilon\}| = 0$$

uniformly in k.

The plan of this paper is as follows. First we will show the existence of an another family of functionals defined on the space l_∞. Then we define a new method of summability of sequences $(x_i) \in l_\infty$ which will be called quasi invariant convergence and we will give a theorem which contains a necessary and sufficient condition for a bounded sequence to be quasi invariant convergent. Next, we shall prove a theorem which shows that if a bounded sequence is invariant convergent to s, then it is quasi invariant convergent to s. Finally we will introduce quasi invariant statistical convergence for sequences in a real normed space and show that if a sequence is invariant statistical convergent to s, then it is quasi invariant statistical convergent to s.

2. Quasi-invariant convergence

Let us define on the space l_∞ the function q by

$$q(x) \equiv q(x_i) = \overline{\lim_{p\to\infty}} \{sup_n \frac{1}{p} \| \sum_{i=0}^{p-1} x_{\sigma^i(np)} \| \} \tag{2}$$

The functional q clearly is real-valued and it satisfies following properties:
(i) $q(x) \geq 0$,
(ii) $q(\alpha x) = |\alpha| q(x)$,
(iii) $q(x + y) \leq q(x) + q(y)$ $(\alpha \in \mathbb{R}; x, y \in l_\infty)$
that is, q is a symmetric convex functional on the space l_∞. According to a corollary of Hahn-Banach theorem there must exist a nontrivial linear functional L on the space l_∞ such that $|L(x_i)| \leq q(x_i)$.

The following lemma is well known in the literature.

Lemma 2.1. *Let X be a real linear space and $q : X \to \mathbb{R}$ be a functional such that the following assertions are valid: $q(x) \geq 0, q(\alpha x) = |\alpha| q(x)$, $q(x + y) \leq q(x) + q(y)$ $(\alpha \in \mathbb{R}; x, y \in l_\infty)$. Then for each $x_0 \in X$, there exists a linear functional L on X such that*

$$(\forall x \in X) \quad |L(x)| \leq q(x), \quad L(x_0) = q(x_0).$$

Denoting now by $\sum$ the family of functionals satisfying the above conditions then for each $s \in X$ we have

$$(\forall L \in \sum) \quad L(x_i - s) = 0 \text{ iff } q(x_i - s) = 0 \quad ((x_i) \in l_\infty). \tag{3}$$

Now we can state following theorem.

Theorem 2.2. *There exists the family of non trivial functionals L defined on the space l_∞ such that for all $\alpha, \beta \in \mathbb{R}$, each $s \in X$ and all $(x_i), (y_i) \in l_\infty$, the following assertions are valid:*

(a) $L(\alpha x_i + \beta y_i) = \alpha L(x_i) + \beta L(y_i)$,

(b) $L(x_{\sigma(i)}) = L(x_i)$,

(c) $|L(x_i)| \leq q(x_i)$,

(d) $L(x_i - s) = 0$ *iff* $q(x_i - s) = 0$.

Having obtained the functionals $L \in \sum$ we can proceed to the investigation of the sequences $(x_i) \in l_\infty$ which all the functionals $L \in \sum$ assigned the same value.

Definition 2.1. A sequence $(x_i) \in l_\infty$ is *quasi invariant convergent* to $s \in X$ or *quasi σ-summable* to s if

$$(\forall L \in \sum) \quad L(x_i - s) = 0. \tag{4}$$

in this case we will write $(Q - \sigma) - lim_{i\to\infty} x_i = s$.

It is easy to see that quasi invariant limit of a sequence defined in such way is unique.

Theorem 2.3. *A bounded sequence (x_i) quasi invariant convergent to $s \in X$ iff*

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| \to 0 \text{ as } p \to +\infty \tag{5}$$

uniformly in n $(= 1, 2, ...)$.

Proof. Suppose for a bounded sequence (x_i), we have $(Q-\sigma) - lim_{i\to\infty} x_i = s$. Then, by (4) and (3), we have $q(x_i - s) = 0$ or, by (2), we have

$$\overline{lim}_{p\to\infty}\{sup_n \frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\|\} = 0.$$

Therefore for any $\epsilon > 0$, there exists an integer $p_0 > 0$ such that for all $p > p_0$ and $n = 1, 2, 3, ...$, we have

$$\frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| < \epsilon$$

Since $\epsilon > 0$, arbitrary, we have

$$\frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| \to 0 \text{ as } p \to \infty$$

uniformly in n, so the condition (5) is necessary. Conversely, let the condition (5) be true. This means that

$$sup_n \frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| \to 0 \text{ as } p \to \infty$$

or

$$q(x_i - s) = \overline{\lim}_{p\to\infty}\{sup_n \frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\|\} = 0$$

hence by (3), we have

$$(\forall L \in \sum) \quad L(x_i - s) = 0,$$

which by (4), means that $(Q-\sigma) - lim_{i\to\infty} x_i = s$, so the condition (5) is sufficient. $\square$

Theorem 2.4. *If a bounded sequence $x = (x_i)$ invariant convergent to $s \in X$, then it is quasi invariant convergent to s.*

Proof. Let bounded sequence $x = (x_i)$ be invariant convergent to $s \in X$. Then by (1) for any $\epsilon > 0$ there exists an integer $p_0 > 0$ such that

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(k)} - s\| < \epsilon \quad (p > p_0, k = 1, 2, 3, ...).$$

hence for $k = np$ $(p > p_0,\ n = 1, 2, 3, ...)$ we have

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| \to 0 \text{ as } p \to \infty$$

uniformly in n which, by (5), means that (x_i) quasi invariant convergent. □

When $\sigma(i) = i + 1$ we have quasi almost convergence which was defined and discussed in [4].

3. Quasi-invariant statistical convergence

Definition 3.1. A sequence (x_i) is said to be *quasi invariant statistically convergent* to $s \in X$ if for each $\epsilon > 0$

$$\lim_{p\to\infty} \frac{1}{p}|\{i \leq p : \|x_{\sigma^i(np)} - s\| \geq \epsilon\}| = 0$$

uniformly in n.

When $\sigma(i) = i + 1$ we have the following definition of quasi almost statistical convergence which have not appeared anywhere by this time.

Definition 3.2. A sequence (x_i) is said to be *quasi almost statistically convergent* to $s \in X$ if for each $\epsilon > 0$

$$\lim_{p\to\infty} \frac{1}{p}|\{i \leq p : \|x_{np+i} - s\| \geq \epsilon\}| = 0$$

uniformly in n.

Theorem 3.1. *If a sequence $x = (x_i) \in X$ invariant statistically convergent to $s \in X$, then it is quasi invariant statistically convergent to s.*

Proof. Let $x = (x_i)$ be invariant statistically convergent to $s \in X$. Then for any $\epsilon > 0$ there exists an integer $p_0 > 0$ such that

$$\frac{1}{p}|\{i \leq p : \|x_{\sigma^i(k)} - s\| \geq \epsilon\}| < \epsilon \ (p > p_0, k = 1, 2, ...).$$

Hence for $k = np$ $(p > p_0, n = 1, 2, ...)$ we have

$$\frac{1}{p}|\{i \leq p : \|x_{\sigma^i(np)} - s\| \geq \epsilon\}| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{p\to\infty} \frac{1}{p}|\{i \leq p : \|x_{\sigma^i(np)} - s\| \geq \epsilon\}| = 0$$

uniformly in n which means that (x_i) is quasi invariant statistically convergent to s convergent. □

We remark that from the comparison of the definitions of invariant statistical convergence and quasi invariant statistical convergence, follows that there is a big possibility that there exist sequences that are quasi invariant statistical convergent, but not invariant statistically convergent. Proof of that is still an open problem. The similar remark also stands for relationship between the quasi invariant convergence and invariant convergence.

References

[1] R.C. Buck, Generalized asymptotic density, *Amer. J. Math.* **75** (1953), 335–346.
[2] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), 241–244.
[3] J.A. Fridy, On statistical convergence, *Analysis* **5** (1985), 301–313.
[4] D. Hajdukovic, Quasi-almost convergence in a normed space, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **13** (2002), 36–41.
[5] F. Nuray and E. Savas, On σ-statistically convergence and lacunary σ-statistically convergence,*Math. Slovaca* **43** (1993), 309–315.
[6] T. Salat, On statistically convergent sequences of real numbers, *Math. Slovaca* **30** (1980), 139–150.
[7] P. Schaefer, Infinite matrices and invariant means, *Proc. Amer. Math. Soc.* **36** (1972), 104–110.
[8] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* **66** (1959), 361–375.

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