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Renormalized solutions for a $p(x)$-Laplacian equation with Neumann nonhomogeneous boundary conditions and L^1-data

Elhoussine Azroul, Abdelkrim Barbara, Mohamed Badr Benboubker, and Stanislas Ouaro

Abstract. An existence result of a renormalized solution for the $p(x)$-Laplacian equation with Neumann nonhomogeneous boundary conditions and L^1 data is established

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1. Introduction

In this paper, we consider the inhomogeneous and nonlinear Neumann boundary value problem:

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u + \alpha(u)|\nabla u|^{p(x)} = f & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \eta} + \gamma(u) = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^N (N \geq 3)$ is a bounded open domain with Lipschitz boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$, α, γ are real functions defined on $\mathbb{R}$ or $\mathbb{R}^N$, $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$.

The operator $-\Delta_{p(x)}u := -div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian which become p-Laplacian when $p(x) \equiv p$ (a constant). It possesses more complicated nonlinearities than the p-Laplacian. As the exponent which appear in (1) depends on the variable x, the functional setting involves Lebesgue and Sobolev spaces with variable exponent $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$. The study of PDEs with variable exponent as experienced a revival of interest over the past few years (see [1, 2, 3, 7, 8, 12, 13, 16, 18, 19, 20, 22, 24] and references therein). The interest of study problem involving variable exponent is due to the fact that they can model various phenomena which arise in the study of elastic mechanics (see [6]), electrorheological fluids (see [21]) or image restauration (see [12]).

In this paper, we study the existence of renormalized solutions of problem (1). The concept of renomalized solution in the context of variable exponent was for the first time studied by wittbold and Zimmerman [23] where they considered an homogeneous Dirichlet boundary condition. In our paper, we consider an inhomogeneous Neumann boundary condition which bring some difficulty to treat the term at the boundary. in order to get our main result, we define a new space which will help us to take into account the boundary condition. This space in the context of variable exponent was for the first time introduced by Ouaro and Tchousso (see [18]).

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The remaining part of the paper is the following: in section 2, we introduce some notations/functional spaces. In section 3, we show some basic assumptions on the data and we define the notion of renormalized solution. We end in section 4 by proving the existence of renormalized solutions for problem (1).

2. Preliminaries

As the exponent $p(x)$ appearing in (1) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents, under the following assumptions on the data:

$$\begin{cases} p(.) : \overline{\Omega} \to I\!R \text{ is a continuous function such that} \\ 1 < p_- \leq p_+ < +\infty, \end{cases} \tag{2}$$

where $p^- := ess \inf_{x\in\Omega} p(x)$ and $p^+ := ess \sup_{x\in\Omega} p(x)$.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \to I\!R$ for which the convex modular

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e., if $p^+ < +\infty$, then the expression

$$||u||_{p(x)} := \inf\{\lambda > 0 : \rho_{p(x)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(x)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(x)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left| \int_\Omega uv\,dx \right| \leq (\frac{1}{p^-} + \frac{1}{p'^-})\|u\|_{p(x)}\|v\|_{p'(x)} \tag{3}$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.
Let

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $(W^{1,p(x)}(\Omega), ||.||_{1,p(x)})$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(x)}$ of the space $L^{p(x)}(\Omega)$. We have the following result :

Proposition 2.1. (see [14, 25]) *If $u_n, u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$, then the following properties hold true:*
(i) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|^{p^-}_{p(x)} < \rho_{p(x)}(u) < \|u\|^{p^+}_{p(x)}$;
(ii) $\|u\|_{p(x)} < 1 \Rightarrow \|u\|^{p^+}_{p(x)} < \rho_{p(x)}(u) < \|u\|^{p^-}_{p(x)}$;
(iii) $\|u\|_{p(x)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{p(x)}(u) < 1$ *(respectively* $= 1; > 1$*);*
(iv) $\|u_n\|_{p(x)} \to 0$ *(respectively* $\to +\infty$*)* $\Leftrightarrow \rho_{p(x)}(u_n) \to +\infty$ *(respectively* $\to +\infty$*) ;*
(v) $\rho_{p(x)}(u/\|u\|_{p(x)}) = 1$.

For a measurable function $u : \Omega \to I\!R$, we introduce the following notation:

$$\rho_{1,p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Proposition 2.2. (see [22, 24]) *If $u \in W^{1,p(x)}(\Omega)$, then the following properties hold true:*
(i) $||u||_{1,p(x)} > 1 \Rightarrow ||u||^{p^-}_{1,p(x)} \leq \rho_{1,p(x)}(u) \leq ||u||^{p^+}_{1,p(x)}$;
(ii) $||u||_{1,p(x)} < 1 \Rightarrow ||u||^{p^+}_{1,p(x)} \leq \rho_{1,p(x)}(u) \leq ||u||^{p^-}_{1,p(x)}$;
(iii) $||u||_{1,p(x)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{1,p(x)}(u) < 1$ *(respectively* $= 1; > 1$*).*

Put

$$p^\partial(x) := (p(x))^\partial := \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)}, \text{ if } p(x) < N \\ \infty, \text{ if } p(x) \geq N. \end{cases}$$

Proposition 2.3. *Let $p \in C(\bar{\Omega})$ and $p^- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition*

$$1 \leq q(x) < p^\partial(x), \ \forall \ x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$.
In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$.

Let us introduce the following notation: given two bounded measurable functions $p(x), q(x) : \Omega \to I\!R$, we write

$$q(x) \ll p(x) \text{ if } \ ess \inf_{x\in\Omega}(p(x) - q(x)) > 0.$$

Proposition 2.4. ([11]) *Let V be a uniformly convex Banach space.*
Let x_n be a sequence in V such that $x_n \to x$ in the weak topology $\sigma(V, V')$ and

$$\limsup \|x_n\| \leq \|x\|.$$

Then $x_n \to x$ strongly.

Lemma 2.5. *Let $\xi, \eta \in I\!R^N$ and let $1 < p < \infty$. We have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi.(\xi - \eta).$$

Proof. We consider the function $f : I\!R^+ \to I\!R$ defined by $x \mapsto x^p - px + (p-1)$. We have

$$f(x) \geq \min_{y\in I\!R^+} f(y) = f(1) = 0 \text{ for all } x \in I\!R^+.$$

Therefore, we take $x = \frac{|\eta|}{|\xi|}$ (if $|\xi| = 0$, the result is obvious) in the inequality above to get the result of the lemma by using Cauchy-Schwarz inequality. □

In the sequel, we need the following two technical lemmas (see [15, 20]).

Lemma 2.6. *Let $(v_n)_{n\in I\!N}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 \ll p(.) \in L^\infty(\Omega)$, then v_n strongly converges to v in $L^1(\Omega)$.*

The second technical lemma is a well known result in measure theory (see [15]):

Lemma 2.7. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < +\infty$. Consider a measurable function $\gamma : X \longrightarrow [0, +\infty]$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \epsilon \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma d\mu < \delta.$$

3. Basic Assumptions on the data and definition of a renormalized solution

In this part, we define the associated renormalized solution to the problem (1). We begin by stating the following assumptions:
(H1) f and g are positive functions such as $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$.
(H2) α and γ are increasing continuous functions defined on $I\!R$ such that $\alpha(0) = \gamma(0) = 0$.

We first recall some notations.
For any $k > 0$, we define the truncation function T_k by $T_k(s) := \max\{-k, \min\{k, s\}\}$.
For any $u \in W^{1,p(x)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. In the sequel, we will identify at the boundary u and $\tau(u)$.
Set

$$\mathcal{T}^{1,p(x)}(\Omega) = \{u : \Omega \to I\!R, \text{ measurable such that } T_k(u) \in W^{1,p(x)}(\Omega), \text{ for any } k > 0\}.$$

As in [9], we can prove the following result.

Proposition 3.1. *Let $u \in \mathcal{T}^{1,p(x)}(\Omega)$. Then, there exists a unique measurable function $v : \Omega \to I\!R^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u.*
Moreover if $u \in W^{1,p(x)}(\Omega)$ then $v \in (L^{p(x)}(\Omega))^N$ and $v = \nabla u$ in the usual sense.

Following [4, 5, 16, 18, 19], we define $\mathcal{T}^{1,p(x)}_{tr}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,p(x)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in I\!N} \subset W^{1,p(x)}(\Omega)$ satisfying the following conditions:
(C_1) $u_n \to u$ a.e. in Ω.
(C_2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^1(\Omega))^N$ for any $k > 0$.
(C_3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. in $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [4, 5]. In the sequel the trace of $u \in \mathcal{T}^{1,p(x)}_{tr}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(x)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}^{1,p(x)}_{tr}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ then $(u-\varphi) \in \mathcal{T}^{1,p(x)}_{tr}(\Omega)$ and $tr(u - \varphi) = tr(u) - tr(\varphi)$.
We can now introduce the notion of renormalized solution of (1).

Definition 3.1. *A measurable function $u : \Omega \to I\!R$* is a renormalized solution of problem (1) if

$$u \in \mathcal{T}^{1,p(x)}_{tr}(\Omega), \tag{4}$$

$$\lim_{h\to+\infty} \frac{1}{h} \int_{\{h<|u|<2h\}} |\nabla u|^{p(x)} = 0, \tag{5}$$

and

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(S(u)\varphi)\, dx + \int_\Omega |u|^{p(x)-2} u S(u)\varphi\, dx + \int_\Omega \alpha(u)|\nabla u|^{p(x)} S(u)\varphi\, dx + \int_{\partial\Omega} \gamma(u)S(u)\varphi\, d\sigma = \int_\Omega f S(u)\varphi\, dx + \int_{\partial\Omega} g S(u)\varphi\, d\sigma, \tag{6}$$

for every $\varphi \in W^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$ and for any smooth function with compact support S in $I\!R$.

4. Existence result

Now we announce the main result of this section.

Theorem 4.1. *Let assumptions (H1)–(H2) hold true. Then there exists at least one renormalized solution u of the elliptic problem (1).*

Proof. The proof of Theorem 4.1 consists into two steps.

Step1. Regularization of the problem. We define the following reflexive space

$$E = W^{1,p(x)}(\Omega) \times L^{p(x)}(\partial\Omega).$$

Let X_0 be the subspace of E defined by

$$X_0 = \{(u, v) \in E : v = \tau(u)\}.$$

In the sequel, we will identify an element $(u, v) \in X_0$ with his representative $u \in W^{1,p(x)}(\Omega)$. We define the operator A_n by

$$A_n u = A_n\langle u, \tau u\rangle = Au + T_n(\alpha(u)|\nabla u|^{p(x)}) + T_n(\gamma(u)), \text{ for all } u \in X_0,$$

where

$$\langle Au, v\rangle = A\langle u, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx + \int_\Omega |u|^{p(x)-2} uv \, dx \quad \forall v \in X_0.$$

Lemma 4.2. *The operator A_n is bounded, hemi-continuous and is of type (M) from X_0 into X_0'. Moreover, A_n is coercive in the following sense*

$$\frac{\langle A_n v, v\rangle}{\|v\|_{1,p(x)}} \longrightarrow +\infty \qquad as \quad \|v\|_{1,p(x)} \longrightarrow +\infty \qquad for\ all \quad v \in W_0^{1,p(x)}(\Omega).$$

Proof of Lemma 4.2. Using the Hölder type inequality, we have

$$\begin{aligned} \langle A_n u, v\rangle &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \, \nabla v \, dx + \int_\Omega |u|^{p(x)-2} u \, v \, dx \\ &\quad + \int_\Omega T_n(\alpha(u)|\nabla u|^{p(x)}) v \, dx + \int_{\partial\Omega} T_n(\gamma(u)) v \, d\sigma \\ &\le C_p \||\nabla u|^{p(x)-1}\|_{L^{p'(x)}(\Omega)} \|\nabla v\|_{L^{p(x)}(\Omega)} + C_p \||u|^{p(x)-1}\|_{L^{p'(x)}(\Omega)} \|v\|_{L^{p(x)}(\Omega)} \\ &\quad + n \int_\Omega |v| \, dx + n \int_{\partial\Omega} |v| \, d\sigma \end{aligned}$$

then A_n is bounded and hemi-continuous. For the coerciveness, thanks to $(H2)$ we have for any X_0,

$$\begin{aligned} \langle A_n u, u\rangle &= \int_\Omega |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{p(x)} \, dx + \int_\Omega T_n(\alpha(u)|\nabla u|^{p(x)}) u \, dx + \int_{\partial\Omega} T_n(\gamma(u)) u \, d\sigma \\ &\ge \int_\Omega |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{p(x)} \, dx. \end{aligned}$$

We deduce that

$$\frac{\langle A_n u, u\rangle}{\|u\|_{1,p(x)}} \longrightarrow +\infty \qquad as \quad \|u\|_{1,p(x)} \longrightarrow +\infty.$$

It remains to show that A_n is of type (M).
Let $(u_k)_k$ a sequence in X_0 such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } X_0, \\ A_n u_k \rightharpoonup \chi & \text{in } X_0', \\ \limsup_{k\to\infty} \langle A_n u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases} \tag{7}$$

We will prove that $\chi = A_n u$.
By (H1), we have $T_n(\alpha(u_k)|\nabla u_k|^{p(x)})u_k \geq 0$, $T_n(\gamma(u_k))u_k \geq 0$, and using the Fatou's lemma, we deduce that

$$\liminf_{k\to\infty} \left(\int_\Omega T_n(\alpha(u_k)|\nabla u_k|^{p(x)})u_k\, dx + \int_{\partial\Omega} T_n(\gamma(u_k))u_k\, d\sigma \right) \geq \int_\Omega T_n(\alpha(u)|\nabla u|^{p(x)})u\, dx + \int_{\partial\Omega} T_n(\gamma(u))u\, d\sigma. \tag{8}$$

On the other hand, thanks to the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} \left(\int_\Omega T_n(\alpha(u_k)|\nabla u_k|^{p(x)})v\, dx + \int_{\partial\Omega} T_n(\gamma(u_k))v\, d\sigma \right) = \int_\Omega T_n(\alpha(u)|\nabla u|^{p(x)})v\, dx + \int_{\partial\Omega} T_n(\gamma(u))v\, d\sigma$$

for any $v \in X_0$, consequently

$$T_n(\alpha(u_k)|\nabla u_k|^{p(x)}) + T_n(\gamma(u_k)) \rightharpoonup T_n(\alpha(u)|\nabla u|^{p(x)}) + T_n(\gamma(u)) \qquad \text{weakly in } X_0'$$

Thus, it follows that

$$Au_k \rightharpoonup \chi - T_n(\alpha(u)|\nabla u|^{p(x)}) + T_n(\gamma(u)) \qquad \text{weakly in } X_0'$$

As the operator A is of type (M), so we have immediately

$$Au = \chi - T_n(\alpha(u)|\nabla u|^{p(x)}) + T_n(\gamma(u)).$$

Therefore we deduce that $A_n u = \chi$.
Hence, the operator A_n is of type (M), which completes the proof of the Lemma 4.2.

Let $F_n = \langle T_n(f), T_n(g)\rangle$ which satisfy (H1); in view of the Lemma 4.2, there exists at least one solution $u_n \in X_0$ (cf. [17]) of the problem

$$\langle A_n u_n, v\rangle = \langle F_n, v\rangle \text{ for all } v \in X_0$$

i.e.

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla v\, dx + \int_\Omega |u_n|^{p(x)-2}u_n \nabla v\, dx + \int_\Omega T_n(\alpha(u_n)|\nabla u_n|^{p(x)})v\, dx + \int_{\partial\Omega} T_n(\gamma(u_n))v\, d\sigma = \int_\Omega T_n(f)v\, dx + \int_{\partial\Omega} T_n(g)v\, d\sigma. \tag{9}$$

Step2. A priori Estimates. **Assertion 1.** $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{p(x)}(\Omega))^N$.
Let $f_n = T_n(f)$ and $g_n = T_n(g)$ for all $n \in \mathbb{N}$, then $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are sequences of bounded functions which converges strongly to $f \in L^1(\Omega)$ and to $g \in L^1(\partial\Omega)$ respectively. Moreover

$$\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)} \text{ and } \|g_n\|_{L^1(\partial\Omega)} \leq \|g\|_{L^1(\partial\Omega)} \text{ for all } n \in \mathbb{N}.$$

We now take $v = T_k(u_n)$ as test function in (9) to get

$$\begin{aligned}&\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx + \int_\Omega |u_n|^{p(x)}\,dx + \int_\Omega T_n(\alpha(u_n)|\nabla u_n|^{p(x)})T_k(u_n)\,dx\\ &+\int_{\partial\Omega} T_n(\gamma(u_n))T_k(u_n)\,d\sigma = \int_\Omega T_n(f)T_k(u_n)\,dx + \int_{\partial\Omega} T_n(g)T_k(u_n)\,d\sigma.\end{aligned} \tag{10}$$

The third and fourth terms in the left-hand side of above equality are nonnegative, then

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx + \int_\Omega |T_k(u_n)|^{p(x)}\,dx \leq k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}). \tag{11}$$

Relation (11) implies, by the Proposition 2.2, that

$$\begin{aligned}\|T_k(u_n)\|^{\gamma}_{1,p(x)} &\leq k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)})\\ &\leq Ck \quad \text{for all} \quad k >,\end{aligned} \tag{12}$$

with

$$\gamma = \begin{cases} p_+ & \text{if} \quad \|T_k(u_n)\|_{1,p(x)} \leq 1,\\ p_- & \text{if} \quad \|T_k(u_n)\|_{1,p(x)} > 1.\end{cases}$$

We deduce that for any $k > 0$, the sequence $(T_k(u_n))_{n\in I\!N}$ is uniformly bounded in $W^{1,p(x)}(\Omega)$. Then, we can assume, up to a subsequence that,

$$T_k(u_n) \rightharpoonup v_k \qquad \text{in} \quad W^{1,p(x)}(\Omega)$$

and by the compact imbedding theorem, we have

$$T_k(u_n) \to v_k \qquad \text{in} \quad L^{p(x)}(\Omega) \qquad \text{and a.e. in } \Omega.$$

Assertion 2. $(u_n)_{n\in I\!N}$ converges in measure to some function u.
To prove this, we show that u_n is a Cauchy sequence in measure.
Let $k > 0$ be large enough. Relation (11) gives

$$\begin{aligned}k\ \text{meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx \leq \int_\Omega |T_k(u_n)|\,dx,\\ &\leq Ck^{\frac{1}{\gamma}}\end{aligned} \tag{13}$$

Therefore

$$\text{meas}\{|u_n| > k\} \leq Ck^{\frac{1}{\gamma}-1} \longrightarrow 0 \qquad \text{as} \quad k \to +\infty. \tag{14}$$

Moreover, for every fixed $t > 0$ and every positive $k > 0$, it is clear that

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\}$$

and hence

$$\begin{aligned}\text{meas }(\{|u_n - u_m| > t\}) \leq \text{meas }(\{|u_n| > k\}) &+ \text{meas }(\{|u_m| > k\})\\ +\text{meas }(\{|T_k(u_n) - T_k(u_m)| > t\}).\end{aligned} \tag{15}$$

Let $\epsilon > 0$, using (14), we choose $k = k(\epsilon)$ such that

$$\text{meas}(\{|u_n| > k\}) \leq \frac{\epsilon}{3} \qquad \text{and} \qquad \text{meas}(\{|u_m| > k\}) \leq \frac{\epsilon}{3}. \tag{16}$$

Since $T_k(u_n)$ converges strongly in $L^{p(x)}(\Omega)$, then it is a Cauchy sequence in $L^{p(x)}(\Omega)$, thus

$$\text{meas }(\{|T_k(u_n) - T_k(u_m)| > t\}) \leq \frac{\epsilon}{3} \qquad \text{for all} \quad n, m \geq n_0(k, t, \epsilon). \tag{17}$$

Finally, from (15), (16) and (17) we obtain

$$\text{meas}(\{|u_n - u_m| > t\}) \leq \epsilon \ \text{ for all } n, m \geq n_0(t, \epsilon) \tag{18}$$

which proves that the sequence $(u_n)_{n\in I\!N}$ is a Cauchy sequence in measure, and then converges almost everywhere to some measurable function u. Therefore,

$$\begin{array}{lll} T_k(u_n) \rightharpoonup T_k(u) & \text{in } W^{1,p(x)}(\Omega), & \\ T_k(u_n) \to T_k(u) & \text{in } \ L^{p(x)}(\Omega) & \text{and a.e. in } \ \Omega. \end{array} \tag{19}$$

Assertion 3. $(\nabla T_k(u_n))_{n\in I\!N}$ converges strongly to $(\nabla T_k(u))$ in $L^{p(x)}(\Omega)$.
To get this result, we need the following lemmas

Lemma 4.3. *For all $k > 0$, we have*

$$\int_{\{u_n\leq T_k(u)\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla(T_k(u_n)-T_k(u)) = \varepsilon_1(n), \tag{20}$$

where $\lim\limits_{n\to+\infty} \varepsilon_1(n) = 0$.

Proof. For the sake of simplicity we will write $\varepsilon_i(n,m)$ for any quantity such that

$$\lim_{m\to+\infty}\lim_{n\to+\infty} \varepsilon_i(n,m) = 0$$

and we will denote by $\varepsilon_i(n)$ for any quantity such that

$$\lim_{n\to+\infty} \varepsilon_i(n) = 0.$$

Let $\delta > 0$, we shall use in (9) the test function $\varphi_\lambda(T_k(u_n)-T_k(u))$, with $\varphi_\lambda(s) = se^{\lambda s^2}$, to get

$$\begin{aligned} &\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla\varphi_\lambda(T_k(u_n)-T_k(u))\,dx \\ &+\int_\Omega |u_n|^{p(x)-2}u_n\varphi_\lambda(T_k(u_n)-T_k(u))dx + \int_\Omega (T_n(\alpha(u_n)|\nabla u_n|^{p(x)})\varphi_\lambda(T_k(u_n)-T_k(u))dx \\ &+\int_{\partial\Omega} T_n(\gamma(u_n))\varphi_\lambda(T_k(u_n)-T_k(u))\,d\sigma \\ &= \int_\Omega T_n(f)\varphi_\lambda(T_k(u_n)-T_k(u))\,dx + \int_{\partial\Omega} T_n(g)\varphi_\lambda(T_k(u_n)-T_k(u))\,d\sigma. \end{aligned} \tag{21}$$

Note that the sign of $\varphi_\lambda(T_k(u_n)-T_k(u))$ is the same as that of u_n in the set $\{|u_n| > k\}$, then

$$\begin{aligned} &\int_{\{|u_n|>k\}} |u_n|^{p(x)-2}u_n\varphi_\lambda(T_k(u_n)-T_k(u))\,dx \\ &+\int_{\{|u_n|>k\}} (T_n(\alpha(u_n)|\nabla u_n|^{p(x)})\varphi_\lambda(T_k(u_n)-T_k(u))\,dx \geq 0 \end{aligned}$$

and

$$\int_{\partial\Omega\cap\{|u_n|>k\}} T_n(\gamma(u_n))\varphi_\lambda(T_k(u_n)-T_k(u)) \geq 0.$$

From (21) we deduce

$$\begin{aligned}
&\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\varphi_\lambda'(T_k(u_n)-T_k(u))\nabla(T_k(u_n)-T_k(u))\,dx\\
&+\int_{\Omega\cap\{|u_n|\le k\}} |T_k(u_n)|^{p(x)-2}T_k(u_n)\varphi_\lambda(T_k(u_n)-T_k(u))\,dx\\
&+\int_{\Omega\cap\{|u_n|\le k\}} T_n(\alpha(T_k(u_n))|\nabla T_k(u_n)|^{p(x)})\varphi_\lambda(T_k(u_n)-T_k(u))\,dx\\
&+\int_{\partial\Omega\cap\{|u_n|\le k\}} T_n(\gamma(u_n))\varphi_\lambda(T_k(u_n)-T_k(u))\,d\sigma\\
&\le \int_\Omega T_n(f)\varphi_\lambda(T_k(u_n)-T_k(u))\,dx+\int_{\partial\Omega} T_n(g)\varphi_\lambda(T_k(u_n)-T_k(u))\,d\sigma.
\end{aligned}\tag{22}$$

For the third term of the left-hand side of (22), we can write

$$\begin{aligned}
&\Big|\int_{\Omega\cap\{|u_n|\le k\}} (T_n(T_k(\alpha(T_k(u_n))|\nabla T_k(u_n)|^{p(x)})\varphi_\lambda(T_k(u_n)-T_k(u))\,dx\Big|\\
&\le M_k\int_\Omega |\nabla T_k(u_n)|^{p(x)}\varphi_\lambda(T_k(u_n)-T_k(u))\,dx
\end{aligned}\tag{23}$$

with $M_k=\sup_{|s|\le k}|\alpha(s)|$ and since

$$\begin{aligned}
&\Big|\int_{\Omega\cap\{|u_n|\le k\}} |T_k(u_n)|^{p(x)-2}T_k(u_n)\varphi_\lambda(T_k(u_n)-T_k(u))\,dx\Big|\\
&\le (k+1)^{p_+-1}\int_{\Omega\cap\{|u_n|\le k\}} |\varphi_\lambda(T_k(u_n)-T_k(u))|\,dx\longrightarrow 0 \qquad \text{as } n\to\infty
\end{aligned}$$

and

$$\lim_{n\to\infty}\int_{\partial\Omega} T_n(\gamma(u_n))\varphi_\lambda(T_k(u_n)-T_k(u))\,d\sigma=0,$$

$$\lim_{n\to\infty}\int_\Omega T_n(f)\varphi_\lambda(T_k(u_n)-T_k(u))\,dx=0,$$

$$\lim_{n\to\infty}\int_{\partial\Omega} T_n(g)\varphi_\lambda(T_k(u_n)-T_k(u))\,d\sigma=0.$$

So, we get from (22),

$$\begin{aligned}
&\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla\varphi_\lambda(T_k(u_n)-T_k(u))\,dx\\
&\qquad\le \varepsilon_1(n)+M_k\int_\Omega |\nabla T_k(u_n)|^{p(x)}\varphi_\lambda(T_k(u_n)-T_k(u))\,dx
\end{aligned}\tag{24}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla(T_k(u_n)-T_k(u))\phi_\lambda(u_n,u)\,dx\\
&=\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(T_k(u_n)-T_k(u))\,dx\\
&-M_k\int_\Omega |\nabla T_k(u_n)|^{p(x)}|\varphi_\lambda(T_k(u_n)-T_k(u))|\,dx\\
&+M_k\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla T_k(u)|\varphi_\lambda(T_k(u_n)-T_k(u))|\,dx\\
&-\int_\Omega |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla(T_k(u_n)-T_k(u))\phi_\lambda(u_n,u)\,dx,
\end{aligned}\tag{25}$$

where $\phi_\lambda(u_n,u)=\varphi'_\lambda(T_k(u_n)-T_k(u))-M_k|\varphi_\lambda(T_k(u_n)-T_k(u))|$
As

$$M_k\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla T_k(u)|\varphi_\lambda(T_k(u_n)-T_k(u))|\,dx=\varepsilon_2(n),$$

and

$$\int_\Omega |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla(T_k(u_n)-T_k(u))\phi_\lambda(u_n,u)\,dx=\varepsilon_3(n),$$

then, according to (24), we get

$$\int_\Omega [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla(T_k(u_n)-T_k(u))\phi_\lambda(u_n,u)\,dx \le \varepsilon_4(n). \tag{26}$$

Choosing $\lambda=\Big(\dfrac{M_k}{2}\Big)^2$, it is well known ([10], lemma 1) that,

$$\varphi'_\lambda(s)-M_k|\varphi_\lambda(s)|\ge\frac{1}{2},\quad \forall s\in I\!R. \tag{27}$$

It follows that,

$$\int_\Omega [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla(T_k(u_n)-T_k(u))\,dx\le 2\varepsilon_4(n). \tag{28}$$

Thus

$$\int_\Omega [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla(T_k(u_n)-T_k(u))\,dx\longrightarrow 0$$

as $n\to\infty$. □

Return now to the proof of assertion 3. According to (28), we have

$$\begin{aligned}\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx &\le 2\varepsilon_4(n)-\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx\\ &+\int_\Omega |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla T_k(u_n)\,dx\\ &+\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla T_k(u)\,dx.\end{aligned} \tag{29}$$

Since $|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\rightharpoonup|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)$ weakly in $L^{p'(x)}(\Omega)$ and $\nabla T_k(u_n)\rightharpoonup|\nabla T_k(u)|$ weakly in $L^{p(x)}(\Omega)$, then we get

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx\le\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx+\varepsilon_6(n).$$

i.e.

$$\limsup_{n\to\infty}\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx\le\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx.$$

Therefore, thanks to the Proposition 2.4, we deduce that

$$\nabla T_k(u_n)\to\nabla T_k(u) \text{ strongly in } (L^{p(x)}(\Omega))^N.$$

Assertion 4. $(u_n)_{n\in I\!N}$ converges a.e. on $\partial\Omega$ to some function v.
We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, then there exists a constant $C_5>0$ such that

$$\|T_k(u_n)-T_k(u)\|_{L^1(\partial\Omega)}\le C_5\|T_k(u_n)-T_k(u)\|_{W^{1,1}(\Omega)}.$$

Therefore,

$$T_k(u_n) \to T_k(u) \text{ in } L^1(\partial\Omega) \text{ and a.e. in } \partial\Omega.$$

we deduce that, there exists $A \subset \partial\Omega$ such that $T_k(u_n)$ converges to $T_k(u)$ on $\partial\Omega\backslash A$ with $\mu(A) = 0$, where μ is the area measure on $\partial\Omega$.
For every $k > 0$, let $A_k = \{x \in \partial\Omega : |T_k(u(x))| < k\}$ and $B = \partial\Omega \setminus \bigcup_{k>0} A_k$.
We have

$$\begin{aligned} \mu(B) = \frac{1}{k}\int_B |T_k(u)| d\sigma &\leq \frac{C_4}{k}\|T_k(u)\|_{W^{1,1}(\Omega)} \\ &\leq \frac{C_6}{k}\|T_k(u)\|_{1,p(x)}. \end{aligned} \tag{30}$$

We know that $\rho_{1,p(.)}(T_k(u_n)) \leq kM$ where M is a positive constant that does not depends on n, then,

$$\int_\Omega |T_k(u_n)|^{p(x)}\, dx + \int_\Omega |\nabla T_k(u_n)|^{p(x)}\, dx \leq kM. \tag{31}$$

We now use the Fatou's lemma in (31) to get

$$\int_\Omega |T_k(u)|^{p(x)}\, dx + \int_\Omega |\nabla T_k(u)|^{p(x)}\, dx \leq kM,$$

which is equivalent to

$$\rho_{1,p(.)}(T_k(u)) \leq kM. \tag{32}$$

According to (32), we deduce that

$$\|T_k(u)\|_{W^{1,p(x)}(\Omega)} \leq C_7\left(k^{\frac{1}{p^-}} + k^{\frac{1}{p^+}}\right).$$

Therefore, we get by letting $k \to +\infty$ in (30) that $\mu(B) = 0$.
Let us now define in $\partial\Omega$ the function v by

$$v(x) = T_k(u(x)) \text{ if } x \in A_k.$$

We take $x \in \partial\Omega\setminus(A \cup B)$; then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_n(x) - v(x) = (u_n(x) - T_k(u_n(x))) + (T_k(u_n(x)) - T_k(u(x))).$$

Since $x \in A_k$, we have $|T_k(u(x))| < k$ and so $|T_k(u_n(x))| < k$, from which we deduce that $|u_n(x)| < k$.
Therefore,

$$u_n(x) - v(x) = (T_k(u_n(x)) - T_k(u(x))) \to 0, \text{ as } n \to +\infty.$$

This means that u_n converges to v a.e. on $\partial\Omega$.

Assertion 5. u is a renormalized solution of the problem (1).
Let $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and let S be a smooth function with compact support in $I\!R$. We take $v = S(u_n)\varphi$ as a test function in (9) to get

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(S(u_n)\varphi)\, dx + \int_\Omega |u_n|^{p(x)-2}u_n(S(u_n)\varphi)\, dx \\ &+ \int_\Omega T_n(\alpha(u_n)|\nabla u_n|^{p(x)})S(u_n)\varphi\, dx + \int_{\partial\Omega} T_n(\gamma(u_n))S(u_n)\varphi\, d\sigma \\ &= \int_\Omega T_n(f)S(u_n)\varphi\, dx + \int_{\partial\Omega} T_n(g)S(u_n)\varphi\, d\sigma. \end{aligned} \tag{33}$$

The function S has compact support, then there exists a positive real number k such that $\mathrm{supp}(S) \subset [-k,k]$ which leads to

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(S(u_n)\varphi)\,dx &= \int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)S(u_n)\nabla\varphi\,dx \\ &\quad + \int_\Omega |\nabla T_k(u_n)|^{p(x)}S'(u_n)\varphi\,dx.\end{aligned} \tag{34}$$

As

$$|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) \rightharpoonup |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \text{ weakly in } (L^{p'(x)}(\Omega))^N,$$

and

$$S(u_n)\nabla\varphi \to S(u)\nabla\varphi \text{ strongly in } L^{p(x)}(\Omega),$$

hence

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)S(u_n)\nabla\varphi\,dx \to \int_\Omega |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)S(u)\nabla\varphi\,dx,$$

and as

$$|\nabla T_k(u_n)|^{p(x)} \to |\nabla T_k(u)|^{p(x)} \text{ in } L^1(\Omega)$$

it follows that

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}S'(u_n)\varphi\,dx \to \int_\Omega |\nabla T_k(u)|^{p(x)}S'(u)\varphi\,dx,$$

Then

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(S(u_n)\varphi)\,dx \to \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(S(u)\varphi)\,dx.$$

In the same way, it is easy to see that

$$\int_\Omega |u_n|^{p(x)-2}u_n(S(u_n)\varphi)\,dx \longrightarrow \int_\Omega |u|^{p(x)-2}u(S(u)\varphi)\,dx,$$

and

$$\int_\Omega T_n(\alpha(u_n)|\nabla u_n|^{p(x)})S(u_n)\varphi\,dx \to \int_\Omega \alpha(u)|\nabla u|^{p(x)}S(u)\varphi\,dx.$$

Moreover, we have

$$u_n \text{ converges to } u \text{ on } \partial\Omega.$$

So, by continuity of γ, it follows that

$$\int_{\partial\Omega} T_n(\gamma(u_n))S(u_n)\varphi\,d\sigma \to \int_{\partial\Omega}\gamma(u)S(u)\varphi\,d\sigma.$$

We can then pass to the limit as $n\to\infty$ in the equality (9), on the basis of results below and the facts that

$$T_n(f) \text{ converges to } f \text{ in } L^1(\Omega)$$

and

$$T_n(g) \text{ converges to } g \text{ in } L^1(\partial\Omega),$$

to concludes that u satisfy relation (6).
According to the Assertions 2,3,4, we deduce that $u \in T^{1,p(x)}_{tr}(\Omega)$.
Now, we claim that

$$\lim_{m\to+\infty}\frac{1}{m}\int_{\{m<|u|<2m\}}|\nabla u|^{p(x)}\,dx = 0. \tag{35}$$

Indeed, by taking $v = S_m(u_n) = T_m(u_n - T_m(u_n))$ in (9), where

$$T_m(s - T_m(s)) = \begin{cases} s - m.sign(s) & \text{if} \quad m < |s| < 2m, \\ m.sign(s) & \text{if} \quad |s| \geq 2m, \\ 0 & \text{if} \quad |s| \leq m, \end{cases}$$

we get,

$$\int_\Omega |\nabla S_m(u_n)|^{p(x)}\, dx \leq \int_\Omega |f| S_m(u_n)\, dx + \int_{\partial\Omega} |g| S_m(u_n)\, d\sigma.$$

Passing to the limit as $n \to \infty$, we obtain

$$\int_\Omega |\nabla S_m(u)|^{p(x)}\, dx \leq \int_\Omega |f| S_m(u)\, dx + \int_{\partial\Omega} |g| S_m(u)\, d\sigma.$$

Then, it follows that

$$\lim_{m\to\infty} \frac{1}{m} \int_\Omega |\nabla S_m(u)|^{p(x)}\, dx \leq 0,$$

which completes the proof of Theorem 4.1. □

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(Elhoussine Azroul, Abdelkrim Barbara, Mohamed Badr Benboubker) Laboratoire d'Analyse Mathématique et Applications (LAMA), Département de Mathématiques, Faculté des Sciences Dhar El Mahraz, B.P 1796 Atlas Fès, Morocco

E-mail address: azroul_elhoussine@yahoo.fr, abdelkrimbarbara@gmail.com, simo.ben@hotmail.com

(Stanislas Ouaro) Laboratoire d'Analyse Mathématique des Equations (LAME), UFR. Sciences Exactes et Appliquées,
Université de Ouagadougou, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso

E-mail address: ouaro@yahoo.fr,souaro@univ-ouaga.bf