Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 42(2), 2015, Pages 254–259
ISSN: 1223-6934

On lacunary strongly almost convergent double sequences of fuzzy numbers

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Abstract. In this paper we introduce the concept of lacunary almost convergence and lacunary strongly almost convergence for double sequence spaces of fuzzy numbers. We obtain some results related to these concepts.

2010 Mathematics Subject Classification. 40A05, 40A30, 40D25.
Key words and phrases. Almost convergence; strongly almost convergence; lacunary sequence; double sequence; fuzzy number.

1. Introduction

The concept of fuzzy sets was first introduced by L.A. Zadeh in 1965. Later on fuzzy logic became an important area of research in various branches of mathematics such as metric and topological spaces by Kaleva and Seikkala [1], Tripathy and Ray [21], Tripathy and Debnath [14], theory of functions by Wu [25], approximation theory by Anastassiou [1], and many others in different directions. Fuzzy set theory also finds its applications for modelling uncertainty and vagueness in various fields of science and engineering, e.g. computer programming by Giles [5], nonlinear dynamical systems by Hong and Sun [7], population dynamics by Barros et.al. [2], control of chaos by Fradkov and Evans [4], quantum physics by Madore [9], and many others. It attracted workers on sequence spaces to introduce different types of classes of sequences of fuzzy numbers.

A fuzzy real number X is a fuzzy set on $\mathbb{R}$, i.e. a mapping $X : \mathbb{R} \to L(= [0,1])$ associating each real number t with its grade of membership. The α - *level set* of a fuzzy real number X, for $0 \leq \alpha \leq 1$ denoted by $[X]^\alpha$ is defined by $[X]^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}$. A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$ then the fuzzy real number X is called *normal.* A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \epsilon))$, for all $a \in L$ is open in the usual topology of $\mathbb{R}$. The set of all upper semi-continuous, normal, convex fuzzy numbers is denoted by $L(\mathbb{R})$. Every real number r can be expressed as a fuzzy real number $\bar{r}$ as follows:

$$\bar{r}(t) = \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{otherwise.} \end{cases}$$

A *fuzzy real valued double sequence* $X = (X_{nk})$ is a double infinite array of fuzzy real numbers, i.e. $X_{nk} \in L(\mathbb{R})$, for all $n,\ k \in \mathbb{N}$.

Let E be the set of all closed bounded intervals $X = [X^L, X^R]$. Let $d(X, Y) = max(|X^L - Y^L|, |X^R - Y^R|)$. Then (E, d) is a complete metric space.

Received April 23, 2013. Revised December 18, 2013.

Let $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\bar{d}(X,Y) = \sup_{0 \le \alpha \le 1} d([X]^{\alpha}, [Y]^{\alpha}), \quad X, Y \in L(\mathbb{R}).$$

Then $\bar{d}$ defines a metric on $L(\mathbb{R})$.

The sequences of fuzzy numbers have been investigated by Tripathy and Baruah [10], Tripathy and Borgohain ([11], [12]), Tripathy and Das [13], Tripathy and Dutta ([15], [16], [17], [18]), Tripathy and Sarma ([22], [24]) and many others.

The initial work on double sequences of real or complex terms is found in Bromwich [3]. The notion of regular convergence of double sequences of real or complex terms is introduced by Hardy [6]. Some works on double sequences on crisp set are due to Tripathy and Sarma [23] and others. Tripathy and Dutta ([15], [16]) Tripathy and Sarma [24] introduced and investigated different types of fuzzy real valued double sequence spaces.

The aim of this present paper is to introduce and investigate lacunary almost convergence and lacunary strongly almost convergence of double sequences of fuzzy numbers and obtain some important results on them. Different classes of lacunary sequences have been introduced and investigated from different aspects by Tripathy and Baruah [10], Tripathy and Dutta [18], Tripathy and Dutta [19], Tripathy and Mahanta [20] and others.

2. Definitions and preliminaries

A double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exists two increasing sequences of integers $\{k_r\}$ and $\{l_s\}$ such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

$$\text{and } l_0 = 0, \hat{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Throughout we denote $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \hat{h}_s$ for all $r, s \in \mathbb{N} \cup \{0\}$; $q_r = \dfrac{k_r}{k_{r-1}}$, $\hat{q}_s = \dfrac{l_s}{l_{s-1}}$, $q_{r,s} = q_r \hat{q}_s$. Intervals determined by $\theta_{r,s}$ are denoted by $I_{r,s} = \{(k,l) : k_{r-1} \le k \le k_r \text{ and } l_{s-1} \le l \le l_s\}$. Also, $\hat{h}_{r,s} = k_r l_s - k_{r-1} l_{s-1}$. $\theta = \hat{\theta}_{r,s}$ is determined by $\hat{I}_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \text{ or } l_{s-1} < l \le l_s\} \setminus (I^1 \cup I^2)$, where $I^1 = \{(k,l) : k_{r-1} < k \le k_r \text{ and } l_s < l < \infty\}$, $I^2 = \{(k,l) : l_{s-1} < l \le l_s \text{ and } k_r < k < \infty\}$.

Denote

$$B_{\alpha,\beta}^{m,n} = \{(i,j) : \alpha < i \le \alpha + m \text{ or } \beta < j \le \beta + n\} \setminus (I_B^1 \cup I_B^2),$$

where $I_B^1 = \{(i,j) : \beta < j \le \beta + n \text{ and } \alpha + m < i < \infty\}$, $I_B^2 = \{(i,j) : \alpha < i \le \alpha + m \text{ and } \beta + m < j < \infty\}$,

$$A_{\alpha,\beta}^{x,y} = \{(i,j) : \alpha + x h_r < i \le \alpha + (x+1) h_r \text{ or } \beta + y \hat{h}_s < j \le \beta + (y+1)\hat{h}_s\} \setminus (I_A^1 \cup I_A^2),$$

where $I_A^1 = \{\alpha + x h_r < i \le \alpha + (x+1) h_r \text{ and } \beta + (y+1)\hat{h}_s < j < \infty\}$, $I_A^2 = \{(i,j) : \beta + y\hat{h}_s < j \le \beta + (y+1)\hat{h}_s \text{ and } \alpha + (x+1)h_r < i < \infty\}$.

Definition 2.1. A fuzzy real valued double sequence $X = \langle X_{nk} \rangle$ is said to be *almost convergent* to the fuzzy real number X_0, if

$$P - \lim_{n,k} \frac{1}{nk} \bar{d}(X_{n+r,k+s}, X_0) = 0, \text{ uniformly in } r, s \ge 0.$$

Definition 2.2. A fuzzy real valued double sequence $X = \langle X_{nk} \rangle$ is said to be *strongly almost convergent* to the fuzzy real number X_0, if

$$P - \lim_{n,k} \frac{1}{nk} \sum_{i,j=1,1}^{n,k} \bar{d}(X_{i+r,j+s}, X_0) = 0, \text{ uniformly in } r, s \geq 0.$$

Throughout, ${}_2\hat{C}^F$, ${}_2\hat{S}^F$ denote the set of all almost convergent and strongly almost convergent fuzzy real valued double sequences respectively. Consider the set

$${}_2S_\theta^F = \{X = (X_{nk}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} \hat{d}(X_{i+r,j+s}, X_0) = 0,$$

$$\text{uniformly in } r, s \geq 0, \text{ for some } X_0 \in L(\mathbb{R})\}.$$

It follows from the following example that ${}_2\hat{S}^F \neq_2 S_\theta^F$.

Example 2.1. Let $\theta_{r,s}$ be such that $\{l_s\}$ is a lacunary sequence of even integers. Consider the sequence (X_{nk}) defined by: for all $n \in \mathbb{N}$ and for all $k = 2n$

$$X_{nk}(t) = \begin{cases} 1 + \dfrac{t}{n+k}, & \text{for } -(n+k) \leq t \leq 0, \\ 1 - \dfrac{t}{n+k}, & \text{for } 0 \leq t \leq n+k, \\ 0, & \text{otherwise,} \end{cases}$$

otherwise $X_{nk} = \bar{0}$.

Then $(X_{nk}) \in {}_2S_\theta^F$ but $(X_{nk}) \notin {}_2\hat{S}^F$.

Now consider the sequence (X_{nk}) defined by: for $n = 1, 2, \ldots\ldots, \sqrt{h_{r,s}}$, $X_{nl} = \bar{n}$, otherwise $X_{nk} = \bar{0}$. Then

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} \bar{d}(X_{i+r,j+s}, \bar{0}) = P - \lim_{h_{r,s}} \frac{1}{h_{r,s}} \frac{\sqrt{h_{r,s}}(\sqrt{h_{r,s}+1})}{2} \neq 0.$$

So $(X_{nk}) \notin {}_2S_\theta^F$ but $(X_{nk}) \in {}_2\hat{S}^F$.

Now we consider the following classes of sequences replacing $h_{r,s}$ and $I_{r,s}$ by $\hat{h}_{r,s}$ and $\hat{I}_{r,s}$ respectively.

Definition 2.3. A fuzzy real valued double sequence $X = \langle X_{nk} \rangle$ is said to be *lacunary almost convergent* to the fuzzy real number X_0, if for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{\hat{h}_{r,s}} |\{(n,k) \in \hat{I}_{r,s} : \bar{d}(X_{n+r,k+s}, X_0)\}| = 0, \text{ uniformly in } r, s \geq 0,$$

where the vertical bars denote the cardinality of the enclosed set.

Definition 2.4. A fuzzy real valued double sequence $X = \langle X_{nk} \rangle$ is said to be *lacunary strongly almost convergent* to the fuzzy real number X_0, if

$$P - \lim_{r,s} \frac{1}{\hat{h}_{r,s}} \sum_{(i,j) \in I_{r,s}} \bar{d}(X_{i+r,j+s}, X_0) = 0, \text{ uniformly in } r, s \geq 0.$$

Let ${}_2\hat{S}_\theta^F$, ${}_2\hat{M}_\theta^F$ denote the sets of all lacunary almost convergent and lacunary strongly almost convergent fuzzy real valued double sequences respectively.

3. Main results

In this section we establish results on lacunary almost convergent and strongly almost convergent sequences of fuzzy numbers.

Lemma 3.1. If for given $\varepsilon > 0$, there exist m_0, n_0, p_0 and q_0 such that

$$\frac{1}{|B_{p,q}^{m,n}|} \sum_{(i,j)\in B_{p,q}^{m,n}} \bar{d}(X_{ij}, X_0) < \varepsilon$$

for $m \geq m_0', n \geq n_0', p \geq p_0, q \geq q_0$ and for some $X_0 \in L(\mathbb{R})$, then $X = \langle X_{nk} \rangle \in {}_2\hat{S}^F$.

Proof. Let $\varepsilon > 0$ be given and there exist m_0', n_0', p_0 and q_0 such that

$$\frac{1}{|B_{p,q}^{m,n}|} \sum_{(i,j)\in B_{p,q}^{m,n}} \bar{d}(X_{ij}, X_0) < \varepsilon \tag{1}$$

for $m \geq m_0', n \geq n_0', p \geq p_0, q \geq q_0$.

We show that for a given $\varepsilon > 0$, there exist m_0'' and n_0'' such that

$$\frac{1}{|B_{p,q}^{m,n}|} \sum_{(i,j)\in B_{p,q}^{m,n}} \bar{d}(X_{ij}, X_0) < \frac{\varepsilon}{2} \tag{2}$$

for $m, n \geq m_0'', n_0''$ and $(p, q) \in D$, where $D = \{(i, j) : 0 \leq i \leq p_0 \text{ and } 0 \leq j \leq q_0\}$. Let $m_0 = \max\{m_0', m_0''\}$ and $n_0 = \max\{n_0', n_0''\}$, then (2) will hold for all $m \geq m_0$, $n \geq n_0$ and for all p, q which gives the result.

Since p_0, q_0 are fixed, we have

$$\sum_{\{(i,j):p\leq i\leq p_0 \cup q\leq j\leq q_0\}} \bar{d}(X_{ij}, X_0) = K \text{ is finite.}$$

Thus $(p, q) \in D$ implies that $p \leq p_0$ and $q \leq q_0$. So for $m \geq p_0$, $n \geq n_0$, we get from (1)

$$\begin{aligned}\frac{1}{|B_{p,q}^{m,n}|} \sum_{(i,j)\in B_{p,q}^{m,n}} \bar{d}(X_{ij}, X_0) \quad &= \frac{1}{|B_{p,q}^{m,n}|} \sum_{\{(i,j):p\leq i\leq p_0 \cup q\leq j\leq q_0\}} \bar{d}(X_{ij}, X_0) \\ &+ \frac{1}{|B_{p,q}^{m,n}|} \textstyle\sum_{(i,j)\in B_{p_0,q_0}^{m,n}} \bar{d}(X_{ij}, X_0) \leq \frac{K}{|B_{p,q}^{m,n}|} + \frac{\varepsilon}{2} < \varepsilon\end{aligned}$$

for sufficiently large m and n. Hence the result. □

Theorem 3.2. *For every* $\hat{\theta}_{r,s}$, ${}_2\hat{M}_\theta^F = {}_2\hat{S}^F$.

Proof. Let $X = \langle X_{nk} \rangle \in {}_2\hat{M}_\theta^F$. Then given $\varepsilon > 0$ there exist r_0, s_0 and $X_0 \in L(R)$ such that

$$\frac{1}{\hat{h}_{r,s}} \sum_{(i,j)\in \bar{I}_{r,s}} \bar{d}(X_{i+r,j+s}, X_0) < \varepsilon, \text{ whenever } r \geq r_0, s \geq s_0.$$

which implies that

$$\frac{1}{\hat{h}_{r,s}} \sum_{(i,j)\in B_{p,q}^{h_r,\bar{h}_s}} \bar{d}(X_{ij}, X_0) < \varepsilon$$

whenever $r \geq r_0, s \geq s_0, p = k_{r-1} + \alpha + 1, q = l_{s-1} + \beta + 1$.

Let $m \geq h_r$ such that $m = \delta_1 h_r + \theta_1$ where δ_1 is an integer and $0 \leq \theta_1 \leq h_r$. Also let

$n \geq \hat{h}_s$ such that $n = \delta_2 \hat{h}_s + \theta_2$ where δ_2 is an integer and $0 \leq \theta_2 \leq \hat{h}_s$.
Since $m \geq h_r$ for $\delta_1 \geq 1$ and $n \geq \hat{h}_s$ for $\delta_2 \geq 1$, we have

$$\begin{aligned}
\frac{1}{|B_{p,q}^{m,n}|} \sum_{(i,j)\in B_{p,q}^{m,n}} \bar{d}(X_{ij}, X_0) &\leq \frac{1}{|B_{p,q}^{m,n}|} \sum_{(i,j)\in B_{p,q}^{(\delta_1+1)h_r,(\delta_2+1)\bar{h}_s}} \bar{d}(X_{ij}, X_0) \\
&= \frac{1}{|B_{p,q}^{m,n}|} \sum_{r=0,s=0,(i,j)\in A_{\alpha,\beta}^{r,s}}^{\delta_1,\delta_2} \bar{d}(X_{ij}, X_0) \\
&\leq \frac{1}{|B_{p,q}^{m,n}|} \sum_{r=0,s=0}^{\delta_1,\delta_2} \hat{h}_{r,s}\varepsilon \leq \frac{(\delta_1+1)(\delta_2+1)}{|B_{p,q}^{m,n}|} \hat{h}_{r,s}\varepsilon = o(1).
\end{aligned}$$

Hence by Lemma 3.1, ${}_2\hat{M}_\theta^F \subseteq {}_2\hat{S}^F$.

Also ${}_2\hat{S}^F \subseteq {}_2\hat{M}_\theta^F$ for every $\bar{\theta}_{r,s}$. Thus ${}_2\hat{M}_\theta^F ={}_2 \hat{S}^F$.
This completes the proof. □

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