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Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data

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Abstract. We study a nonlinear elliptic problem governed by a general Leray-Lions operator with variable exponents and diffuse Radon measure data that does not charge the sets of zero $p(.)$-capacity. We prove a decomposition theorem for these measures (more precisely, as a sum of a function in $L^1(\Omega)$ and of a measure in $W^{-1,p'(\cdot)}(\Omega)$) and an existence and uniqueness result of entropy solution.

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1. Introduction and main results

Our aim is to study the existence and uniqueness of a solution for the nonlinear boundary value problem of the form

$$P(\beta,\mu):\begin{cases} -\nabla\cdot a(x,\nabla u)+\beta(u)\ni\mu & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where β is a maximal monotone graph on $\mathbb{R}$ such that $0\in\beta(0)$, the vector field a is a Leray-Lions operator with variable exponent, μ is a bounded Radon diffuse measure and $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain ($N\geq 1$).
In [15], the authors studied the following problem

$$\begin{cases} -\Delta u+g(u)=\mu & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{1.1}$$

where μ is a bounded measure in Ω and $g:(-\infty,1)\to\mathbb{R}$ is a continuous nondecreasing function such that $g(0)=0$ and

$$\lim_{t\uparrow 1} g(t)=+\infty.\tag{1.2}$$

It is well-known that (see, e.g., [7]) a solution of (1.1), whenever it exists, is unique. It has also been proved by Boccardo [4] (in the spirit of Brezis-Strauss [10]) that, for every $\mu\in L^1(\Omega)$, problem (1.1) has a solution. Moreover, Boccardo also shows that problem (1.1) has no solution if μ is a Dirac mass δ_a, with $a\in\Omega$. Consequently, in [15], the authors introduced the notion of good measure. They said that μ is a good measure (relative to g) if problem (1.1) has a solution u.

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We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measures in Ω, equipped with its standard norm $\|.\|_{\mathcal{M}_b(\Omega)}$. Given $\nu \in \mathcal{M}_b(\Omega)$, we say that ν is diffuse with respect to the capacity $W_0^{1,p}(\Omega)$ (p-capacity for short) if $\nu(E) = 0$, for every set E such that $\mathrm{Cap}_p(E,\Omega) = 0$. The p-capacity of every subset E with respect to Ω is defined as

$$\mathrm{Cap}_p(E,\Omega) = \inf\left\{\int_\Omega |\nabla u|^p\,dx\right\}$$

and the infimum is taken on all functions $u \in W_0^{1,p}(\Omega)\cap C_0(\Omega)$ such that $u = 1$ almost everywhere on E, $u \geq 0$ almost everywhere on Ω. The set of bounded Radon diffuse measures in the constant exponent setting is denoted by $\mathcal{M}_b^p(\Omega)$.

Moreover, in [15], the authors introduced for the problem (1.1) the notion of reduced measure denoted by μ^* associated with μ. It corresponds to the right measure that we can associate with μ such that problem (1.1) with μ replaced by μ^* has a unique solution. Indeed, by using natural approximation scheme (keep μ fixed and approximate g or keep g fixed and approximate μ) and passing to the limit in the equation they characterized the right part of μ for which the problem (1.1) is well-posed. This approach was deeply analyzed and studied in the literature for the laplacian (see [7], [8], [9] and [15]).

In [18], the authors used a different approach to study the problem $P(\beta,\mu)$ where the vector field a is a Leray-Lions operator with constant exponent. For the maximal monotone graph β, the authors set in [18] the following.

$$\mathrm{int}(\mathrm{dom}\beta) = (m,M) \quad \text{with } -\infty \leq m \leq 0 \leq M \leq +\infty.$$

Recall that a Leray-Lions operator with constant exponent (see [21]) is a Carathéodory function $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ (i.e. $a(x,\xi)$ is measurable in $x \in \Omega$ for every $\xi \in \mathbb{R}^N$ and continuous in $\xi \in \mathbb{R}^N$, for almost every $x \in \Omega$) such that

- there exists $\lambda > 0$ such that $\forall \xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$,
$$a(x,\xi).\xi \geq \lambda|\xi|^p; \tag{1.3}$$
- for any $(\xi,\eta) \in \mathbb{R}^N\times\mathbb{R}^N$ with $\xi \neq \eta$ and a.e. $x \in \Omega$,
$$\Big(a(x,\xi) - a(x,\eta)\Big).\Big(\xi-\eta\Big) > 0; \tag{1.4}$$
- there exists $\Lambda > 0$ such that for a.e. $x \in \Omega$ and for any $\xi \in \mathbb{R}^N$,
$$\Big|a(x,\xi)\Big| \leq \Lambda\Big(j_1(x) + |\xi|^{p-1}\Big) \tag{1.5}$$

where j_1 is a nonnegative function in $L^{p'}(\Omega)$ with $p' = \dfrac{p}{p-1}$.

Indeed, as a consequence of the preceding arguments in [15], it is clear that the standard notion of weak solution neither standard renormalized/entropy solution is not the natural one for $P(\beta,\mu)$ when μ is a Radon measure. Indeed, the singular part of μ with respect to Lebesgue measure creates an obstruction to the existence of such kind of solutions. This is related to the fact that passing to the limit in the approximation scheme, singular parts may appear in the equation and need to be treated. In [18], the authors analyzed and studied the main feature of these quantities in the case of $\mu \in \mathcal{M}_b^p(\Omega)$ and of maximal monotone graph β. Handling these parts give the right notion of solutions for $P(\beta,\mu)$ when $\mu \in \mathcal{M}_b^p(\Omega)$. This notion of solution is such that any measure in $\mathcal{M}_b^p(\Omega)$ is a good measure for $P(\beta,\mu)$.

Recall that, taking the nonlinearity g satisfying (1.2), the authors of [15] have shown that, there exists a diffuse measure μ with respect to the capacity $H^1(\Omega)$ such that the problem (1.1) has no weak solution. So, in general, bounded Radon diffuse measures are not good measures for problem (1.1) with respect to the standard notion of weak solution. But, it is a good measure for (1.1) with respect to the notion of solution introduced in [18]. Moreover, in [18], the authors proved that if

$$\text{int}(\text{dom}\beta) = (m, M) \quad \text{with } -\infty < m \leq 0 \leq M < +\infty,$$

then the reduced measure and the good measure coincides i.e. any measure in $\mathcal{M}_b^p(\Omega)$ is a reduced measure in the sense that the solution is unique.
In this paper, we generalized the work in [18] for the case of variable exponents. Indeed, in [18], the authors used the following famous decomposition theorem of measure in $\mathcal{M}_b^p(\Omega)$.

Theorem 1.1. (see [5], Theorem 2.1) *Let p be a real number such that $1 < p < +\infty$. Let μ be an element of $\mathcal{M}_b(\Omega)$. Then $\mu \in L^1(\Omega)+W^{-1,p'}(\Omega)$ if and only if $\mu \in \mathcal{M}_b^p(\Omega)$.*

In this paper we prove an equivalent theorem for the variable exponent setting. Given $\nu \in \mathcal{M}_b(\Omega)$, we say that ν is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$ ($p(.)$-capacity for short) if $\nu(E) = 0$ for every set E such that $\text{Cap}_{p(.)}(E, \Omega) = 0$. The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$.
We firstly prove a decomposition theorem for measures in $\mathcal{M}_b^{p(.)}(\Omega)$. More precisely, we prove the following.

Theorem 1.2. *Let $p(.) : \overline{\Omega} \to (1, +\infty)$ be a continuous function and $\mu \in \mathcal{M}_b(\Omega)$. Then $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$ if and only if $\mu \in L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.*

To give our notion of solution and the main results, we set

$$\text{int}(\text{dom}\beta) = (m, M) \quad \text{with } -\infty < m \leq 0 \leq M < +\infty.$$

For any $r \in \mathbb{R}$ and any measurable function u on Ω, $[u = r]$, $[u \leq r]$ and $[u \geq r]$ denote, respectively, the sets $\{x \in \Omega : u(x) = r\}$, $\{x \in \Omega : u(x) \leq r\}$ and $\{x \in \Omega : u(x) \geq r\}$.
Our main result is stated as follows:

Theorem 1.3. *For any $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$, the problem $P(\beta, \mu)$ has at least one solution (u, w) in the sense that $(u, w) \in W_0^{1,p(.)}(\Omega) \times L^1(\Omega)$, $u \in dom(\beta)$ $\mathcal{L}^N -$ a.e. in Ω, $w \in \beta(u)$ $\mathcal{L}^N -$ a.e. in Ω, there exists $\nu \in \mathcal{M}_b^{p(.)}(\Omega)$ such that $\nu \perp \mathcal{L}^N$,*

$$\nu^+ \text{ is concentred on } [u = M], \ \nu^- \text{ is concentred on } [u = m]$$

and

$$\int_\Omega a(x, \nabla u).\nabla\xi dx + \int_\Omega w\xi dx + \int_\Omega \xi d\nu = \int_\Omega \xi d\mu, \tag{1.6}$$

for any $\xi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$. Moreover

$$\nu^+ \leq \mu_s\lfloor\ [u = M] \tag{1.7}$$

and

$$\nu^- \leq -\mu_s\lfloor\ [u = m]. \tag{1.8}$$

The connexion between our notion of solution and the entropic formulation (see [3]) of the solution is given in the following theorem.

Theorem 1.4. *1. If (u,w) is a solution of $P(\beta,\mu)$ in the sense of Theorem 1.3, then (u,w) is a solution in the following sense : for any $\xi \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ such that $\xi \in dom\beta$,*

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-\xi)dx + \int_\Omega wT_k(u-\xi)dx \le \int_\Omega T_k(u-\xi)d\mu, \quad \text{for any } k>0. \tag{1.9}$$

2. The solution of $P(\beta,\mu)$ is unique.

In particular, this equivalent formulation of entropy solution and the notion of solution in Theorem 1.3 is very useful to prove the uniqueness of solution of problem $P(\beta,\mu)$. Recalling that the notion of entropy solution was used in [2], to get the existence of solutions for the problem

$$\begin{cases} -\text{div}(a(x,u,\nabla u)) + g(x,u,\nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

under the assumption that μ is a measure in $L^1(\Omega)+W^{-1,p'(.)}(\Omega)$.
In [23], the authors proved the existence and uniqueness of entropy solution of $P(\beta,\mu)$, where $\mu \in L^1(\Omega)$. This work also generalizes the ones done in [2], [23].
The paper is organized as follows. In section 2, we state some basic results, we prove the Theorem 1.2 in section 3 and finally, in section 4, we deal with the proofs of Theorem 1.3 and Theorem 1.4.

2. Assumptions and preliminary

We study the problem $P(\beta,\mu)$ for a continuous variable exponent $p(.)$. More precisely, we assume that

$$p(.):\Omega \to (1,+\infty) \text{ is continuous such that } 1<p_- \le p_+ < +\infty, \tag{2.1}$$

where $p_- := ess \inf_{x\in\Omega} p(x)$ and $p_+ := ess \sup_{x\in\Omega} p(x)$.
We assume that the vector field $a:\Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that:

- There exists a positive constant C_1 such that

$$|a(x,\xi)| \le C_1(j(x)+|\xi|^{p(x)-1}) \tag{2.2}$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^N$ where j is a non negative function in $L^{p'(.)}(\Omega)$.

- For almost every $x\in\Omega$ and for every $\xi,\eta\in\mathbb{R}^N$ with $\xi\neq\eta$,

$$(a(x,\xi)-a(x,\eta)).(\xi-\eta)>0. \tag{2.3}$$

- There exists a positive constant C_2 such that

$$a(x,\xi).\xi \ge C_2\,|\xi|^{p(x)} \tag{2.4}$$

for almost every $x\in\Omega$, $C_2>0$ and for every $\xi\in\mathbb{R}^N$.
As the exponent $p(.)$ appearing in (2.2) and (2.4) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(.)} = \inf\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space.
Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.
Finally, we have the Hölder type inequality:

$$\left| \int_\Omega uvdx \right| \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-} \right) |u|_{p(.)} \, |v|_{p'(.)} \,, \tag{2.5}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.
Let

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equiped with the following norm

$$\|u\|_{1,p(.)} = |u|_{p(.)} + |\nabla u|_{p(.)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|.\|_{1,p(.)})$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result [16]:

Proposition 2.1. *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold true:*
(i) $|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p_-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_+}$;
(ii) $|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p_+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_-}$;
(iii) $|u|_{p(.)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow \rho_{p(.)}(u) < 1$ (*respectively* $= 1; > 1$);
(iv) $|u_n|_{p(.)} \to 0$ (*respectively* $\to +\infty$) $\Leftrightarrow \rho_{p(.)}(u_n) \to 0$ (*respectively* $\to +\infty$);
(v) $\rho_{p(.)} \left(u / |u|_{p(.)} \right) = 1$.

Next, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ under the norm

$$\|u\| := |\nabla u|_{p(.)} \,.$$

The space $(W_0^{1,p(\cdot)}(\Omega), \|.\|)$ is a separable and reflexive Banach space. For more details about Lebesgue and Sobolev spaces with variable exponent, we refer to [14], [19] and the references therein.
We now recall some notations. For any given $l, k > 0$, we define the function h_l by $h_l(r) = \min((l + 1 - |r|)^+, 1)$ and the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = \max\{-k, \min(k, s)\}$.
We set

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R}, \text{ measurable such that } T_k(u) \in W_0^{1,p(\cdot)}(\Omega), \text{ for any } k > 0 \right\}.$$

For any $l_0 > 0$, we define $h_0 = h_{l_0}$ by

$$\begin{cases} h_0 \in C_c^1(\mathbb{R}), h_0(r) \geq 0 \quad \forall r \in \mathbb{R}, \\ h_0(r) = 1 \text{ if } |r| \leq l_0 \text{ and } h_0(r) = 0 \text{ if } |r| \geq l_0 + 1. \end{cases}$$

If γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ, i.e.

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s) \text{ if } \gamma(s) \neq \emptyset, \\ \\ +\infty \text{ if } [s, +\infty) \cap D(\gamma) = \emptyset, \\ \\ -\infty \text{ if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

We give some useful convergence results.

Lemma 2.2. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x, y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$. We suppose that: $\forall n \geq 1$, $w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then*

$$z \in dom(\beta).$$

To prove Lemma 2.2, we need the "biting lemma of Chacon" [12]. Let us recall it.

Lemma 2.3. ***(biting lemma of Chacon).***
Let $\Omega \subset \mathbb{R}^N$ be an open bounded set of $\mathbb{R}^N$ and $(f_n)_{n\in\mathbb{N}^*}$ a bounded sequence in $L^1(\Omega)$. Then there exist $f \in L^1(\Omega)$, a sequence $(f_{n_k})_{k\in\mathbb{N}^*}$ and a sequence of measurable sets $(E_j)_{j\in\mathbb{N}^*}$, $E_j \subset \Omega$, $\forall j \in \mathbb{N}$ with $E_{j+1} \subset E_j$ and $\lim_{j\to+\infty} |E_j| = 0$, such that for any $j \in \mathbb{N}$, $f_{n_k} \rightharpoonup f$ in $L^1(\Omega \backslash E_j)$.

Proof. (Proof of Lemma 2.2.) Since the sequence $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$, using the "biting lemma of Chacon", there exist $w \in L^1(\Omega)$, a subsequence $(w_{n_k})_{k\geq 1}$ and a sequence of measurable sets $(E_j)_{j\in\mathbb{N}}$ in Ω such that $E_{j+1} \subset E_j$, $\forall j \in \mathbb{N}$, $\lim_{j\to+\infty} |E_j| = 0$ and $\forall j \in \mathbb{N}$, $w_{n_k} \rightharpoonup w$ in $L^1(\Omega \backslash E_j)$. Since $z_{n_k} \to z$ in $L^1(\Omega)$ and so in $L^1(\Omega \backslash E_j)$, $\forall j \in \mathbb{N}$ and $\beta_{n_k} \to \beta$ in the sense of graphs, we have $w \in \beta(z)$ a.e. in $\Omega \backslash E_j$. Thus $z \in dom(\beta)$ a.e. in $\Omega \backslash E_j$. Finally, we obtain $z \in dom(\beta)$ a.e. in Ω. $\square$

For $E \subset \Omega$, we denote

$$S_{p(.)}(E) = \{u \in W_0^{1,p(.)}(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } E, \ u \geq 0 \text{ on } \Omega\}.$$

The Sobolev $p(.)-$capacity of E is defined by

$$\text{Cap}_{p(.)}(E, \Omega) = \inf_{u \in S_{p(.)}(E)} \int_\Omega |\nabla u|^{p(x)} dx.$$

In the case $S_{p(.)}(E) = \emptyset$, we set $\text{Cap}_{p(.)}(E, \Omega) = +\infty$.
If a property $P(x)$ holds for all $x \in \Omega$ excepted for a set of zero $p(.)-$capacity, we say that the property $P(x)$ holds quasi-everywhere on Ω and we note $P(x)$ holds q.e..
We say that a function $g : \Omega \longrightarrow \mathbb{R}$ is quasi continuous if for every $\epsilon > 0$, there exists an open set $A \subset \Omega$ with $\text{Cap}_{p(.)}(A, \Omega) < \epsilon$ such that g is continuous on $\Omega \backslash A$.
Every $u \in W^{1,p(.)}(\Omega)$ has a quasi continuous representative denoted by $\tilde{u}$ which is essentially unique. In fact if φ_1 and φ_2 are quasi continuous, and $\varphi_1 = \varphi_2$ a.e. (with

respect to Lebesgue measure), then $\varphi_1 = \varphi_2$ q.e..
We say that a sequence of functions $(g_n)_{n\geq 0}$ converges to g q.e. if $\lim\limits_{n\to+\infty} g_n(x) = g(x)$ q.e.. For more details about $p(.)$-capacity, we refer to [14], [17] and the references therein.

3. Proof of the decomposition theorem of a measure in $\mathcal{M}_b^{p(.)}(\Omega)$

Before we prove the Theorem 1.2, we need the following result.

Proposition 3.1. *For any nonnegative measure $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$ there exist a non negative Radon measure γ with $\gamma \in W^{-1,p'(.)}(\Omega)$ and a nonnegative function h with $h \in L^1(\Omega,\gamma)$ such that $\mu = h\gamma$.*

For the proof of Proposition 3.1 we need the following lemma.

Lemma 3.2. *If $(u_n)_{n\in\mathbb{N}}$ is a sequence in $W^{1,p(.)}(\Omega)$ which converges to u in $W^{1,p(.)}(\Omega)$ then there exits a subsequence $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ of $(\tilde{u}_n)_{n\in\mathbb{N}}$ which converges to $\tilde{u}$ q.e.*

Proof. According to Proposition 2.1, the proof is similar to the proof of Theorem 4 in [22] (see also [13] for more details). □

Proof. (Proof of Proposition 3.1) Let $F : W^{1,p(.)}(\Omega) \longrightarrow [0,+\infty)$ be defined by $F(u) = \displaystyle\int_\Omega \max\{\tilde{u}, 0\} d\mu$ where $\tilde{u}$ denotes the quasi continuous representative of u. The function F is convex and lower semicontinuous on $W^{1,p(.)}(\Omega)$ (the lower semicontinuity follows from Fatou's Lemma and Lemma 3.2). Since $W^{1,p(.)}(\Omega)$ is separable, the function F is the supremum of a countable family of continuous affine functions. Therefore there exit a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $W^{-1,p'(.)}(\Omega)$ and a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $F(u) = \sup\limits_{n\in\mathbb{N}} \left[\left\langle \gamma_n, u\right\rangle + a_n\right]$, for every $u \in W^{1,p(.)}(\Omega)$.
Since $F(0) = 0$, we have $a_n \leq 0$, hence

$$F(u) \leq \sup_{n\in\mathbb{N}} \left\langle \gamma_n, u\right\rangle. \tag{3.1}$$

For every $t > 0$ and for every $u \in W^{1,p(.)}(\Omega)$, we have,

$$t\left\langle \gamma_n, u\right\rangle + a_n \leq F(tu) = tF(u).$$

Dividing by t and letting t goes to $+\infty$, we get $\left\langle \gamma_n, u\right\rangle \leq F(u)$. Hence, from (3.1) it follows that

$$F(u) = \sup_{n\in\mathbb{N}} \left\langle \gamma_n, u\right\rangle. \tag{3.2}$$

For every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$, we have

$$-\left\langle \gamma_n, \varphi\right\rangle = \left\langle \gamma_n, -\varphi\right\rangle \leq F(-\varphi) = 0,$$

thus $\left\langle \gamma_n, \varphi\right\rangle \geq 0$. By the Riesz representation Theorem, there exits a nonnegative Radon measure on Ω still denoted by γ_n, such that

$$\left\langle \gamma_n, \varphi\right\rangle = \int_\Omega \varphi d\gamma_n,$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$.

For every Borel set, $B \subset \Omega$ we define $\gamma(B) = \sum_{n=1}^{+\infty} b_n \gamma_n(B)$ where $b_n = 2^{-n}\|\gamma_n\|^{-1}_{W^{-1,p'(\cdot)}(\Omega)}$. Since $\gamma_n(B) \leq \|\gamma_n\|_{W^{-1,p'(\cdot)}(\Omega)}\mathrm{Cap}_{p(.)}(B,\Omega)$, it follows that γ is a nonnegative Radon measure on Ω. As the series $\sum_{n=1}^{+\infty} b_n\gamma_n$ converges in $W^{-1,p'(\cdot)}(\Omega)$ then $\gamma \in W^{-1,p'(\cdot)}(\Omega)$. For every $n \in \mathbb{N}$, the measure γ_n is absolutely continuous with respect to γ, thus, there exists a non negative Borel function $h_n : \Omega \longrightarrow \mathbb{R}$ such that

$$\gamma_n(B) = \int_B h_n d\gamma,$$

for every Borel set $B \subset \Omega$. From (3.2), it follows that

$$\sup_{n\in\mathbb{N}} \int_\Omega \varphi h_n d\gamma = \int_\Omega \varphi d\mu, \tag{3.3}$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. Hence

$$\int_B h_n d\gamma \leq \mu(B), \tag{3.4}$$

for every Borel set $B \subset \Omega$.
Let $h = \sup_{n\in\mathbb{N}} h_n$ and denote $f_k = \sup_{n\leq k} h_n$. We have $f_k \uparrow h$ as $k \to +\infty$. Then, using the Fatou's Lemma yields from (3.4) that

$$\int_B h d\gamma \leq \mu(B),$$

for every Borel set $B \subset \Omega$. Therefore

$$\int_\Omega \varphi d\mu = \sup_{n\in\mathbb{N}} \int_\Omega \varphi h_n d\gamma \leq \int_\Omega \varphi h d\gamma$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. This implies that

$$\mu(B) = \int_B h d\gamma, \tag{3.5}$$

for every Borel set $B \subset \Omega$; that is $\mu = h\gamma$.
Since μ is a bounded Radon measure on Ω, setting $B = \Omega$ in (3.5), one sees that $h \in L^1(\Omega, \gamma)$ □

Proof. (Proof of Theorem 1.2.) **(i)** We first prove that if $\mu \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$ then $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$.
If $\mu \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$ then there exist $f \in L^1(\Omega)$ and $F \in \left(L^{p'(\cdot)}(\Omega)\right)^N$ such that $\mu = f - \mathrm{div}(F)$ in $\mathcal{D}'(\Omega)$. Consider a subset E of Ω such that $\mathrm{Cap}_{p(.)}(E,\Omega) = 0$. Then there exists $u_0 \in S_{p(.)}(E)$ such that $\int_\Omega |\nabla u_0|^{p(x)} dx = 0$.
Let us consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^{(p_-)'}(\Omega)$ such that $f_n \longrightarrow f$ in $L^1(\Omega)$. We set $\mu_n = f_n - \mathrm{div}(F)$. One has $\mu_n \longrightarrow \mu$ in the sense of measures and then $\lim_{n\to+\infty} \mu_n(E) = \mu(E)$.
Furthermore

$$\mu_n(E) = \int_E d\mu_n = \int_E u_0 d\mu_n = \int_E f_n u_0 dx - \int_E \mathrm{div}(F) u_0 dx.$$

Hence, we get

$$|\mu_n(E)| = \left|\int_E f_n u_0 dx - \int_E \operatorname{div}(F) u_0 dx\right| \leq \int_E |f_n| u_0 dx + \int_E |\operatorname{div}(F)|\, u_0 dx.$$

Which implies

$$|\mu_n(E)| \leq \int_\Omega |f_n u_0| dx + \int_\Omega |\operatorname{div}(F)|\, u_0 dx. \tag{3.6}$$

The second term of the right hand side of (3.6) leads

$$\begin{aligned}
\int_\Omega |\operatorname{div}(F)|\, u_0 dx &= \int_\Omega \operatorname{sign}_0(\operatorname{div}(F))\operatorname{div}(F) u_0 dx \\
&= -\int_\Omega F.\nabla\left(\operatorname{sign}_0(\operatorname{div}(F)) u_0\right) dx \\
&\leq \left|\int_\Omega F.\nabla\left(\operatorname{sign}_0(\operatorname{div}(F)) u_0\right) dx\right| \\
&\leq \int_\Omega \left|F.\nabla\left(\operatorname{sign}_0(\operatorname{div}(F)) u_0\right)\right| dx \\
&= \int_A |F.\nabla u_0|\, dx, \text{ where } A := \{x \in \Omega : \operatorname{div}(F(x)) \neq 0\} \\
&\leq \int_\Omega |F.\nabla u_0|\, dx.
\end{aligned}$$

Then, it follows that

$$|\mu_n(E)| \leq \int_\Omega |f_n| u_0 dx + \int_\Omega |F.\nabla u_0|\, dx. \tag{3.7}$$

For the second term of the right hand side of (3.7), we have

$$\int_\Omega |F.\nabla u_0| dx \leq C(p^-)|F|_{p'(.)}|\nabla u_0|_{p(.)}. \tag{3.8}$$

Having in mind that $u_0 \in S_{p(.)}(E)$, we get $u_0 \in L^\infty(\Omega)$ and the first term of the right hand side of (3.7) gives

$$\int_\Omega |f_n| u_0 dx \leq \|f_n\|_{L^{(p^-)'}(\Omega)} \|u_0\|_{L^{p^-}(\Omega)}.$$

Since $p^- > 1,\ L^{p(.)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ continuously then there exists $C > 0$ such that

$$\|u_0\|_{L^{p^-}(\Omega)} \leq C|u_0|_{p(.)}.$$

Now, since $p(.)$ is continuous, by Poincaré's inequality, we get

$$|u_0|_{p(.)} \leq C|\nabla u_0|_{p(.)}.$$

Thus,

$$\int_\Omega |f_n| u_0 dx \leq C\|f_n\|_{L^{(p^-)'}(\Omega)}|\nabla u_0|_{p(.)}. \tag{3.9}$$

Relations (3.7), (3.8) and (3.9) allow us to write

$$|\mu_n(E)| \leq |\nabla u_0|_{p(.)}\Big(C\|f_n\|_{L^{(p^-)'}(\Omega)} + C(p^-)|F|_{p'(.)}\Big). \tag{3.10}$$

Since $\int_\Omega |\nabla u_0|^{p(x)} dx = 0$, we get $|\nabla u_0|_{p(.)} = 0$ and the relation (3.10) reduces to $|\mu_n(E)| \leq 0$. Hence $|\mu(E)| = \lim_{n\to+\infty} |\mu_n(E)| \leq 0$. Finally, one has $\mu(E) = 0$, therefore

$\mu \in \mathcal{M}_b^{p(.)}(\Omega)$.

(ii) The converse is proven as follows: if $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$ then $\mu \in L^1(\Omega)+W^{-1,p'(.)}(\Omega)$. The proof is done in three steps.

Step 1: We first observe that if $\mathrm{Cap}_{p(.)}(E,\Omega)=0$, then, both $\mu^+(E)$ and $\mu^-(E)$ are equal to zero. This is a consequence of the definition of μ^+ and μ^- and the monotonicity of the $p(.)$-capacity. Recalling that $\mu^+(E)=\sup\{\mu(B) : B \text{ borelian}, B\subset E\}$, then we may assume μ to be positive in the next steps.

Step 2: By Proposition 3.1, we write μ as $\mu = h\gamma$, with $\gamma \in W^{-1,p'(.)}(\Omega),\ \gamma \geq 0$ and $h \in L^1(\Omega,\gamma),\ h\geq 0$.
Let $(K_n)_{n\geq 0}$ be an increasing sequence of compact sets contained in Ω such that $\bigcup_{n=0}^{+\infty} K_n = \Omega$ and let us denote $\mu_n^{(1)} = T_n(h\chi_{K_n})\gamma$.

Claim 3.1 *$\left(\mu_n^{(1)}\right)_{n\geq 0}$ is an increasing sequence of positive measures in $W^{-1,p'(.)}(\Omega)$ with compact support in Ω.*

The fact that $(\mu_n^{(1)})_{n\geq 0}$ is an increasing sequence of positive measures with compact support in Ω is obvious. Then, we prove that $\forall n\geq 0,\ \mu_n^{(1)} \in W^{-1,p'(.)}(\Omega)$. For any $n\geq 0$ and for any $\phi \in W^{1,p(.)}(\Omega)$, we have

$$\begin{aligned}
0 \leq \left|\int_\Omega \phi d\mu_n^{(1)}\right| &= \left|\int_\Omega \phi T_n(h\chi_{K_n})d\gamma\right| \\
&\leq \|T_n(h\chi_{K_n})\|_{L^\infty(\Omega,\gamma)} \int_\Omega |\phi|\, d\gamma \\
&\leq C(\gamma)\|T_n(h\chi_{K_n})\|_{L^\infty(\Omega,\gamma)}\|\phi\|_{W^{1,p(.)}(\Omega)} \quad \text{since } \gamma\in W^{-1,p'(.)}(\Omega) \\
&\leq C(n,h,\gamma)\|\phi\|_{W^{1,p(.)}(\Omega)}
\end{aligned}$$

which means that, $\forall n\geq 0,\ \mu_n^{(1)} \in W^{-1,p'(.)}(\Omega)$. Let $\mu_0 = \mu_0^{(1)}$ and for any $n\geq 1,\ \mu_n = \mu_n^{(1)} - \mu_{n-1}^{(1)}$. The series $\sum_{n\geq 0}\mu_n$ converges strongly in $\mathcal{M}_b(\Omega)$ to μ so that $\mu = \sum_{n=0}^{+\infty}\mu_n$. In particular $\sum_{n=0}^{+\infty}\|\mu_n\|_{\mathcal{M}_b(\Omega)} < +\infty$. Recalling that for $Z\in\mathcal{M}_b(\Omega),\ Z\geq 0,\ \|Z\|_{\mathcal{M}_b(\Omega)} = Z(\Omega)$.

Step 3: Let ρ be a function in $C_0^\infty(\Omega)$ such that $\rho(x)\geq 0,\ \forall x\in\Omega$ and $\int_\Omega \rho(x)dx = 1$.
Let $(\rho_n)_{n\geq 0}$ be the sequence of mollifiers associated to ρ that is $\rho_n(x) = n^N\rho(nx),\ \forall x\in\Omega$. For $n\geq 0$, if μ_n is the measure defined in Step 2, $(\mu_n*\rho_m)_{m\geq 0}$ converges to μ_n in $W^{-1,p'(.)}(\Omega)$ as m tends to infinity. By properties on μ_n and ρ_m, $\mu_n*\rho_m$ belongs to $C_0^\infty(\Omega)$ if m is large enough.
Choose $m=m_n$ such that $\mu_n*\rho_{m_n}$ belongs to $C_0^\infty(\Omega)$ and $\|\mu_n*\rho_{m_n}-\mu_n\|_{W^{-1,p'(.)}(\Omega)} \leq 2^{-n}$. We know that $\mu_n = f_n+g_n$ with $f_n = \mu_n*\rho_{m_n}$ and $g_n = \mu_n-\mu_n*\rho_{m_n}$. Thanks to the choice of m_n, the series $\sum_{n\geq 0} g_n$ converges in $W^{-1,p'(.)}(\Omega)$ and so $g = \sum_{n\geq 0}^{+\infty} g_n$ belongs to $W^{-1,p'(.)}(\Omega)$.

Since $\|f_n\|_{L^1(\Omega)} = \|\mu_n * \rho_{m_n}\|_{L^1(\Omega)} \leq \|\mu_n\|_{\mathcal{M}_b(\Omega)}$, by Step 2, the series $\sum_{n\geq 0} f_n$ converges absolutely in $L^1(\Omega)$ and so $f = \sum_{n\geq 0}^{+\infty} f_n$ belongs to $L^1(\Omega)$. Thus, the three series $\sum_{n\geq 0}\mu_n$, $\sum_{n\geq 0} g_n$ and $\sum_{n\geq 0} f_n$ converge in the sense of distribution. Therefore

$$\mu = f + g \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$$

□

4. Proof of the main results

Throughout this section, $\mu \in \mathcal{M}_b^{p(\cdot)}(\Omega)$. By Theorem 1.2, we set $\mu = f - \operatorname{div}(F)$, with $f \in L^1(\Omega)$ and $F \in \left(L^{p'(\cdot)}(\Omega)\right)^N$. This section is devoted to the proof of Theorem 1.3 and Theorem 1.4.

For every $\epsilon > 0$, we consider the Yosida regularization β_ϵ of β (see[11]), given by

$$\beta_\epsilon = \frac{1}{\epsilon}\left(I - (I + \epsilon\beta)^{-1}\right).$$

Thanks to [11], there exists a non negative, convex and l.s.c. function j defined on $\mathbb{R}$, such that

$$\beta = \partial j.$$

To regularize β, we consider

$$j_\epsilon(s) = \min_{r\in\mathbb{R}}\left\{\frac{1}{2\epsilon}|s-r|^2 + j(r)\right\},\ \forall s \in \mathbb{R},\ \forall \epsilon > 0.$$

By Proposition 2.11 in [11] we have

$$\begin{cases} \operatorname{dom}(\beta) \subset \operatorname{dom}(j) \subset \overline{\operatorname{dom}(j)} = \overline{\operatorname{dom}(\beta)}, \\ j_\epsilon(s) = \frac{\epsilon}{2}|\beta_\epsilon(s)|^2 + j(J_\epsilon(s)) \text{ where } J_\epsilon := (I+\epsilon\beta)^{-1}, \\ j_\epsilon \text{ is a convex, Frechet-differentiable function and } \beta_\epsilon = \partial j_\epsilon, \\ j_\epsilon \uparrow j \text{ as } \epsilon \downarrow 0. \end{cases}$$

Moreover, for any $\epsilon > 0$, β_ϵ is a nondecreasing and Lipschitz-continuous function. To regularize μ, for any $\epsilon > 0$, we define the functions

$$f_\epsilon(x) = T_{\frac{1}{\epsilon}}(f(x)) \quad \text{for any } x \in \Omega$$

and

$$\mu_\epsilon = f_\epsilon - \nabla\cdot F \quad \text{for any } \epsilon > 0.$$

Then, we consider the following approximating scheme problem

$$P_\epsilon(\beta_\epsilon,\mu_\epsilon)\begin{cases} -\nabla\cdot a(x,\nabla u_\epsilon) + \beta_\epsilon(u_\epsilon) = \mu_\epsilon \text{ in } \Omega, \\ \\ u_\epsilon = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Theorem 4.1. *The problem $P_\epsilon(\beta_\epsilon,\mu_\epsilon)$ admits a unique weak solution u_ϵ in the sense that $u_\epsilon \in W_0^{1,p(\cdot)}(\Omega)$, $\beta_\epsilon(u_\epsilon) \in L^1(\Omega)$ and $\forall\varphi \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$,*

$$\int_\Omega a(x,\nabla u_\epsilon).\nabla\varphi dx + \int_\Omega \beta_\epsilon(u_\epsilon)\varphi dx = \int_\Omega f_\epsilon\varphi dx + \int_\Omega F.\nabla\varphi dx. \tag{4.1}$$

Proof. The techniques of this proof follow the proof of the Theorem 3.2 in [24] (see also [6]).By the Theorem 3.1 in [20], for any $k > 0$, if g is a continuous nondecreasing function with $g(0) = 0$, the following problem

$$P(T_k(g), \Upsilon) \begin{cases} -\nabla \cdot a(x, \nabla u) + T_k(g(u)) = \Upsilon \text{ in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

admits at least one weak solution $u_k \in W_0^{1,p(.)}(\Omega)$ such that $\forall \varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, \nabla u_k).\nabla\varphi dx + \int_\Omega T_k(g(u_k))\varphi dx = \int_\Omega \Upsilon\varphi dx, \tag{4.2}$$

where $\Upsilon \in L^\infty(\Omega)$.
Furthermore

$$\forall k > \|\Upsilon\|_\infty, \ |g(u_k)| \le \|\Upsilon\|_\infty \text{ a.e. in } \Omega. \tag{4.3}$$

Let us fix $k > \|\Upsilon\|_\infty$, we get the existence of solution of problem $P(g, \Upsilon)$ for any g and Υ as above.
The proof of (4.3) and the uniqueness proof are detailed in [24] (see also [6]). So, we can set $g = \beta_\epsilon$ and $\Upsilon = \mu_\epsilon$ to get the result of Theorem 4.1. □

The sequence $(u_\epsilon)_{\epsilon>0}$ satisfies the following lemma.

Lemma 4.2. *Let u_ϵ be a weak solution of $P_\epsilon(\beta_\epsilon, \mu_\epsilon)$, then*

$$meas\{|u_\epsilon| > k\} \le \frac{C(\mu, \Omega)}{k^{p_- - 1}} \tag{4.4}$$

and

$$meas\{|\nabla u_\epsilon| > k\} \le \frac{C(\mu, \Omega)}{k^{\frac{1}{(p_-)'}}}. \tag{4.5}$$

The proof of this lemma follow the proof of Proposition 4.7 and Proposition 4.8 in [6].
We have the following results.

Proposition 4.3.
(*i*) *There exists* $0 < C < +\infty$ *such that for any* $k > 0$,

$$\int_{[|u_\epsilon| \le k]} |\nabla u_\epsilon|^{p(x)} dx \le Ck. \tag{4.6}$$

(*ii*) *The sequence $(\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*
(*iii*) *For any $k > 0$, the sequence $(\beta_\epsilon(T_k(u_\epsilon)))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*

Proof. (*i*) For any $k > 0$, taking $\varphi = T_k(u_\epsilon)$ in (4.1), we get

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla T_k(u_\epsilon) dx + \int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx = \int_\Omega T_k(u_\epsilon) d\mu_\epsilon. \tag{4.7}$$

Using assumption (2.4), we obtain

$$C_2 \int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)} dx + \int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \le \int_\Omega T_k(u_\epsilon) d\mu_\epsilon \tag{4.8}$$

From (4.8), it yields

$$C_2 \int_{[|u_\epsilon| \le k]} |\nabla u_\epsilon|^{p(x)} dx + \int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \le k|\mu|(\Omega). \tag{4.9}$$

All the terms in (4.9) are non negative so that we have

$$C_2 \int_{[|u_\epsilon|\le k]} |\nabla u_\epsilon|^{p(x)} dx \le k|\mu|(\Omega) \tag{4.10}$$

and

$$\int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \le k|\mu|(\Omega). \tag{4.11}$$

Relation (4.6) follows from (4.10).
(ii) Dividing the terms in (4.11) by $k > 0$ and letting k goes to 0, we get

$$\lim_{k\to 0} \frac{1}{k} \int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \le |\mu|(\Omega)$$

which gives

$$\int_\Omega \beta_\epsilon(u_\epsilon) \mathrm{sign}_0(u_\epsilon) dx = \int_\Omega |\beta_\epsilon(u_\epsilon)| dx \le |\mu|(\Omega).$$

Therefore, (ii) follows.
(iii) Assertion (iii) follows from (ii) and the fact for any $k > 0$,

$$\int_\Omega |\beta_\epsilon(T_k(u_\epsilon))| dx \le \int_\Omega |\beta_\epsilon(u_\epsilon)| dx.$$

□

Proposition 4.4. *There exists $u \in W_0^{1,p(.)}(\Omega) \subset \mathcal{T}_0^{1,p(.)}(\Omega)$ such that $u \in dom(\beta)$ a.e. in Ω and*

$$u_\epsilon \longrightarrow u \text{ in measure and a.e. in } \Omega \text{ as } \epsilon \to 0. \tag{4.12}$$

Proof. For $k > 0$, the sequence $(\nabla T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $\left(L^{p(.)}(\Omega)\right)^N$, hence the sequence $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W_0^{1,p(.)}(\Omega)$. Then, up to a subsequence we can assume that for any $k > 0$, $(T_k(u_\epsilon))_{\epsilon>0}$ converges weakly to σ_k in $W_0^{1,p(.)}(\Omega)$ and so $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly to σ_k in $L^{p_-}(\Omega)$.
Let $s > 0$ and define

$$E_1 := [|u_{\epsilon_1}| > k],\ E_2 := [|u_{\epsilon_2}| > k] \text{ and } E_3 := [|T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})| > s],$$

where $k > 0$ is to be fixed.
We have

$$[|u_{\epsilon_1} - u_{\epsilon_2}| > s] \subset E_1 \cup E_2 \cup E_3$$

and hence

$$\mathrm{meas}\{[|u_{\epsilon_1} - u_{\epsilon_2}| > s]\} \le \mathrm{meas}(E_1) + \mathrm{meas}(E_2) + \mathrm{meas}(E_3). \tag{4.13}$$

Let $\epsilon > 0$, using Lemma 4.2, we choose $k = k(\epsilon)$ such that

$$\mathrm{meas}(E_1) \le \epsilon/3 \text{ and } \mathrm{meas}(E_2) \le \epsilon/3. \tag{4.14}$$

Since $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence in $L^{p_-}(\Omega)$.
Thus,

$$\mathrm{meas}(E_3) \le \frac{1}{s^{p_-}} \int_\Omega |T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})|^{p_-} dx \le \frac{\epsilon}{3}, \tag{4.15}$$

for all $\epsilon_1, \epsilon_2 \ge n_0(s, \epsilon)$.
Finally, we obtain

$$\mathrm{meas}\{[|u_{\epsilon_1} - u_{\epsilon_2}| > s]\} \le \epsilon \text{ for all } \epsilon_1, \epsilon_2 \ge n_0(s, \epsilon). \tag{4.16}$$

Hence, the sequence $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure and there exists a function u on Ω such that $u_\epsilon \to u$ in measure. We can then extract a subsequence still denoted $(u_\epsilon)_{\epsilon>0}$ such that $u_\epsilon \to u$ a.e. in Ω.
As for $k > 0$, T_k is continuous, then $T_k(u_\epsilon) \to T_k(u)$ a.e. in Ω and $\sigma_k = T_k(u)$ a.e. in Ω.
Finally, using Lemma 2.2 we deduce that for all $k > 0$, $T_k(u) \in dom(\beta)$ a.e. in Ω. Since $T_k(u) \in dom(\beta)$, we get $u \in dom(\beta)$ a.e. in Ω and as $dom(\beta)$ is bounded, then $u \in W_0^{1,p(.)}(\Omega)$. □

The following convergence results hold, for any $k > 0$.

Proposition 4.5. *(i)* $a(x, \nabla T_k(u_\epsilon)) \rightharpoonup a(x, \nabla T_k(u))$ *weakly in* $\left(L^{p'(.)}(\Omega)\right)^N$.
(ii) $\nabla T_k(u_\epsilon) \longrightarrow \nabla T_k(u)$ *a.e. in* Ω.
(iii) $a(x, \nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \longrightarrow a(x, \nabla T_k(u)).\nabla T_k(u)$ *a.e.in* Ω *and strongly in* $L^1(\Omega)$.
(iv) $\nabla T_k(u_\epsilon) \longrightarrow \nabla T_k(u)$ *strongly in* $\left(L^{p(.)}(\Omega)\right)^N$.

Proof. (i) For any $k > 0$, the sequence $(a(x, \nabla T_k(u_\epsilon)))$ is bounded in $\left(L^{p'(.)}(\Omega)\right)^N$.
We can extract a subsequence such that $a(x, \nabla T_k(u_\epsilon)) \rightharpoonup \Phi_k$ in $\left(L^{p'(.)}(\Omega)\right)^N$.
Now, we show that $\Phi_k(x) = a(x, \nabla T_k(u))$ a.e. $x \in \Omega$. The proof consists of four steps.

Step 1: We prove that for every function $h \in W^{1,+\infty}(\Omega)$, $h \geq 0$ with a compact support, $supp(h) \subset [-l, l] \subset \mathbb{R}$,

$$\limsup_{\epsilon\to 0} \int_\Omega a(x, \nabla u_\epsilon)\nabla[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))]\, dx \leq 0. \tag{4.17}$$

Let us take $\varphi = h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))$ as a test function in (4.1).
We have

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))]\, dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))\, dx \\ \displaystyle = \int_\Omega h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))d\mu_\epsilon. \end{array} \right. \tag{4.18}$$

For any $r > 0$, sufficiently small, we consider

$$u_r = (u \wedge (M - r)) \vee (m + r).$$

For any $k > 0$, $T_k(u_r) \in W_0^{1,p(.)}(\Omega)$. Since

$$\int_\Omega h(u_\epsilon)(\beta_\epsilon(u_\epsilon) - \beta_\epsilon(u_r))(T_k(u_\epsilon) - T_k(u_r))dx \geq 0,$$

we have

$$\begin{aligned} \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))dx \quad &\geq \quad \int_\Omega h(u_\epsilon)\beta_\epsilon(u_r)(T_k(u_\epsilon) - T_k(u_r))dx \\ &\quad + \int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)(T_k(u_r) - T_k(u))dx. \end{aligned}$$

Note that

$$m + r \leq u_r \leq M - r,$$

so that

$$\beta_\epsilon(m + r) \leq \beta_\epsilon(u_r) \leq \beta_\epsilon(M - r).$$

Using Lebesgue dominated convergence Theorem, we get

$$\limsup_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\beta_\epsilon(u_r)(T_k(u_\epsilon)-T_k(u_r))dx=\int_\Omega h(u)\beta_0(u_r)(T_k(u)-T_k(u_r))dx.$$

Consider now the term

$$I:=\int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)(T_k(u_r)-T_k(u))dx.$$

We have

$$\begin{aligned} I=&\int_\Omega f_\epsilon h(u_\epsilon)(T_k(u_r)-T_k(u))dx-\int_\Omega a(x,\nabla u_\epsilon).\nabla\Big[h(u_\epsilon)(T_k(u_r)-T_k(u))\Big]dx\\ &+\int_\Omega F.\nabla\Big[h(u_\epsilon)(T_k(u_r)-T_k(u))\Big]dx\\ =&\int_\Omega f_\epsilon h(u_\epsilon)(T_k(u_r)-T_k(u))dx-\int_\Omega h(u_\epsilon)a(x,\nabla u_\epsilon).\nabla(T_k(u_r)-T_k(u))dx\\ &-\int_\Omega h'(u_\epsilon)(T_k(u_r)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx+\int_\Omega F.\nabla\Big[h(u_\epsilon)(T_k(u_r)-T_k(u))\Big]dx. \end{aligned}$$

Note that $f_\epsilon h(u_\epsilon)(T_k(u_r)-T_k(u))\to 0$ a.e. in Ω as $r\to 0$, $|f_\epsilon h(u_\epsilon)(T_k(u_r)-T_k(u))|\leq C(k)\,|f_\epsilon|\in L^1(\Omega)$, where $C(k)\,|f_\epsilon|$ depends only on k and ϵ. Then, by Lebesgue dominated convergence Theorem, we get

$$\lim_{r\to 0}\int_\Omega f_\epsilon h(u_\epsilon)(T_k(u_r)-T_k(u))dx=0.$$

As $h(u_\epsilon)a(x,\nabla u_\epsilon)=h(u_\epsilon)a(x,\nabla T_l(u_\epsilon))$ is uniformly bounded in $\left(L^{p'(.)}(\Omega)\right)^N$ (by assumption (2.2)) and $\nabla\left[T_k(u_r)-T_k(u)\right]\rightharpoonup 0$ as $r\to 0$, then

$$\lim_{r\to 0}\int_\Omega h(u_\epsilon)a(x,\nabla u_\epsilon).\nabla(T_k(u_r)-T_k(u))dx=0.$$

For the third term of I, we have

$$\begin{aligned} &\left|\int_\Omega h'(u_\epsilon)(T_k(u_r)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx\right|\\ &\qquad= \left|\int_\Omega h'(u_\epsilon)(T_k(u_r)-T_k(u))a(x,\nabla T_l(u_\epsilon)).\nabla T_l(u_\epsilon)dx\right|\\ &\qquad\leq rC_h\int_\Omega a(x,\nabla T_l(u_\epsilon)).\nabla T_l(u_\epsilon)dx\\ &\qquad\leq rC_h\left[\int_\Omega T_l(u_\epsilon)d\mu_\epsilon-\int_\Omega \beta_\epsilon(u_\epsilon)T_l(u_\epsilon)dx\right]\\ &\qquad\leq rC(h,l,\Omega,\mu), \end{aligned}$$

where $C(h,l,\Omega,\mu)$ is a constant depending on h,l,Ω and μ.
Then, we get

$$\lim_{r\to 0}\int_\Omega h'(u_\epsilon)(T_k(u_r)-T_k(u))a(x,\nabla u_\epsilon).\nabla u_\epsilon dx=0.$$

For the last term of I, we have

$$\int_\Omega F.\nabla\Big[h(u_\epsilon)(T_k(u_r)-T_k(u))\Big]dx = \int_\Omega h(u_\epsilon)F.\nabla\Big(T_k(u_r)-T_k(u)\Big)dx + \int_\Omega h'(u_\epsilon)\Big(T_k(u_r)-T_k(u)\Big)F.\nabla T_l(u_\epsilon)dx.$$

Using the results above, one sees that the last term of I goes to zero as $r \to 0$. Therefore, we obtain

$$\lim_{r\to 0}\int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)(T_k(u_r)-T_k(u))dx = 0.$$

Now, let us see that

$$h(u)\beta_0(u_r)(T_k(u)-T_k(u_r)) \geq 0.$$

Indeed,

$$h(u)\beta_0(u_r)(T_k(u)-T_k(u_r)) = h(u)\beta_0(M-r)(T_k(u)-T_k(M-r))\chi_{\{M-r\leq u\leq M\}} + h(u)\beta_0(m+r)(T_k(u)-T_k(m+r))\chi_{\{m\leq u\leq m+r\}} \geq 0,$$

since $0 \in \beta(0)$ and $m+r \leq 0 \leq M-r$.
It follows that

$$\limsup_{\epsilon\to 0}\int_\Omega \beta_\epsilon(u)h(u_\epsilon)(T_k(u_\epsilon)-T_k(u)) \geq 0.$$

We also have $h(u_\epsilon)(T_k(u_\epsilon)-T_k(u)) \to 0$ a.e. in Ω, $|h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))| \leq C(h,k) \in L^1(\Omega)$, where $C(h,k)$ is a constant which depends only on k and h. By the Lebesgue dominated convergence Theorem, we deduce that $h(u_\epsilon)(T_k(u_\epsilon)-T_k(u)) \to 0$ strongly in $L^1(\Omega)$. As $\mu_\epsilon \rightharpoonup \mu$ weakly $*$ in sense of measure, then

$$\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))\,d\mu_\epsilon = 0.$$

Passing to the limit in (4.18) and using the results above, we obtain (4.17).
Step 2: We prove that

$$\limsup_{l\to+\infty}\limsup_{\epsilon\to 0}\int_{\{l<|u_\epsilon|<l+1\}} a(x,\nabla u_\epsilon)\nabla u_\epsilon\,dx \leq 0. \tag{4.19}$$

Let us take for $l > 0$, $\varphi = T_1(u_\epsilon - T_l(u_\epsilon))$ as a test function in (4.1).
We have

$$\left\{\begin{array}{l}\displaystyle\int_\Omega a(x,\nabla u_\epsilon).\nabla T_1(u_\epsilon - T_l(u_\epsilon))\,dx + \int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_l(u_\epsilon))\,dx\\ \displaystyle = \int_\Omega T_1(u_\epsilon - T_l(u_\epsilon))\,d\mu_\epsilon.\end{array}\right. \tag{4.20}$$

The term $\int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_l(u_\epsilon))\,dx$ is nonnegative.

We also have $\int_\Omega a(x,\nabla u_\epsilon).\nabla T_1(u_\epsilon - T_l(u_\epsilon))\,dx = \int_{\{l<|u_\epsilon|<l+1\}} a(x,\nabla u_\epsilon)\nabla u_\epsilon\,dx.$

As in Step 1, we show that $\lim_{\epsilon\to 0}\int_\Omega T_1(u_\epsilon - T_l(u_\epsilon))\,d\mu_\epsilon = \int_\Omega T_1(u - T_l(u))\,d\mu.$

Since $T_1(u - T_l(u)) \to 0$ a.e. in Ω as $l \to +\infty$, by using Lebesgue dominated convergence Theorem, we obtain $\lim_{l\to+\infty}\int_\Omega T_1(u - T_l(u))\,d\mu = 0$ which implies that

$$\lim_{l\to+\infty}\lim_{\epsilon\to 0}\int_\Omega T_1(u_\epsilon - T_l(u_\epsilon))\,d\mu_\epsilon = 0.$$

Passing to the limit as $\epsilon \to 0$ and to the limit as $l \to +\infty$ in (4.20), we deduce (4.19).
Step 3: We prove that, for every $k > 0$,

$$\limsup_{\epsilon\to 0} \int_\Omega a(x, \nabla u_\epsilon)\nabla[T_k(u_\epsilon) - T_k(u)]\,dx \leq 0. \tag{4.21}$$

For $\nu > k$, we have,

$$\begin{cases} \displaystyle\int_\Omega a(x, \nabla u_\epsilon).\nabla[h_\nu(u_\epsilon)(T_k(u_\epsilon) - T_k(u))]\,dx \\ \displaystyle = \int_{\{|u_\epsilon|\leq k\}} h_\nu(u_\epsilon)a(x, \nabla T_k(u_\epsilon)).\nabla[T_k(u_\epsilon) - T_k(u)]\,dx \\ \displaystyle + \int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x, \nabla u_\epsilon).\nabla[-T_k(u)]\,dx \\ \displaystyle + \int_\Omega h'_\nu(u_\epsilon)[T_k(u_\epsilon) - T_k(u)]a(x, \nabla u_\epsilon).\nabla u_\epsilon\,dx. \end{cases}$$

Since $\nu > k$, on the set $\{|u_\epsilon| \leq k\}$, it follows that $h_\nu(u_\epsilon) = 1$ and we get

$$\begin{aligned} &\int_{\{|u_\epsilon|\leq k\}} h_\nu(u_\epsilon)a(x, \nabla T_k(u_\epsilon)).\nabla[T_k(u_\epsilon) - T_k(u)]\,dx \\ &\quad = \int_{\{|u_\epsilon|\leq k\}} a(x, \nabla T_k(u_\epsilon)).\nabla[T_k(u_\epsilon) - T_k(u)]\,dx \\ &\quad = \int_\Omega a(x, \nabla T_k(u_\epsilon)).\nabla[T_k(u_\epsilon) - T_k(u)]\,dx, \end{aligned}$$

as when $\{|u_\epsilon| > k\}$, then $\{|u| \geq k\}$.
We can also write that

$$-\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x, \nabla u_\epsilon).\nabla T_k(u)\,dx = -\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x, \nabla T_{\nu+1}(u_\epsilon)).\nabla T_k(u)\,dx.$$

Using Lebesgue dominated convergence Theorem, we deduce that

$$h_\nu(u_\epsilon)\chi_{\{|u_\epsilon|>k\}}\nabla T_k(u) \to h_\nu(u)\chi_{\{|u|\geq k\}}\nabla T_k(u) \text{ strongly in } L^{p(\cdot)}(\Omega).$$

The sequence $(a(x, \nabla T_{\nu+1}(u_\epsilon)))_{\epsilon>0}$ is bounded in $\left(L^{p'(\cdot)}(\Omega)\right)^N$, then it converges weakly in $\left(L^{p'(\cdot)}(\Omega)\right)^N$ to $\Gamma_{\nu+1}$.
By Lebesgue dominated convergence Theorem, we find

$$\lim_{\epsilon\to 0}\left(-\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x, \nabla T_{\nu+1}(u_\epsilon)).\nabla T_k(u)\,dx\right) = -\int_{\{|u|\geq k\}} h_\nu(u)\Gamma_{\nu+1}.\nabla T_k(u)\,dx = 0.$$

We also have

$$\begin{aligned} &\left(-\int_\Omega h'_\nu(u_\epsilon)[T_k(u_\epsilon) - T_k(u)]a(x, \nabla u_\epsilon).\nabla u_\epsilon\,dx\right) \\ &\quad \leq \left|\int_\Omega h'_\nu(u_\epsilon)[T_k(u_\epsilon) - T_k(u)]a(x, \nabla u_\epsilon).\nabla u_\epsilon\,dx\right| \\ &\quad \leq 2k\int_{\{\nu<|u_\epsilon|<\nu+1\}} a(x, \nabla u_\epsilon).\nabla u_\epsilon\,dx. \end{aligned}$$

Using the result in Step 2, we find that

$$\limsup_{\nu\to+\infty}\limsup_{\epsilon\to 0}\left(-\int_\Omega h'_\nu(u_\epsilon)[T_k(u_\epsilon) - T_k(u)]a(x, \nabla u_\epsilon).\nabla u_\epsilon\,dx\right) \leq 0.$$

Applying (4.17) with h replaced by h_ν, $\nu > k$, it follows that

$$\limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla T_k(u_\epsilon))\nabla[T_k(u_\epsilon)-T_k(u)]\,dx$$
$$\le \limsup_{\nu\to+\infty}\limsup_{\epsilon\to 0}\left(-\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x,\nabla T_{\nu+1}(u_\epsilon)).\nabla T_k(u)\,dx\right)=0.$$

Therefore, (4.21) follows.

Step 4: Now, we prove, by standard monotonicity arguments, that, for all $k > 0$, $\Phi_k = a(.,\nabla T_k(u))$ a.e. in Ω. Let $\varphi \in \mathcal{D}(\Omega)$ and $\lambda \in \mathbb{R}^*$. Using (4.21), we get

$$\begin{aligned}
\lambda \lim_{\epsilon\to 0}&\int_\Omega a(x,\nabla T_k(u_\epsilon))\nabla\varphi\,dx\\
&\ge \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla T_k(u_\epsilon))\,[\nabla T_k(u_\epsilon)-\nabla T_k(u)+\nabla(\lambda\varphi)]\;dx\\
&\ge \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla(T_k(u)-\lambda\varphi))\,[\nabla T_k(u_\epsilon)-\nabla T_k(u)+\nabla(\lambda\varphi)]\;dx\\
&\ge \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla(T_k(u)-\lambda\varphi))\nabla(\lambda\varphi)\,dx \quad \text{since } \nabla T_k(u_\epsilon)\rightharpoonup \nabla T_k(u)\\
&\ge \lambda\int_\Omega a(x,\nabla(T_k(u)-\lambda\varphi))\nabla\varphi\,dx.
\end{aligned}$$

Dividing by $\lambda > 0$ and by $\lambda < 0$, passing the limt with $\lambda \to 0$, it follows that

$$\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla T_k(u_\epsilon))\nabla\varphi\,dx = \int_\Omega a(x,\nabla(T_k(u)))\nabla\varphi\,dx.$$

This means that, $\forall\ k > 0$, $\int_\Omega \Phi_k\nabla\varphi\,dx = \int_\Omega a(x,\nabla(T_k(u)))\nabla\varphi\,dx$. Hence $\Phi_k = a(.,\nabla(T_k(u)))$ a.e. in Ω and we have $a(x,\nabla T_k(u_\epsilon)) \rightharpoonup a(x,\nabla(T_k(u)))$ weakly in $\left(L^{p'(.)}(\Omega)\right)^N$.

(ii) From (4.21), we deduce that for all $k > 0$

$$\lim_{\epsilon\to 0}\int_\Omega [a(x,\nabla T_k(u_\epsilon))-a(x,\nabla(T_k(u)))]\,.\,[\nabla T_k(u_\epsilon)-\nabla T_k(u)]\;dx = 0.$$

Now, set $g_\epsilon(.) = [a(.,\nabla T_k(u_\epsilon)) - a(.,\nabla(T_k(u)))]\,.\,[\nabla T_k(u_\epsilon)-\nabla T_k(u)] \ge 0$.
$g_\epsilon(.) \to 0$ strongly in $L^1(\Omega)$. Up to a subsequence, $g_\epsilon(.) \to 0$ a.e. in Ω, which means that there exists $\omega \subset \Omega$ such that meas$(\omega) = 0$ and $g_\epsilon(.) \to 0$ in $\Omega\backslash\omega$. Let $x \in \Omega\backslash\omega$. Using assumptions (2.2) and (2.4), it follows that the sequence $(\nabla T_k(u_\epsilon(x)))_{\epsilon>0}$ is bounded in $\mathbb{R}^N$ and so we can extract a subsequence which converges to some θ in $\mathbb{R}^N$.
Passing to the limit in the expression of $g_\epsilon(x)$, it follows that

$$0 = [a(x,\theta) - a(x,\nabla(T_k(u))]\,.\,[\theta - \nabla T_k(u)]$$

and this leads $\theta = \nabla T_k(u)$, $\forall x \in \Omega\backslash\omega$.
As the limit does not depend on the subsequence, the whole sequence $(\nabla T_k(u_\epsilon(x)))_{\epsilon>0}$ converges to θ in $\mathbb{R}^N$. This means that $\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ a.e. in Ω.
(iii) The continuity of $a(x,\xi)$ with respect to $\xi \in \mathbb{R}^N$ gives us

$$a(x,\nabla T_k(u_\epsilon)) \to a(x,\nabla T_k(u)) \text{ a.e. in } \Omega.$$

Therefore,

$$a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u)).\nabla T_k(u) \text{ a.e. in } \Omega.$$

Setting $z_\epsilon = a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon)$ and $z = a(x,\nabla T_k(u)).\nabla T_k(u)$, we have

$$\begin{cases} z_\epsilon \geq 0, z_\epsilon \to z \text{ a.e. in } \Omega, z \in L^1(\Omega), \\ \displaystyle\int_\Omega z_\epsilon\, dx \to \int_\Omega z\, dx \end{cases}$$

and as $\int_\Omega |z_\epsilon - z|\, dx = 2\int_\Omega (z-z_\epsilon)^+\, dx + \int_\Omega (z_\epsilon - z)\, dx$ and $(z_\epsilon - z)^+ \leq z$, it follows, by using Lebesgue dominated convergence Theorem, that

$$\lim_{\epsilon\to 0}\int_\Omega |z_\epsilon - z|\, dx = 0,$$

which means that

$$a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u)).\nabla T_k(u) \text{ strongly in } L^1(\Omega).$$

(iv) By (2.4), we have $|\nabla T_k(u_\epsilon)|^{p(x)} \leq C_1 a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon)$. Using the L^1-convergence of (iii) we obtain (iv). □

Lemma 4.6. *For any $h \in C_c^1(\mathbb{R})$ and $\xi \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$,*

$$\nabla[h(u_\epsilon)\xi] \longrightarrow \nabla[h(u)\xi] \textit{ strongly in } \left(L^{p_-}(\Omega)\right)^N \textit{ as } \epsilon \to 0.$$

Proof. For any $h \in C_c^1(\mathbb{R})$ and $\xi \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\begin{aligned} \nabla[h(u_\epsilon)\xi] &= h(u_\epsilon)\nabla\xi + h'(u_\epsilon)\xi\nabla u_\epsilon \\ &= h(u_\epsilon)\nabla\xi + h'(u_\epsilon)\xi\nabla T_l(u_\epsilon) \text{ for } l>0 \text{ such that } \operatorname{supp}(h)\subset(-l,+l). \end{aligned}$$

Using Lebesgue dominated convergence Theorem, we get

$$h(u_\epsilon)\nabla\xi \longrightarrow h(u)\nabla\xi \text{ strongly in } \left(L^{p_-}(\Omega)\right)^N \text{ as } \epsilon \to 0.$$

Moreover, since $|h'(u_\epsilon)\xi\nabla T_l(u_\epsilon)| \leq C|\nabla T_l(u_\epsilon)|$, then using generalized convergence Theorem and Proposition 4.5-(iv), we deduce that

$$h'(u_\epsilon)\xi\nabla T_l(u_\epsilon) \longrightarrow h'(u)\xi\nabla T_l(u) = h'(u)\xi\nabla u \text{ strongly in } \left(L^{p_-}(\Omega)\right)^N \text{ as } \epsilon \to 0.$$

So Lemma 4.6 follows. □

Now, we pass to the limit in $\beta_\varepsilon(u_\varepsilon)$. Since, for any $k>0$, $(h_k(u_\epsilon)z_\epsilon)_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k \in \mathcal{M}_b(\Omega)$, such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon) \overset{*}{\rightharpoonup} z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, for any $\xi \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_\Omega \xi\, dz_k = \int_\Omega \xi h_k(u)\, d\mu - \int_\Omega a(x,\nabla u)\cdot\nabla(h_k(u)\xi)dx,$$

which implies that $z_k \in \mathcal{M}_b^{p(.)}(\Omega)$ and, for any $k \leq l$,

$$z_k = z_l \quad \text{ on } [|T_k(u)|<k].$$

Let us consider the Radon measure z defined by

$$\begin{cases} z = z_k, & \text{on } [|T_k(u)|<k] \text{ for } k \in \mathbb{N}^*, \\ z = 0 & \text{on } \displaystyle\bigcap_{k\in\mathbb{N}^*}[|T_k(u)|=k]. \end{cases} \tag{4.22}$$

For any $h \in \mathcal{C}_c(\mathbb{R})$, $h(u) \in L^\infty(\Omega, d|z|)$ and

$$\int_\Omega h(u)\xi \, dz = -\int_\Omega a(x, \nabla u) \cdot \nabla(h(u)\xi) dx + \int_\Omega h(u)\xi d\mu,$$

for any $\xi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$. Indeed, let $k_0 > 0$ be such that $\text{supp}(h) \subseteq [-k_0, k_0]$,

$$\begin{aligned}
\int_\Omega h(u)\xi \, dz &= \int_\Omega h(u)\xi \, dz_{k_0} \\
&= -\lim_{\epsilon\to 0} \int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla(h(u_\epsilon)\xi) dx + \lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\xi d\mu_\epsilon \\
&= -\lim_{\epsilon\to 0} \int_\Omega a(x, \nabla T_{k_0}(u_\epsilon)) \cdot \nabla(h(u_\epsilon)\xi) dx + \lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\xi d\mu_\epsilon \\
&= -\int_\Omega a(x, \nabla u) \cdot \nabla(h(u)\xi) dx + \int_\Omega h(u)\xi d\mu.
\end{aligned}$$

Moreover, we have

Lemma 4.7. *The Radon-Nikodym decomposition of the measure z given by* (4.22) *with respect to $\mathcal{L}^N$,*

$$z = w\,\mathcal{L}^N + \nu \text{ with } \nu \perp \mathcal{L}^N$$

satisfies the following properties

$$\begin{cases}
w \in \beta(u)\ \mathcal{L}^N - a.e. \text{ in } \Omega,\ w \in L^1(\Omega),\ \nu \in \mathcal{M}_b^{p(.)}(\Omega), \\
\nu^+ \text{ is concentrated on } [u = M], \\
\nu^- \text{ is concentrated on } [u = m].
\end{cases}$$

Proof. Since, for any $\epsilon > 0$, $z_\epsilon \in \partial j_\epsilon(u_\epsilon)$, we have

$$j(t) \geq j_\epsilon(t) \geq j_\epsilon(u_\epsilon) + (t - u_\epsilon) z_\epsilon\ \mathcal{L}^N - \text{a.e. in } \Omega, \quad \forall t \in \mathbb{R}.$$

Then, for any $h \in \mathcal{C}_c(\mathbb{R})$, $h \geq 0$ and $k > 0$ such that $\text{supp}(h) \subseteq [-k, k]$, we have

$$\xi h(u_\epsilon) j(t) \geq \xi h(u_\epsilon) j_\epsilon(u_\epsilon) + (t - u_\epsilon)\xi h(u_\epsilon) h_k(u_\epsilon) z_\epsilon.$$

In addition, for any $0 < \epsilon < \tilde{\epsilon}$, we have

$$\xi h(u_\epsilon) j(t) \geq \xi h(u_\epsilon) j_{\tilde{\epsilon}}(u_\epsilon) + (t - u_\epsilon)\xi h(u_\epsilon) h_k(u_\epsilon) z_\epsilon$$

and, integrating over Ω gives

$$\int_\Omega \xi h(u_\epsilon) j(t) dx \geq \int_\Omega \xi h(u_\epsilon) j_{\tilde{\epsilon}}(u_\epsilon) dx + \int_\Omega (t - u_\epsilon)\xi h(u_\epsilon) h_k(u_\epsilon) z_\epsilon dx.$$

As $\epsilon \to 0$, we get by using Fatou's Lemma

$$\int_\Omega \xi h(u) j(t) dx \quad \geq \quad \int_\Omega \xi h(u) j_{\tilde{\epsilon}}(u) dx + \liminf_{\epsilon\to 0} \int_\Omega (t - u_\epsilon)\xi h(u_\epsilon) h_k(u_\epsilon) z_\epsilon dx.$$

Now, for any $\xi \in C_c^1(\Omega)$ and $t \in \mathbb{R}$, setting

$$\tilde{h}(r) = (t - r)h(r),$$

we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega (t-u_\epsilon)h(u_\epsilon)\xi h_k(u_\epsilon)z_\epsilon dx &= \lim_{\epsilon\to 0}\int_\Omega \tilde{h}(u_\epsilon)\xi h_k(u_\epsilon)z_\epsilon dx\\ &= \int_\Omega (t-u)h(u)\xi dz_k\\ &= \int_\Omega (t-u)h(u)\xi dz.\end{aligned}$$

So,

$$\int_\Omega \xi h(u)j(t)dx \geq \int_\Omega \xi h(u)j_{\tilde{\epsilon}}(u)dx + \int_\Omega \xi(t-u)h(u)dz.$$

As $\tilde{\epsilon}\to 0$, we get by using again Fatou's Lemma

$$\int_\Omega \xi h(u)j(t)dx \geq \int_\Omega \xi h(u)j(u)dx + \int_\Omega \xi(t-u)h(u)dz.$$

From the inequality above, we have

$$h(u)j(t) \geq h(u)j(u) + (t-u)h(u)z, \quad \text{ in } \mathcal{M}_b(\Omega),\ \forall t\in\mathbb{R}. \tag{4.23}$$

Using the Radon-Nikodym decomposition of z we have $z = w\mathcal{L}^N + \nu$ with $\nu \perp \mathcal{L}^N, w \in L^1(\Omega)$, then comparing the regular part and the singular part of (4.23), for any $h \in \mathcal{C}_c(\mathbb{R})$, we obtain

$$h(u)j(t) \geq h(u)j(u) + (t-u)h(u)w\ \mathcal{L}^N - \text{a.e. in } \Omega,\ \forall t\in\mathbb{R} \tag{4.24}$$

and

$$(t-u)h(u)\nu \leq 0 \text{ in } \mathcal{M}_b(\Omega),\ \forall\ t\in\overline{\text{dom}(j)}. \tag{4.25}$$

From (4.24) we get

$$j(t) \geq j(u) + (t-u)w\ \mathcal{L}^N - \text{a.e. in } \Omega,\ \forall t\in\mathbb{R},$$

so that $w\in\partial j(u)\ \mathcal{L}^N$ – a.e in Ω. As to (4.25), this implies that for any $t\in\overline{\text{dom}(j)}$,

$$\nu \geq 0 \text{ in } [u\in(t,\infty)\cap\text{supp}(h)] \tag{4.26}$$

and

$$\nu \leq 0 \text{ in } [u\in(-\infty,t)\cap\text{supp}(h)]. \tag{4.27}$$

In particular, this implies that

$$\nu([m<u<M]) = 0.$$

Then (4.27) (resp. (4.26)) implies that

$$\nu^- \text{ is concentrated on } [u=m] \quad (\text{resp. } \nu^+ \text{ is concentrated on } [u=M]).$$

So the proof of the Lemma 4.7 is finished. □

To end the proof of Theorem 1.3, we consider $\xi\in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ and $h\in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\xi$ as test function in (4.1) to get

$$\int_\Omega a(x,\nabla u_\epsilon).\nabla[h(u_\epsilon)\xi]dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx = \int_\Omega h(u_\epsilon)\xi f_\epsilon dx + \int_\Omega F.\nabla[h(u_\epsilon)\xi]dx. \tag{4.28}$$

Using Lemma 4.6, it is not hard to see that

$$\lim_{\epsilon\to 0}\left(\int_\Omega h(u_\epsilon)\xi f_\epsilon dx + \int_\Omega F.\nabla[h(u_\epsilon)\xi]dx\right) = \int_\Omega h(u)\xi d\mu.$$

The first term of (4.28) can be written as

$$\int_\Omega a(x,\nabla u_\epsilon).\nabla[h(u_\epsilon)\xi]dx = \int_\Omega a(x,\nabla T_{l_0+1}(u_\epsilon)).\nabla[h(u_\epsilon)\xi]dx,$$

for some $l_0 > 0$ so that, by Proposition 4.5-(i) and Lemma 4.6, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla[h(u_\epsilon)\xi]dx &= \lim_{\epsilon\to 0}\int_\Omega a(x,\nabla T_{l_0+1}(u_\epsilon)).\nabla[h(u_\epsilon)\xi]dx \\ &= \int_\Omega a(x,\nabla T_{l_0+1}(u)).\nabla[h(u)\xi]dx \\ &= \int_\Omega a(x,\nabla u).\nabla[h(u)\xi]dx.\end{aligned}$$

From the convergence result of Lemma 4.6, Proposition 4.5-(i) and using (4.28), we get

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx &= \int_\Omega h(u)\xi d\mu - \int_\Omega a(x,\nabla u).\nabla[h(u)\xi]dx \\ &= \int_\Omega h(u)\xi dz \\ &= \int_\Omega h(u)w\xi dx + \int_\Omega h(u)\xi d\nu.\end{aligned}$$

Letting ϵ goes to 0 in (4.28), we have:

$$\int_\Omega a(x,\nabla u).\nabla[h(u)\xi]dx + \int_\Omega wh(u)\xi dx + \int_\Omega h(u)\xi d\nu = \int_\Omega h(u)\xi d\mu. \tag{4.29}$$

Since (4.29) holds for any $h \in C_c^1(\mathbb{R})$, we can take $h = h_{l_0}$ with $[m,M] \subset [-l_0,+l_0]$ so that (1.6) holds.
Let us show (1.7) and (1.8) to conclude that (u,w) is a solution of $P(\beta,\mu)$.
To this end we prove the following lemma.

Lemma 4.8. *Let* $\eta \in W_0^{1,p(.)}(\Omega)$, $Z \in \mathcal{M}_b^{p(.)}(\Omega)$ *and* $\lambda \in \mathbb{R}$ *be such that*

$$\begin{cases} \eta \le \lambda \text{ a.e. in } \Omega \text{ (resp. } \eta \ge \lambda), \\ Z = -div\, a(x,\nabla\eta) \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{4.30}$$

Then

$$\int_{[\eta=\lambda]} \xi dZ \ge 0, \tag{4.31}$$

(resp.)

$$\int_{[\eta=\lambda]} \xi dZ \le 0, \tag{4.32}$$

for any $\xi \in C_c^1(\Omega)$, $\xi \ge 0$.

Proof. The proof of this lemma follows the same steps of [1]. For seek of completeness, let us give the arguments. For $n \ge 1$, let $\varphi_n(r) = \inf(1,(nr+1-n\lambda)^+)$. Note that $\varphi_n(r)$ converges to $\chi_{[\lambda,\infty)}(r)$ for every $r \in \mathbb{R}$, so $\varphi_n(\eta(x))$ converges to $\chi_{[\lambda,\infty)}(\eta(x))$ at every x where $\eta(x)$ is defined. As η is defined quasi everywhere and $\chi_{[\lambda,\infty)} \circ \eta = \chi_{\{x\in\Omega:\eta(x)=\lambda\}}$, then the convergence of $\varphi_n(\eta)$ to $\chi_{[\lambda,\infty)}(\eta)$ is quasi everywhere.

Therefore, since Z is diffuse, then $\varphi_n(\eta)$ converges to $\chi_{\{x\in\Omega:\eta(x)=\lambda\}}$, $Z-$a.e. in Ω. Next, we use the Lebesgue dominated convergence theorem and (2.4) to get

$$\begin{aligned}\int_{\eta=\lambda}\xi dZ &= \lim_{n\to+\infty}\int_\Omega \xi\varphi_n(\eta)dZ\\ &= \lim_{n\to+\infty}\int_\Omega a(x,\nabla\eta)\nabla\left(\xi\varphi_n(\eta)\right)dx\\ &\geq \lim_{n\to+\infty}\int_\Omega a(x,\nabla\eta)\varphi_n(\eta)\nabla\xi dx\\ &\geq -\|\nabla\xi\|_\infty \lim_{n\to+\infty}\int_{\left\{x\in\Omega:\lambda-\frac{1}{n}\leq\eta(x)\leq\lambda\right\}}|a(x,\nabla\eta)|\,dx\\ &\geq -\|\nabla\xi\|_\infty\int_\Omega |a(x,0)|\,dx\\ &= 0,\end{aligned}$$

since $a(x,0)=0$ for a.e. $x\in\Omega$. Indeed for $x\in\Omega$ fixed, denote $z=a(x,0)\in\mathbb{R}^N$. By the continuity of $a(x,.)$, we have $\lim\limits_{\xi\to0}a(x,\xi)=z$. Suppose now that $z\neq0$ and choose $\xi_0=-sz$ with $s>0$ used to tend toward 0; then $a(x,\xi_0).\xi_0=-s(z+\epsilon(s)).z\leq -s\,|z|^2+s\,|z|\,|\epsilon(s)|$, where $\lim\limits_{s\to0}|\epsilon(s)|=0$. Therefore, for s sufficiently small, $-s\,|z|^2+s\,|z|\,|\epsilon(s)|<0$, which is a contradiction by assumption (2.4). Thus, $z=0$. Finally, if $\eta\geq\lambda$, we do the same calculus with $\widetilde{\eta}=-\eta$, $\widetilde{\lambda}=-\lambda$ and $\widetilde{a}(x,\eta)=-a(x,-\eta)$ to get the result. □

Since

$$\nu=\text{div }a(x,\nabla u)-w\mathcal{L}^N+\mu,$$

we have

$$\mu-\nu-w\mathcal{L}^N=-\text{div }a(x,\nabla u).$$

By Lemma 4.8, for any $\xi\in C_c^1(\Omega)$, $\xi\geq0$, we have

$$\int_{[u=M]}\xi d\nu^+\leq\int_{[u=M]}\xi d\mu-\int_{[u=M]}\xi w dx$$

and

$$\int_{[u=m]}\xi d\nu^-\leq-\int_{[u=m]}\xi d\mu+\int_{[u=m]}\xi w dx.$$

The first inequality implies that

$$\int_\Omega\xi d\nu^+\leq\int_\Omega\xi d\mu\lfloor[u=M]-\int_\Omega\xi w\chi_{[u=M]}dx.$$

Consequently (1.7) holds. Similarly we get (1.8). □

Proof of Theorem 1.4.

1. If (u,w) is a solution of $P(\beta,\mu)$ in the sense of Theorem 1.3, for any $\xi\in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ with $\xi\in\text{dom}(\beta)$ and for any $k>0$ the function $T_k(u-\xi)$ belongs to $W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ and then this can be used as test function in (1.6) to get

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-\xi)dx+\int_\Omega wT_k(u-\xi)dx+\int_\Omega T_k(u-\xi)d\nu=\int_\Omega T_k(u-\xi)d\mu. \quad (4.33)$$

We split the third term in (4.33) as

$$\begin{aligned}\int_{\Omega} T_k(u-\xi)d\nu &= \int_{[u=M]} T_k(u-\xi)d\nu^{+} - \int_{[u=m]} T_k(u-\xi)d\nu^{-} \\ &= \int_{[u=M]} T_k(M-\xi)d\nu^{+} - \int_{[u=m]} T_k(m-\xi)d\nu^{-} \\ &\geq 0.\end{aligned}$$

Then from (4.33), we have (1.9).

2. Suppose that $(u_1, w_1), (u_2, w_2)$ are two solutions of $P(\beta, \mu)$. For u_1, we choose $\xi = u_2$ as test function in (1.9) to get

$$\int_{\Omega} a(x, \nabla u_1).\nabla T_k(u_1 - u_2)dx + \int_{\Omega} w_1 T_k(u_1 - u_2)dx \leq \int_{\Omega} T_k(u_1 - u_2)d\mu.$$

Similarly we get for u_2 by taking $\xi = u_1$ as test function in (1.9),

$$\int_{\Omega} a(x, \nabla u_2).\nabla T_k(u_2 - u_1)dx + \int_{\Omega} w_2 T_k(u_2 - u_1)dx \leq \int_{\Omega} T_k(u_2 - u_1)d\mu.$$

Adding these two last inequalities yields

$$\int_{\Omega} \Big(a(x, \nabla u_1) - a(x, \nabla u_2)\Big).\nabla T_k(u_1 - u_2)dx + \int_{\Omega} (w_1 - w_2)\, T_k(u_1 - u_2)dx \leq 0. \quad (4.34)$$

For any $k > 0$, from (4.34) it yields

$$\int_{\Omega} \Big(a(x, \nabla u_1) - a(x, \nabla u_2)\Big).\nabla T_k(u_1 - u_2)dx = 0. \quad (4.35)$$

From (4.35), it follows that there exists a constant c such that $u_1 - u_2 = c$ a.e. in Ω. Using the fact that $u_1 = u_2 = 0$ on $\partial\Omega$ we get $c = 0$. Thus, $u_1 = u_2$ a.e. in Ω. At last, let us see that $w_1 = w_2$ a.e. in Ω and $\nu_1 = \nu_2$. Indeed for any $\varphi \in \mathcal{D}(\Omega)$, taking φ as test function in (1.6) for the solutions (u_1, w_1) and (u_1, w_2), after substraction of these equalities we get

$$\int_{\Omega} (w_1 - w_2)\varphi dx + \int_{\Omega} \varphi d(\nu_1 - \nu_2) = 0.$$

Hence

$$\int_{\Omega} w_1 \varphi dx + \int_{\Omega} \varphi d\nu_1 = \int_{\Omega} w_2 \varphi dx + \int_{\Omega} \varphi d\nu_2.$$

Therefore

$$w_1 \mathcal{L}^N + \nu_1 = w_2 \mathcal{L}^N + \nu_2.$$

Since the Radon-Nikodym decomposition of a measure is unique, we get $w_1 = w_2$ a.e. in Ω and $\nu_1 = \nu_2$. □

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