Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 41(1), 2014, Pages 6–14
ISSN: 1223-6934

Some further results on belonging of trigonometric series to Orlicz space

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Abstract. Here in this paper we have introduced a new condition which is not worse than the condition that satisfy numerical sequences of Rest Bounded Variation Mean Sequences. This condition is used to obtain some integrability conditions of the functions $g(x)$ and $f(x)$ (which denote formal sine and cosine trigonometric series respectively) such that these functions are going to belong to the Orlicz space. This study may be considered as a continuation of the investigations previously done by L. Leindler [5] and S. Tikhonov [14].

2010 Mathematics Subject Classification. Primary 42A32; Secondary 46E30.
Key words and phrases. Trigonometric series, Integrability, Orlicz space.

1. Introduction

Many authors have studied the integrability of the formal series

$$g(x) := \sum_{n=1}^{\infty} \lambda_n \sin nx \tag{1}$$

and

$$f(x) := \sum_{n=1}^{\infty} \lambda_n \cos nx \tag{2}$$

imposing certain conditions on the coefficients λ_n (see for example [2], [3], [8], [9], and [11]–[13]).

As initial example, R. P. Boas in [1] proved the following result for (1):

Theorem 1.1. *If $\lambda_n \downarrow 0$ then for $0 \leq \gamma \leq 1$, $x^{-\gamma} g(x) \in L[0, \pi]$ if and only if $\sum_{n=1}^{\infty} n^{\gamma-1}\lambda_n$ converges.*

This result had previously been proved for $\gamma = 0$ by W.H. Young [15] and it was later extended by P. Heywood [4] for $1 < \gamma < 2$.

Later on the monotonicity condition on the coefficients λ_n was replaced to more general ones by S.M. Shah [12] and L. Leindler [7].

Recently, S. Tikhonov [14] has proved two theorems giving sufficient conditions of belonging of $g(x)$ and $f(x)$ to Orlicz space. Before we state his theorems we shall recall some notions and notations.

L. Leindler [7] introduced a class of numerical sequences which has an interesting property and useful in many applications. A sequence $c := \{c_n\}$ of positive numbers

Received May 15, 2013.

tending to zero is of rest bounded variation, or briefly R_0^+BVS, if it possesses the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(c)c_m \tag{3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (decreasing) if there exists constant $C := C(\gamma) \ge 1$ such that

$$C\gamma_n \ge \gamma_m \quad (\gamma_n \le C\gamma_m)$$

holds for any $n \ge m$.

Here and further C, C_i denote positive constants that are not necessarily the same at each occurrence, and also we use the notion $u \ll w$ $(u \gg w)$ at inequalities if there exists a positive constant C such that $u \le Cw$ $(u \ge Cw)$ holds.

We will denote (see [10]) by $\triangle(p,q)$, $(0 \le q \le p)$ the set of all nonnegative functions $\Phi(x)$ defined on $[0,1)$ such that $\Phi(0) = 0$ and $\Phi(x)/x^p$ is nonincreasing and $\Phi(x)/x^q$ is nondecreasing. It is clear that $\triangle(p,q) \subset \triangle(p,0)$, $(0 < q \le p)$. As an example, $\triangle(p,0)$ contains the function $\Phi(x) = \log(1+x)$.

Here and in the sequel, a function $\gamma(x)$ is defined by the sequence γ in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n$, $n \in \mathbb{N}$ and there exist positive constants C_1 and C_2 such that $C_1\gamma_{n+1} \le \gamma(x) \le C_2\gamma_n$ for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$.

A locally integrable almost everywhere positive function $\gamma(x) : [0,\pi] \to [0,\infty)$ is said to be a weight function. Let $\Phi(t)$ be a nondecreasing continuous function defined on $[0,\infty)$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = +\infty$. For a weight $\gamma(x)$ the weighted Orlicz space $L(\Phi,\gamma)$ is defined by

$$L(\Phi,\gamma) = \left\{ h : \int_0^{\pi} \gamma(x)\Phi(\varepsilon|h(x)|)dx < \infty \quad \text{for some} \quad \varepsilon > 0 \right\}. \tag{4}$$

Tikhonov's results now can be read as follows:

Theorem 1.2. *Let $\Phi(x) \in \triangle(p,0)$, $0 \le p$. If $\lambda_n \in R_0^+BVS$, and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}\Phi(n\lambda_n) < \infty \Rightarrow \psi(x) \in L(\Phi,\gamma), \tag{5}$$

where a function $\psi(x)$ is either a sine or cosine series.

Theorem 1.3. *Let $\Phi(x) \in \triangle(p,q)$, $0 \le q \le p$. If $\lambda_n \in R_0^+BVS$, and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-(1+q)+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}}\Phi(n^2\lambda_n) < \infty \Rightarrow g(x) \in L(\Phi,\gamma). \tag{6}$$

A null-sequence c of nonnegative numbers possessing the property

$$\sum_{n=2m}^{\infty} |c_n - c_{n+1}| \le \frac{K(c)}{m} \sum_{\nu=m}^{2m-1} c_\nu \tag{7}$$

is called a sequence of mean rest bounded variation, in symbols, $c \in MRBVS$.

In [5] L. Leindler extended Theorem 1.2 and Theorem 1.3 so that the sequence $\{\lambda_n\}$ belongs the class $MRBVS$ instead of the class R_0^+BVS. His results are formulated as follows:

Theorem 1.4. *Theorems 1.2 and 1.3 can be improved when the condition* $\lambda_n \in R_0^+BVS$ *is replaced by the assumption* $\lambda_n \in MRBVS$*. Furthermore the conditions of (5) and (6) may be modified as follows:*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2} \Phi\left(\sum_{\nu=n}^{2n-1} \lambda_\nu\right) < \infty \Rightarrow \psi(x) \in L(\Phi,\gamma), \tag{8}$$

and

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}} \Phi\left(n \sum_{\nu=n}^{2n-1} \lambda_\nu\right) < \infty \Rightarrow g(x) \in L(\Phi,\gamma), \tag{9}$$

respectively.

Let $C := C_n := \frac{1}{n+1}\sum_{i=0}^{n} c_k$, where c_k is a sequence of nonnegative numbers. Very recently, R. N. Mohapatra and B. Szal [16] introduced the following class of sequences of nonnegative numbers:

If $C \in RBVS$, i.e.

$$\sum_{k=m}^{\infty} |C_k - C_{k+1}| \le K(c) C_m, \tag{10}$$

then it is said that C is of rest bounded variation means sequence, briefly denoted by $C \in RBVMS$.

Aiming to prove the counterparts of Theorem 1.2 and Theorem 1.3 so that the sequence $\{\lambda_n\}$ belongs the class $RBVMS$ instead of the classes $MRBVS$ or R_0^+BVS, we were not in able. However, we have proved two theorems, when not a worse condition than (10) will be fulfilled. Indeed, we have required that the sequence $\{\lambda_n\}$ satisfies condition (obviously not worse than condition (10))

$$\sum_{k=n}^{\infty} k|V_k - V_{k+1}| \le K V_n, \quad (n = 1, 2, \dots), \tag{11}$$

where $V_k := \frac{1}{k}\sum_{j=1}^{k} \lambda_j$.

To prove our main results we need some helpful statements given in next section.

2. Auxiliary lemmas

We shall use the following lemmas for the proof of the main results.

Lemma 2.1 ([6])**.** *If* $a_n \ge 0$*,* $b_n > 0$*, and if* $p \ge 1$*, then*

$$\sum_{n=1}^{\infty} b_n \left(\sum_{v=1}^{n} a_v\right)^p \le C \sum_{n=1}^{\infty} b_n^{1-p} a_n^p \left(\sum_{v=n}^{\infty} a_v\right)^p.$$

Lemma 2.2 ([10])**.** *Let* $\Phi \in \triangle(p,q)$*,* $0 \le q \le p$*, and* $t_j \ge 0$*,* $j = 1, 2, \dots, n$*,* $n \in \mathbb{N}$*. Then*

(1) $\theta^p \Phi(t) \le \Phi(\theta t) \le \theta^q \Phi(t)$*,* $0 \le \theta \le 1$*,* $t \ge 0$*,*

(2) $\Phi\left(\sum_{j=1}^{n} t_j\right) \le \left(\sum_{j=1}^{n} \Phi^{1/p*}(t_j)\right)^{p*}$*,* $p* := \max(1, p)$*.*

Lemma 2.3. *Let* $\Phi \in \triangle(p,q)$*,* $0 \le q \le p$*. If* $\rho_n > 0$*,* $\lambda_n \ge 0$*, and if*

$$V_{\nu+j} \ll V_\nu, \quad V_\nu := \frac{1}{\nu}\sum_{j=1}^{\nu} \lambda_j \tag{12}$$

holds for all $j, \nu \in \mathbb{N}, j \leq \nu$, then

$$\sum_{n=1}^{\infty} \rho_n \Phi\left(n \sum_{k=1}^{n} V_k\right) \ll \sum_{n=1}^{\infty} \rho_n n^p \Phi(nV_n) \left(\frac{\sum_{v=n}^{\infty} v^p \rho_v}{\rho_n n^{p+1}}\right)^{p*},$$

where $p* := \max(1, p)$.

Proof. Let be ξ be an integer such that $2^{\xi} \leq n < 2^{\xi+1}$. Then based on (12) we have

$$\sum_{v=1}^{n} V_k \leq \sum_{m=0}^{\xi} \sum_{v=2^m}^{2^{m+1}-1} V_v + \sum_{v=2^{\xi}}^{n} V_v \leq C_1 \sum_{m=0}^{\xi} 2^m V_{2^m}.$$

Lemma 2.2 and the properties of Φ imply

$$\begin{aligned} \Phi\left(n \sum_{k=1}^{n} V_k\right) &\leq \Phi\left(C_1 2^{\xi+1} \sum_{m=0}^{\xi} 2^m V_{2^m}\right) \leq (C_1 2^{\xi+1})^p \Phi\left(\sum_{m=0}^{\xi} 2^m V_{2^m}\right) \\ &\leq Cn^p \Phi\left(\sum_{m=0}^{\xi} 2^m V_{2^m}\right) \leq Cn^p \left(\sum_{m=0}^{\xi} \Phi^{\frac{1}{p*}}\left(2^m V_{2^m}\right)\right)^{p*} \\ &\leq Cn^p \left(\sum_{m=0}^{n} \frac{\Phi^{\frac{1}{p*}}(mV_m)}{m}\right)^{p*}. \end{aligned}$$

Finally by Lemma 2.1, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \rho_n \Phi\left(n \sum_{k=1}^{n} V_k\right) &\leq C \sum_{n=1}^{\infty} \rho_n n^p \left(\sum_{m=0}^{n} \frac{\Phi^{\frac{1}{p*}}(mV_m)}{m}\right)^{p*} \\ &\leq C \sum_{n=1}^{\infty} \rho_n n^p \Phi(nV_n) \left(\frac{\sum_{v=n}^{\infty} v^p \rho_v}{\rho_n n^{p+1}}\right)^{p*}. \end{aligned}$$

□

Lemma 2.4. *Let $\lambda_k \to 0$ as $k \to \infty$. Then the following representations of $g(x)$ and $f(x)$ hold true:*

$$g(x) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \lambda_j\right) [\sin kx - \sin (k+1)x]$$

and

$$f(x) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \lambda_j\right) [\cos kx - \cos (k+1)x].$$

Proof. These equalities are immediate results of the summation by parts. This is why we omit the proof. □

3. Main results

First, we establish the following.

Theorem 3.1. *Let $\Phi(x) \in \triangle(p, 0)$, $p \geq 0$. If λ_n satisfies condition (11) and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1+p+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2(1-p)}} \Phi\left(\frac{1}{n} \sum_{j=1}^{n} \lambda_j\right) < \infty \Longrightarrow \psi(x) \in L(\Phi, \gamma), \tag{13}$$

where a function $\psi(x)$ is either a sine or cosine series.

Proof. Let $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right]$. Based on Lemma 2.4 and applying the summation by parts we obtain

$$\begin{aligned} |g(x)| &\leq 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right) + \left|\sum_{k=n}^{\infty}\left(\sum_{j=1}^{k}\lambda_j\right)[\sin kx - \sin(k+1)x]\right| \\ &\leq 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right) + \sum_{k=n}^{\infty}|V_k - V_{k+1}|\left|\widehat{D}_k(x)\right| + V_n\left|\widehat{D}_{n-1}(x)\right|, \end{aligned}$$

where V_k and $\widehat{D}_k(x)$ are defined by

$$V_k := \frac{1}{k}\sum_{j=1}^{k}\lambda_j,$$

and

$$\widehat{D}_k(x) := \frac{\cos\frac{x}{2} - \cos\left(k+\frac{3}{2}\right)x - 2k\sin\frac{x}{2}\sin(k+1)x}{2\sin\frac{x}{2}}, \quad k \in \mathbb{N},$$

respectively.

Subsequently, taking into account that $|\widehat{D}_k(x)| = O\left(\frac{k+1}{x}\right)$, $(0,\pi]$, and $\{\lambda_n\}$ satisfies (11) we have that

$$\begin{aligned} |g(x)| &\ll \sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right) + \sum_{k=n}^{\infty}|V_k - V_{k+1}|\left|\widehat{D}_k(x)\right| + V_n\left|\widehat{D}_{n-1}(x)\right| \\ &\ll \sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right) + n\sum_{k=n}^{\infty}k|V_k - V_{k+1}| + n^2V_n \\ &\ll \sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right) + \sum_{j=1}^{n}\lambda_j + n^2V_n. \end{aligned}$$

Moreover,

$$V_n \gg \sum_{k=n}^{\infty}k|V_k - V_{k+1}| \geq |nV_n + V_{n+1} + V_{n+2} + \cdots| \geq nV_n \Longrightarrow n^2V_n \ll \sum_{j=1}^{n}\lambda_j,$$

and since the condition

$$\sum_{k=n}^{\infty}|V_k - V_{k+1}| \leq KV_n$$

also holds, then

$$KV_n \geq \sum_{k=n}^{\infty}|V_k - V_{k+1}| \geq \sum_{k=m}^{\infty}|V_k - V_{k+1}| \geq V_m \Longrightarrow V_m \ll V_n \quad \forall m \geq n.$$

Therefore

$$|g(x)| \ll n\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right) + \frac{1}{n}\sum_{j=1}^{n}\lambda_j\sum_{k=1}^{n}1 \ll n\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right).$$

Similarly, we have:

$$\begin{aligned}|f(x)| &\leq 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right)+\left|\sum_{k=n}^{\infty}\left(\sum_{j=1}^{k}\lambda_j\right)[\cos kx-\cos(k+1)x]\right|\\ &\leq 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right)+\sum_{k=n}^{\infty}|V_k-V_{k+1}|\left|\widetilde{D}_k(x)\right|+V_n\left|\widetilde{D}_{n-1}(x)\right|\\ &\ll \sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right)+n\sum_{k=n}^{\infty}k|V_k-V_{k+1}|+n^2V_n\\ &\ll \sum_{k=1}^{n}\left(\sum_{j=1}^{k}\lambda_j\right)+\sum_{j=1}^{n}\lambda_j+n^2V_n\ll n\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right),\end{aligned}$$

where $\widetilde{D}_k(x)$ are defined by

$$\widetilde{D}_k(x):=\frac{\sin\left(k+\frac{1}{2}\right)x-\sin\frac{x}{2}-2k\sin\frac{x}{2}\cos(k+1)x}{2\sin\frac{x}{2}},\quad k\in\mathbb{N}.$$

Thus

$$|\psi(x)|\ll n\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right),\tag{14}$$

where a function $\psi(x)$ is either a $f(x)$ or a $g(x)$.

According to Lemma 2.3 (the condition (12) is satisfied), and using (14) we obtain

$$\begin{aligned}\int_0^{\pi}\gamma(x)\Phi(|\psi(x)|)dx &\ll \sum_{n=1}^{\infty}\Phi\left(n\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)\right)\int_{\pi/(n+1)}^{\pi/n}\gamma(x)dx\\ &\ll \sum_{n=1}^{\infty}\frac{\gamma_n}{n^2}\Phi\left(n\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)\right)\\ &\ll \sum_{n=1}^{\infty}\Phi\left(\sum_{j=1}^{n}\lambda_j\right)\frac{\gamma_n}{n^{2-p}}\left(\frac{n^{1-p}}{\gamma_n}\sum_{\nu=n}^{\infty}\frac{\gamma_\nu}{\nu^{2-p}}\right)^{p*}\\ &\ll \sum_{n=1}^{\infty}\frac{\gamma_n}{n^{2(1-p)}}\Phi\left(\frac{1}{n}\sum_{j=1}^{n}\lambda_j\right)\left(\frac{n^{1-p}}{\gamma_n}\sum_{\nu=n}^{\infty}\frac{\gamma_\nu}{\nu^{2-p}}\right)^{p*},\end{aligned}$$

where $p*:=\max(1,p)$.

Finally, by the assumption on $\{\gamma_n\}$, we get

$$\frac{n^{1-p}}{\gamma_n}\sum_{\nu=n}^{\infty}\frac{\gamma_\nu}{\nu^{1-p-\varepsilon}}\nu^{-\varepsilon-1}\ll 1$$

which together with above inequality immediately imply (13). The proof is completed. □

Theorem 3.2. *Let $\Phi(x) \in \triangle(p,q)$, $0 \le q \le p$. If λ_n satisfies (11) and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1-2q+p+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2(1+q)-p}} \Phi\left(n^2 \sum_{j=1}^{n} \lambda_j\right) < \infty \Longrightarrow g(x) \in L(\Phi,\gamma). \tag{15}$$

Proof. Let $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right]$. Then

$$\begin{aligned}
|g(x)| &\ll \sum_{k=1}^{n} kx \left(\sum_{j=1}^{k} \lambda_j\right) + \left|\sum_{k=n}^{\infty} \left(\sum_{j=1}^{k} \lambda_j\right) [\sin kx - \sin (k+1)x]\right| \\
&\ll \frac{1}{n}\sum_{k=1}^{n} k \left(\sum_{j=1}^{k} \lambda_j\right) + \sum_{k=n}^{\infty} |V_k - V_{k+1}| \left|\widehat{D}_k(x)\right| + V_n \left|\widehat{D}_{n-1}(x)\right| \\
&\ll \frac{1}{n}\sum_{k=1}^{n} k \left(\sum_{j=1}^{k} \lambda_j\right) + \sum_{j=1}^{n} \lambda_j + n^2 V_n \\
&\ll \frac{1}{n}\sum_{k=1}^{n} k \left(\sum_{j=1}^{k} \lambda_j\right) + \frac{1}{n(n+1)} \sum_{j=1}^{n} \lambda_j \sum_{k=1}^{n} k \\
&\ll \frac{1}{n}\sum_{k=1}^{n} k \left(\sum_{j=1}^{k} \lambda_j\right). \qquad (16)
\end{aligned}$$

By Lemmas 2.2–2.3 and the estimate (16) we have

$$\begin{aligned}
\int_0^{\pi} \gamma(x)\Phi(|g(x)|)dx &\ll \sum_{n=1}^{\infty} \Phi\left(\frac{1}{n}\sum_{k=1}^{n} k \left(\sum_{j=1}^{k} \lambda_j\right)\right) \int_{\pi/(n+1)}^{\pi/n} \gamma(x)dx \\
&\ll \sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2(1+q)}} \Phi\left(n \sum_{k=1}^{n} k \left(\sum_{j=1}^{k} \lambda_j\right)\right) \qquad (17) \\
&\ll \sum_{n=1}^{\infty} \Phi\left(n^2 \sum_{j=1}^{n} \lambda_j\right) \frac{\gamma_n}{n^{2(1+q)-p}} \left(\frac{n^{1+2q-p}}{\gamma_n} \sum_{\nu=n}^{\infty} \frac{\gamma_\nu}{\nu^{2(1+q)-p}}\right)^{p*},
\end{aligned}$$

where $p* := \max(1,p)$.

By the assumption on $\{\gamma_n\}$, we get

$$\frac{n^{1+2q-p}}{\gamma_n} \sum_{\nu=n}^{\infty} \frac{\gamma_\nu}{\nu^{1+2q-p-\varepsilon}} \nu^{-\varepsilon-1} \ll 1,$$

and hence (17) takes its form

$$\int_0^{\pi} \gamma(x)\Phi(|g(x)|)dx \ll \sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2(1+q)-p}} \Phi\left(n^2 \sum_{j=1}^{n} \lambda_j\right),$$

which proves (15). With this the proof of theorem is finished. □

4. Applications to $L^p(0,\pi)$ $(p \geq 0)$ spaces

Let $\Phi(t) = t$. Then the following are true.

Corollary 4.1. *Let $p \geq 0$. If λ_n satisfies condition (11) and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1+p+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} n^{2p-3}\gamma_n \sum_{j=1}^{n} \lambda_j < \infty \Longrightarrow \psi(x) \in L(0,\pi),$$

where a function $\psi(x)$ is either a sine or cosine series.

Corollary 4.2. *Let $0 \leq q \leq p$. If λ_n satisfies (11) and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1-2q+p+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} n^{p-2q}\gamma_n \sum_{j=1}^{n} \lambda_j < \infty \Longrightarrow g(x) \in L(0,\pi).$$

Let $\Phi(t) = t^p$. Then:

Corollary 4.3. *Let $p \geq 0$. If λ_n satisfies condition (11) and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1+p+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} n^{p-2}\gamma_n \left(\sum_{j=1}^{n} \lambda_j\right)^p < \infty \Longrightarrow \psi(x) \in L^p(0,\pi),$$

where a function $\psi(x)$ is either a sine or cosine series.

Corollary 4.4. *Let $0 \leq q \leq p$. If λ_n satisfies (11) and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1-2q+p+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} n^{3p-2q-2}\gamma_n \left(\sum_{j=1}^{n} \lambda_j\right)^p < \infty \Longrightarrow g(x) \in L^p(0,\pi).$$

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