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Localization of divisible residuated lattices

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Abstract. The aim of the present paper is to define the localization of a divisible residuated lattices L with respect to a topology $\mathcal{F}$ on L. In the last part of the paper is proved that the maximal divisible residuated lattice of quotients (defined in [15]) and the divisible residuated lattice of fractions relative to an $\wedge-$ closed system (defined in [3]) are divisible residuated lattices of localization.

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Basic Fuzzy logic (BL from now on) is the many-valued residuated logic introduced by Hájek in [11] to cope with the logic of continuous t-norms and their residua.

A remarkable construction in ring theory is the *localization ring* $A_{\mathcal{F}}$ associated with a Gabriel topology $\mathcal{F}$ on a ring A.

Using the model of localization ring, in [10], G. Georgescu defined for a bounded distributive lattice L *the localization lattice* $L_{\mathcal{F}}$ *of* L *with respect to a topology* $\mathcal{F}$ *on* L and prove that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of regular ideals); analogous results we have for lattices of fractions of bounded distributive lattices relative to $\wedge-$ closed systems.

The main aim of this paper is to develop a theory of localization for divisible residuated lattices. Since $BL-$ algebras are particular classes of divisible residuated lattices, the results of this paper generalize a part of the results from [2] for $BL-$ algebras.

1. Definitions and preliminaries

Definition 1.1. A *residuated lattice* ([1], [19]) is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:
(a_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, whose order is $\leq$;
(a_2) $(L, \odot, 1)$ is a commutative ordered monoid;
(a_3) $(\odot, \rightarrow)$ is an adjoint pair, i.e. $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$ for every $x, y, z \in L$.

The class $\mathcal{RL}$ of residuated lattices is equational (see [12]). For examples of residuated lattices see [3] and [19].

In this section by L we denote the universe of a residuated lattice. For $x \in L$, we denote $x^* = x \rightarrow 0$ and $(x^*)^* = x^{**}$.

We review some rules of calculus for residuated lattices L used in this paper:

Theorem 1.1. ([1], [19]) *Let $x, y, z \in L$. Then we have the following:*

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(c_1) $1 \to x = x, x \to x = 1, y \le x \to y, x \odot (x \to y) \le y, x \to 1 = 1, 0 \to x = 1, x \odot 0 = 0$;
(c_2) $x \le y$ *iff* $x \to y = 1$;
(c_3) $x \le y$ *implies* $x \odot z \le y \odot z, z \to x \le z \to y$ *and* $y \to z \le x \to z$;
(c_4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$, *so* $(x \odot y)^* = x \to y^* = y \to x^*$;
(c_5) $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \le y^*$;
(c_6) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(c_7) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

By $B(L)$ we denote the set of all complemented elements in the lattice $(L, \wedge, \vee, 0, 1)$. Complements are generally not unique, unless the lattice is distributive; in the case of residuated lattices, however, although the underlying lattices need not be distributive, the complements are unique ([9]). Also, if b is the complement of a, then a is the complement of b, $b = a^*, a^2 = a$ and $a^{**} = a$ ([1], [3]). So, $B(L)$ is a Boolean subalgebra of L, called the *Boolean center* of L.

Theorem 1.2. ([3]) *For* $e \in L$ *the following assertions are equivalent:*
(i) $e \in B(L)$;
(ii) $e \vee e^* = 1$.

Theorem 1.3. ([3]) *If* $e, f \in B(L)$ *and* $x, y \in L$, *then:*
(c_8) $e \odot x = e \wedge x$;
(c_9) $x \odot (x \to e) = e \wedge x, e \odot (e \to x) = e \wedge x$;
(c_{10}) $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)]$;
(c_{11}) $x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)]$.

Definition 1.2. ([11]) *A divisible residuated lattice* is a residuated lattice satisfying the divisibility equation*:*
(d) $x \odot (x \to y) = x \wedge y$.

The variety of divisible residuated lattices will be denoted by $\mathcal{RL}_d$. For examples of divisible residuated lattices see [4, 15, 16, 19].

Proposition 1.4. ([15]) *For a residuated lattice* L, *the following conditions are equivalent:*
(i) $L \in \mathcal{RL}_d$;
(ii) *For every* $x, y \in L$ *with* $x \le y$ *there exists* $z \in L$ *such that* $x = y \odot z$;
(iii) *For every* $x, y, z \in L$ *we have:*
(c_{12}) $x \to (y \wedge z) = (x \to y) \odot [(x \wedge y) \to z]$.

Corollary 1.5. ([4]) *Let* $L \in \mathcal{RL}_d$. *Then for every* $x, y, z \in L$ *we have:*
(c_{13}) $(x^{**} \to x)^* = 0, (x \to y)^{**} = x^{**} \to y^{**}, (x \odot y)^{**} = x^{**} \odot (x^{**} \wedge y^*)^*, (x \wedge y)^{**} = x^{**} \wedge y^{**}$;
(c_{14}) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;
(c_{15}) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(c_{16}) $y^* \le x \Rightarrow x \to (x \odot y)^{**} = y^{**}$.

Definition 1.3. Let $(P, \le)$ an ordered set. A nonempty subset I of P is called *order ideal (or decreasing set)* if, whenever $x \in I, y \in P$ and $y \le x$, we have $y \in I$; we denote by $I(P)$ the set of all order ideals of P.

For a divisibile residuated lattice L we denote by $Id(L)$ the set of all ideals of the lattice $(L, \wedge, \vee)$.

Remark 1.6. Clearly, $Id(L) \subseteq I(L)$ and if $I_1, I_2 \in I(L)$, then $I_1 \cap I_2 \in I(L)$. Also, if $I \in I(L)$, then $0 \in I$.

2. Topologies on a divisible residuated lattice

In what follows, by L we denote the universe of a divisible residuated lattice.

Definition 2.1. A non-empty set $\mathcal{F}$ of elements $I \in I(L)$ will be called a *topology* on L if the following axioms hold:
(a_4) If $I_1 \in \mathcal{F}, I_2 \in I(L)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$ (hence $L \in \mathcal{F}$);
(a_5) If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$.

Remark 2.1. 1. $\mathcal{F}$ is a topology on L if and only if $\mathcal{F}$ is a filter of the lattice of power set of L; for this reason a topology on $I(L)$ is usually called a Gabriel filter on $I(L)$.
2. Clearly, if $\mathcal{F}$ is a topology on L, then $(L, \mathcal{F} \cup \{\emptyset\})$ is a topological space.

Any intersection of topologies on L is a topology; so, the set $T(L)$ of all topologies of L is a complete lattice with respect to inclusion.

Example 2.1. If $I \in I(L)$, then the set $\mathcal{F}(I) = \{I' \in I(L) : I \subseteq I'\}$ is a topology on L.

Definition 2.2. ([15]) A non-empty set $I \subseteq L$ will be called *regular* if for every $x, y \in A$ such that $e \wedge x = e \wedge y$ for every $e \in I \cap B(A)$, then $x = y$.

Example 2.2. If we denote $R(L) = \{I \subseteq L : I$ is a regular subset of $L\}$, then $I(L) \cap R(L)$ is a topology on L.

Example 2.3. A nonempty set $I \subseteq L$ will be called *dense* (see [10]) if for $x \in L$ such that $e \wedge x = 0$ for every $e \in I \cap B(L)$, then $x = 0$. If we denote by $D(L)$ the set of all dense subsets of L, then $R(L) \subseteq D(L)$ and $\mathcal{F} = I(L) \cap D(L)$ is a topology on L.

Definition 2.3. ([3]) A subset $S \subseteq L$ is called $\wedge-$*closed* if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

Example 2.4. For any $\wedge-$ closed subset S of L, the set

$$\mathcal{F}_S = \{I \in I(L) : I \cap S \cap B(L) \neq \oslash\}$$

is a topology on L.

1. If S is a $\wedge-$closed systems of L such that $0 \in S$ we have $I \cap S \cap B(L) \neq \oslash$ for every $I \in I(L)$, so $\mathcal{F}_S = I(L)$.
2. If $0 \notin S$ then $\mathcal{F}_S = \{L\}$ (because, if $I \in I(L)$ and $1 \in I$ implies $I = L$).

3. $\mathcal{F}$-multipliers and localization divisible residuated lattices

Let $\mathcal{F}$ be a topology on a $MTL-$algebra L and we consider the relation $\theta_{\mathcal{F}}$ of L defined in the following way: $(x, y) \in \theta_{\mathcal{F}} \Leftrightarrow$ there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for any $e \in I \cap B(L)$.

Lemma 3.1. *$\theta_{\mathcal{F}}$ is a congruence on L.*

Proof. As in [2] for the case of $BL-$ algebras. □

We shall denote by $a/\theta_{\mathcal{F}}$ the congruence class of an element $a \in L$ and by $p_{\mathcal{F}} : L \to L/\theta_{\mathcal{F}}$ the canonical morphism of residuated lattices.

Proposition 3.2. *For $a \in L$, $a/\theta_{\mathcal{F}} \in B(L/\theta_{\mathcal{F}})$ if and only if there exists $I \in \mathcal{F}$ such that $a \vee a^* \geq e$ for every $e \in I \cap B(L)$. So, if $a \in B(L)$, then $a/\theta_{\mathcal{F}} \in B(L/\theta_{\mathcal{F}})$.*

Proof. Using Theorem 1.2, for $a \in L$, we have

$$\begin{aligned} a/\theta_{\mathcal{F}} \in B(L/\theta_{\mathcal{F}}) &\Leftrightarrow a/\theta_{\mathcal{F}} \vee (a/\theta_{\mathcal{F}})^{*} = 1/\theta_{\mathcal{F}} \Leftrightarrow (a \vee a^{*})/\theta_{\mathcal{F}} = 1/\theta_{\mathcal{F}} \\ &\Leftrightarrow \text{there exist } I \in \mathcal{F} : (a \vee a^{*}) \wedge e = 1 \wedge e = e, \\ &\quad\ \text{for every } e \in I \cap B(L) \\ &\Leftrightarrow a \vee a^{*} \geq e, \text{ for every } e \in I \cap B(L). \end{aligned}$$

If $a \in B(L)$, then for every $I \in \mathcal{F}$, $1 = a \vee a^{*} \geq e$, for every $e \in I \cap B(L)$, hence $a/\theta_{\mathcal{F}} \in B(L/\theta_{\mathcal{F}})$. □

Corollary 3.3. *If* $\mathcal{F} = I(L) \cap R(L)$, *then for* $a \in L$, $a/\theta_{\mathcal{F}} \in B(L/\theta_{\mathcal{F}})$ *if and only if* $a \in B(L)$.

Definition 3.1. Let $\mathcal{F}$ be a topology on L. A $\mathcal{F}-$ *multiplier* is a mapping $f : I \to L/\theta_{\mathcal{F}}$ where $I \in \mathcal{F}$ and for every $x \in I$ and $e \in B(L)$ the following axioms are fulfilled:
(a_6) $f(e \odot x) = e/\theta_{\mathcal{F}} \wedge f(x) = e/\theta_{\mathcal{F}} \odot f(x)$;
(a_7) $f(x) \leq x/\theta_{\mathcal{F}}$.

By $dom(f) \in \mathcal{F}$ we denote the domain of f; if $dom(f) = L$, we called f *total.*

To simplify language, we will use $\mathcal{F}-$ *multiplier* instead *partial* $\mathcal{F}-$ *multiplier,* using *total* to indicate that the domain of a certain $\mathcal{F}-$ multiplier is L.

If $\mathcal{F} = \{L\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of L so a $\mathcal{F}-$ multiplier is a total multiplier.

The maps $\mathbf{0}, \mathbf{1} : L \to L/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in L$ are $\mathcal{F}-$ multipliers in the sense of Definition 3.1.

Also, for $a \in B(L)$, $f_a : L \to L/\theta_{\mathcal{F}}$ defined by $f_a(x) = a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$ for every $x \in L$, is a $\mathcal{F}-$ multiplier. If $dom(f_a) = L$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$ and $\overline{f_1} = \mathbf{1}$.

We shall denote by $M(I, L/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}$ and $M(L/\theta_{\mathcal{F}}) = \bigcup_{I \in \mathcal{F}} M(I, L/\theta_{\mathcal{F}})$. If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$ we have a canonical mapping $\varphi_{I_1,I_2} : M(I_2, L/\theta_{\mathcal{F}}) \to M(I_1, L/\theta_{\mathcal{F}})$ defined by $\varphi_{I_1,I_2}(f) = f_{|I_1}$ for $f \in M(I_2, L/\theta_{\mathcal{F}})$. Let us consider the directed system of sets

$\langle \{M(I, L/\theta_{\mathcal{F}})\}_{I \in \mathcal{F}},\ \{\varphi_{I_1,I_2}\}_{I_1,I_2 \in \mathcal{F}, I_1 \subseteq I_2} \rangle$ and denote by $L_{\mathcal{F}}$ the inductive limit (in the category of sets) $L_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, L/\theta_{\mathcal{F}})$. For any $\mathcal{F}-$ multiplier $f : I \to L/\theta_{\mathcal{F}}$ we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $L_{\mathcal{F}}$.

Remark 3.4. If $f_i : I_i \to L/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}-$ multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $L_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.

Proposition 3.5. *If* $I_1, I_2 \in \mathcal{F}$ *and* $f_i \in M(I_i, L/\theta_{\mathcal{F}}), i = 1, 2,$ *then*
(c_{17}) $f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] = f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)]$, *for every* $x \in I_1 \cap I_2$.

Proof. Using (d), for $x \in I_1 \cap I_2$ we have $f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{(a_7)}{=} (x/\theta_{\mathcal{F}} \wedge f_1(x)) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{(d)}{=} x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \odot (x/\theta_{\mathcal{F}} \to f_2(x)) = [x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_2(x))] \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \overset{(a_7)}{=} f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)]$. □

Let $f_i : I_i \to L/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), $\mathcal{F}-$multipliers. Let us consider the mappings $f_1 \wedge f_2, f_1 \vee f_2, f_1 \otimes f_2, f_1 \rightsquigarrow f_2 : I_1 \cap I_2 \to L/\theta_{\mathcal{F}}$ defined by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x), (f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$

$$(f_1 \otimes f_2)(x) = f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{(c_{17})}{=} f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)],$$
$$(f_1 \rightsquigarrow f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)],$$

for any $x \in I_1 \cap I_2$, and let

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = (\widehat{I_1 \cap I_2, f_1 \wedge} f_2), \widehat{(I_1, f_1)} \curlyvee \widehat{(I_2, f_2)} = (\widehat{I_1 \cap I_2, f_1 \vee} f_2),$$
$$\widehat{(I_1, f_1)} \otimes \widehat{(I_2, f_2)} = (\widehat{I_1 \cap I_2, f_1 \otimes} f_2), \widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)} = (\widehat{I_1 \cap I_2, f_1 \rightsquigarrow} f_2).$$

Clearly, the definitions of the operations $\curlywedge, \curlyvee, \otimes$ and $\longmapsto$ on $L_{\mathcal{F}}$ are correct.

As in the case of $BL-$algebras(see[2]) we deduce

Lemma 3.6. $f_1 \wedge f_2, f_1 \vee f_2, f_1 \otimes f_2, f_1 \rightsquigarrow f_2 \in M(I_1 \cap I_2, L/\theta_{\mathcal{F}})$.

Proposition 3.7. $(L_{\mathcal{F}}, \curlywedge, \curlyvee, \otimes, \longmapsto, \mathbf{0} = \widehat{(L, \mathbf{0})}, \mathbf{1} = \widehat{(L, \mathbf{1})})$ *is a divisible residuated lattice.*

Proof. We verify the axioms of divisible residuated lattices.

(a_1). Obviously $(L_{\mathcal{F}}, \curlywedge, \curlyvee, \mathbf{0} = \widehat{(L, \mathbf{0})}, \mathbf{1} = \widehat{(L, \mathbf{1})})$ is a bounded lattice, where the order on $L_{\mathcal{F}}$ is given by $\widehat{(I_1, f_1)} \le \widehat{(I_2, f_2)}$ iff there is $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_1(x) \le f_2(x)$ for every $x \in I$.

(a_2). As in the case of $BL-$ algebras (see [2]), by using (c_{17}).

(a_3). Let $f_i \in M(I_i, L/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1, 2, 3$.

If $\widehat{(I_1, f_1)} \le \widehat{(I_2, f_2)} \mapsto \widehat{(I_3, f_3)}$, then $\widehat{(I_1, f_1)} \le (\widehat{I_2 \cap I_3, f_2 \rightsquigarrow} f_3)$, so there is $I \in \mathcal{F}$ such that $I \subseteq I_1 \cap I_2 \cap I_3$ and for every $x \in I$, we have $f_1(x) \le (f_2 \rightsquigarrow f_3)(x) \Rightarrow f_1(x) \le x/\theta_{\mathcal{F}} \odot [f_2(x) \to f_3(x)]$. So, by (c_3), $f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \le x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \odot [f_2(x) \to f_3(x)] \overset{(d)}{\Rightarrow} f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \le f_2(x) \odot [f_2(x) \to f_3(x)] \le f_3(x) \Rightarrow (f_1 \otimes f_2)(x) \le f_3(x)$, that is, $\widehat{(I_1, f_1)} \otimes \widehat{(I_2, f_2)} \le \widehat{(I_3, f_3)}$. Conversely, if $\widehat{(I_1, f_1)} \otimes \widehat{(I_2, f_2)} \le \widehat{(I_3, f_3)}$, then there is $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2 \cap I_3$ such that for every $x \in I$, $(f_1 \otimes f_2)(x) \le f_3(x) \Rightarrow f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)] \le f_3(x)$. Obviously for $x \in I$, $x/\theta_{\mathcal{F}} \to f_1(x) \le f_2(x) \to f_3(x) \overset{(c_3)}{\Rightarrow} x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \le x/\theta_{\mathcal{F}} \odot (f_2(x) \to f_3(x)) \Rightarrow f_1(x) \le (f_2 \rightsquigarrow f_3)(x)$. So, $\widehat{(I_1, f_1)} \le \widehat{(I_2, f_2)} \longmapsto \widehat{(I_3, f_3)}$. Then $\widehat{(I_1, f_1)} \le \widehat{(I_2, f_2)} \longmapsto \widehat{(I_3, f_3)}$ iff $\widehat{(I_2, f_2)} \otimes \widehat{(I_1, f_1)} \le \widehat{(I_3, f_3)}$. Since the divisibility (d) is proved as in the case of $BL-$ algebras (see [2]), we deduce that $(L_{\mathcal{F}}, \curlywedge, \curlyvee, \otimes, \longmapsto, \mathbf{0} = \widehat{(L, \mathbf{0})}, \mathbf{1} = \widehat{(L, \mathbf{1})})$ is a divisible residuated lattice. □

Definition 3.2. The divisible residuated lattice $L_{\mathcal{F}}$ will be called *the localization divisible residuated lattice of L with respect to the topology* $\mathcal{F}$.

Remark 3.8. If divisible residuated lattice $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a $BL-$ algebra in [2] $(L_{\mathcal{F}}, \curlywedge, \curlyvee, \otimes, \longmapsto, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ called the localization BL-algebra of L with respect to the topology $\mathcal{F}$.

Lemma 3.9. *Let the map* $v_{\mathcal{F}} : B(L) \to L_{\mathcal{F}}$ *defined by* $v_{\mathcal{F}}(a) = \widehat{(L, \overline{f_a})}$ *for every* $a \in B(L)$. *Then:*

(i) $v_{\mathcal{F}}$ *is a morphism of residuated lattices;*

(ii) *For* $a \in B(L)$, $\widehat{(L, \overline{f_a})} \in B(L_{\mathcal{F}})$;

(iii) $v_{\mathcal{F}}(B(L)) \in R(L_{\mathcal{F}})$.

Proof. $(i), (iii)$. As in the case of $BL-$ algebras (see [2]).

(ii). For $a \in B(L)$ we have $a \vee a^* = 1$, hence $(a \wedge x) \vee [x \odot (a \wedge x)^*] = (a \wedge x) \vee [x \odot (a^* \vee x^*)] \overset{(c_6)}{=} (a \wedge x) \vee [(x \odot a^*) \vee (x \odot x^*)] \overset{(c_5)}{=} (a \wedge x) \vee [(x \odot a^*) \vee 0) \overset{(c_8)}{=}$

$(a \wedge x) \vee (x \wedge a^*) \overset{(c_{15})}{=} x \wedge (a \vee a^*) = x \wedge 1 = x$, for every $x \in L$. Since $L \in \mathcal{F}$ we deduce that $(a \wedge x)/\theta_{\mathcal{F}} \vee [x/\theta_{\mathcal{F}} \odot ((a \wedge x)/\theta_{\mathcal{F}})^*] = x/\theta_{\mathcal{F}}$ hence $\overline{f_a} \vee (\overline{f_a})^* = \mathbf{1}$, that is, $\widehat{(L, \overline{f_a})} \curlyvee \widehat{(L, \overline{f_a})}^* = \widehat{(L, \mathbf{1})}$, so $\widehat{(L, \overline{f_a})} \in B(L_{\mathcal{F}})$. □

4. Applications

In the following we describe the localization divisible residuated lattice $L_{\mathcal{F}}$ in some special instances.

1. If $I \in I(L)$, and $\mathcal{F}$ is the topology $\mathcal{F}(I) = \{I' \in I(L) : I \subseteq I'\}$ (see Example 2.1), then $L_{\mathcal{F}}$ is isomorphic with $M(I, L/\theta_{\mathcal{F}})$ and $v_{\mathcal{F}} : B(L) \to L_{\mathcal{F}}$ is defined by $v_{\mathcal{F}}(a) = \overline{f_{a|I}}$ for every $a \in B(L)$.

If I is a regular subset of L, then $\theta_{\mathcal{F}}$ is the identity, hence $L_{\mathcal{F}}$ is isomorphic with $M(I, L)$, which in generally is not a Boolean algebra. For example, if $I = L = [0, 1]$ is the *Lukasiewicz structure* (see [16]) then $L_{\mathcal{F}}$ is not a Boolean algebra (see [2]).

2. Main remark. To obtain *the maximal divisible residuated lattice of quotients* $Q(L)$ (introduced in [15]) as a localization relative to a topology $\mathcal{F}$, we have to develop another theory of multipliers (meaning we add new axioms for $\mathcal{F}$-multipliers).

Definition 4.1. Let $\mathcal{F}$ be a topology on L. A *strong - $\mathcal{F}-$ multiplier* is a mapping $f : I \to L/\theta_{\mathcal{F}}$ (where $I \in \mathcal{F}$) which verifies the axioms $(a_6), (a_7)$ (see Definition 3.1) and

(a_8) If $e \in I \cap B(L)$, then $f(e) \in B(L/\theta_{\mathcal{F}})$;

(a_9) $(x/\theta_{\mathcal{F}}) \wedge f(e) = (e/\theta_{\mathcal{F}}) \wedge f(x)$, for every $e \in I \cap B(L)$ and $x \in I$.

Remark 4.1. If $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a divisible residuated lattice, the maps $\mathbf{0}, \mathbf{1} : L \to L/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in L$ are strong - $\mathcal{F}-$ multipliers. We recall that if $f_i : I_i \to L/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$) are $\mathcal{F}-$multipliers $f_1 \wedge f_2, f_1 \vee f_2, f_1 \otimes f_2, f_1 \rightsquigarrow f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$ defined by $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x), (f_1 \vee f_2)(x) = f_1(x) \vee f_2(x), (f_1 \otimes f_2)(x) = f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{(c_{17})}{=} f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)], (f_1 \rightsquigarrow f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)]$, for any $x \in I_1 \cap I_2$ are $\mathcal{F}-$multipliers. If f_1, f_2 are strong - $\mathcal{F}-$ multipliers then the multipliers $f_1 \wedge f_2, f_1 \vee f_2, f_1 \otimes f_2, f_1 \rightsquigarrow f_2$ are also strong - $\mathcal{F}-$ multipliers (the proof is as in the case of $BL-$algebras, see [2]).

Remark 4.2. Analogous as in the case of $\mathcal{F}-$ multipliers if we work with strong-$\mathcal{F}-$ multipliers we obtain a divisible residuated lattice of $L_{\mathcal{F}}$ denoted by $s - L_{\mathcal{F}}$ which will be called the *strong-localization divisible residuated lattice of L with respect to the topology $\mathcal{F}$.*

So, if $\mathcal{F} = I(L) \cap R(L)$ is the topology of regular ideals, then $\theta_{\mathcal{F}}$ is the identity congruence of L and we obtain the definition for multipliers on L, so

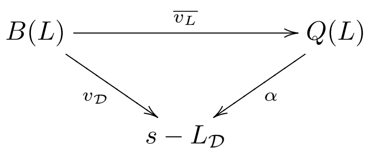
$$s - L_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} (s - M(I, L)),$$

where $s - M(I, L)$ is the set of strong multipliers of L having the domain I.

In this situation we obtain:

Proposition 4.3. *In the case $\mathcal{F} = I(L) \cap R(L)$, $L_{\mathcal{F}}$ is exactly the maximal divisible residuated lattice $Q(L)$ of quotients of L (introduced in* [15]*) which is a Boolean algebra. If divisible residuated lattice L is a $BL-$ algebra, $L_{\mathcal{F}}$ is exactly the maximal BL-algebra $Q(L)$ of quotients of L.*

3. Denoting by $\mathcal{D}$ the topology of dense ordered ideals of L, then (since $R(L) \subseteq D(L)$) there exists a morphism of residuated lattices $\alpha : Q(L) \to s{-}L_{\mathcal{D}}$ such that the diagram



is commutative (i.e. $\alpha \circ \overline{v_A} = v_{\mathcal{D}}$). Indeed, if $[f, I] \in Q(L)$ (with $I \in I(L) \cap R(L)$ and $f : I \to L$ a strong multiplier) we denote by $f_{\mathcal{D}}$ the strong - $\mathcal{D}-$multiplier $f_{\mathcal{D}} : I \to L/\theta_{\mathcal{D}}$ defined by $f_{\mathcal{D}}(x) = f(x)/\theta_{\mathcal{D}}$ for every $x \in I$. Thus, α is defined by $\alpha([f, I]) = [f_{\mathcal{D}}, I]$.

4. Let $S \subseteq L$ a $\wedge$-closed system of divisible residuated lattice L. Consider the following congruence on $L : (x, y) \in \theta_S \Leftrightarrow$ there exists $e \in S \cap B(L)$ such that $x \wedge e = y \wedge e$ (see [3]). $L[S] = L/\theta_S$ will be called the *divisible residuated lattice of fractions of L relative to the $\wedge-$closed system S.*

As in the case of $BL-$algebras we obtain the following result:

Proposition 4.4. *If $\mathcal{F}_S$ is the topology associated with a $\wedge$-closed system $S \subseteq L$, then the divisible residuated lattice $s - L_{\mathcal{F}_S}$ is isomorphic with $B(L[S])$.*

Remark 4.5. In the proof of Proposition 4.4 the axiom (a_9) is not necessarily.

Concluding remarks

Since in particular a $BL-$ algebra is a divisible residuated lattice we obtain a part of the results about localization of $BL-$ algebras (see [2]), so we deduce that the main results of this paper are generalization of the analogous result relative to $BL-$ algebras from [2].

We use in the construction of localization divisible residuated lattice $L_{\mathcal{F}}$ the Boolean center $B(L)$ of divisible residuated lattice L; as a consequence of this fact, $s - L_{\mathcal{F}}$ is a Boolean algebra in some particular cases.

A very interesting subject for future research would be a treatment of the localization for divisible residuated lattice or residuated lattices without use the Boolean center.

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