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Additional results for an anisotropic problem with variable exponents

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Abstract. We treat a nonlinear elliptic problem with constant Dirichlet condition involving $\overrightarrow{p}(\cdot)$-Laplace type operators. More exactly, we discuss the uniqueness the multiplicity of the solution, under different hypotheses.

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1. Introduction and preliminaries

In the present paper we consider $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ to be a rectangular-like domain, that is, a union of finitely many rectangular domains (or cubes) with edges parallel to the coordinate axes. We are interested in problems involving $\overrightarrow{p}(\cdot)$-Laplace type operators, where $p : \overline{\Omega} \to \mathbb{R}^N$ is the vectorial function

$$\overrightarrow{p}(x) = (p_1(x), \ldots, p_N(x)),$$

with $p_i \in C_+(\overline{\Omega})$for all $i \in \{1, \ldots, N\}$ and

$$C_+(\overline{\Omega}) = \{r \in C(\overline{\Omega}; \mathbb{R}) : \quad \inf_{x\in\Omega} r(x) > 1\}.$$

We introduce the following notations to help us simplify our writting.

$$p_M(x) = \max\{p_1(x), \ldots, p_N(x)\},$$

$$p_m(x) = \min\{p_1(x), \ldots, p_N(x)\}$$

and

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^{N} 1/p_i(x)}.$$

In addition, for $r \in C_+(\overline{\Omega})$, we denote

$$r^+ = \sup_{x\in\Omega} r(x), \quad r^- = \inf_{x\in\Omega} r(x)$$

and

$$r^\star(x) = \begin{cases} Nr(x)/[N - r(x)] & \text{if} \quad r(x) < N, \\ \infty & \text{if} \quad r(x) \geq N. \end{cases}$$

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Our goal is to continue the study started in [6] concerning the following class of anisotropic problems with variable exponents,

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) + b(x)|u|^{p_M(x)-2}u = \lambda f(x,u), & \text{for} \quad x \in \Omega \\ u(x) \equiv \text{constant}, & \text{for} \quad x \in \partial\Omega \end{cases} \tag{1}$$

where the functions $b : \Omega \to \mathbb{R}, a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}, f : \Omega \times \mathbb{R} \to \mathbb{R}$ are fulfilling the following hypotheses for every $i \in \{1, \ldots, N\}$.

(B): $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.

(A0): a_i is a Carathéodory function.

(A1): There exists a positive constant $\bar{c}_i$ such that a_i fulfills

$$|a_i(x,s)| \leq \bar{c}_i \left(d_i(x) + |s|^{p_i(x)-1}\right),$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$, where $d_i \in L^{p_i'(\cdot)}(\Omega)$ (with $1/p_i(x) + 1/p_i'(x) = 1$) is a nonnegative function.

(A2): There exists $k_i > 0$ such that

$$k_i|s|^{p_i(x)} \leq a_i(x,s)s \leq p_i(x)\ A_i(x,s),$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$.

(A3): The monotonicity condition

$$[a_i(x,s) - a_i(x,t)](s-t) > 0$$

takes place for all $x \in \Omega$ and all $s,\, t \in \mathbb{R}$ with $s \neq t$.

(A4): $a_i(x,0) = 0$ for all $x \in \partial\Omega$.

(F0): f is a Carathéodory function.

(F1): There exist $k > 0$ and $q \in C_+(\overline{\Omega})$ with $p_M^+ < q^- < q^+ < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$, such that f verifies

$$|f(x,s)| \leq k\left(1 + |s|^{q(x)-1}\right)$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$.

(F2): There exist $\gamma > p_M^+$ and $s_0 > 0$ such that the Ambrosetti-Rabinowitz condition

$$0 < \gamma F(x,s) \leq s f(x,s)$$

holds for all $x \in \Omega$ and for all $s \in \mathbb{R}$ with $|s| > s_0$.

(F3): $\displaystyle\lim_{|s|\to 0} \frac{f(x,s)}{|s|^{p_M^+-1}} = 0$ for all $x \in \Omega$.

Under the above hypotheses, the existence of problem (1) was established in ([6, Theorem 4]). By adding hypothesis

(F4): f is fulfilling

$$(f(x,s) - f(x,t))(s-t) < 0,$$

for all $x \in \overline{\Omega}$ and $s, t \in \mathbb{R}$ with $s \neq t$.

we will show the uniqueness of this solution.

On the other hand, if instead of (F4) we add hypotheses

(A5): A_i is even in s, that is, $A_i(x,-s) = A_i(x,s)$ for all $x \in \Omega$

and

(F5): f is odd in V, that is, $f(x,-s) = -f(x,s)$ for all $x \in \Omega$, we infer the existence of infinitely many weak solutions to problem (1).

The interest in treating problem (1) is not purely mathematical. It is important to note that having such a general operator as

$$\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) \tag{2}$$

can lead to a large scale of applications since we can obtain various operators from it. We refer here to two well known operators that are particular cases of (2). When taking

$$a_i(x,s) = |s|^{p_i(x)-2}s \text{ for all } i \in \{1,\dots,N\},$$

(2) becomes in particular the $\overrightarrow{p}(\cdot)$ - Laplace operator

$$\Delta_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right). \tag{3}$$

This is why the operators (3) are often known as generalized $\overrightarrow{p}(\cdot)$ - Laplace type operators. At the same time, when choosing

$$a_i(x,s) = (1+|s|^2)^{(p_i(x)-2)/2} s \quad \text{for all } i \in \{1, ..., N\},$$

we are led to the anisotropic mean curvature operator with variable exponent

$$\sum_{i=1}^{N} \partial_{x_i} \left[\left(1 + |\partial_{x_i} u|^2\right)^{(p_i(x)-2)/2} \partial_{x_i} u \right].$$

Working with variable exponents, hence working in the framework of variable exponent spaces, opens the door for multiple applications.

We refer here to the electrorheological fluids and to the thermorheological fluids that have multiple applications to hydraulic valves and clutches, brakes, shock absorbers, robotics, space technology, tactile displays etc (see for example [1, 18, 21, 22, 23]). In addition, the variable exponent spaces are involved in studies that provide other types of applications, like the ones in elastic materials [25], image restoration [8], contact mechanics [4] etc. We recall in what follows the definition of the variable exponent Lebesgue space, that is,

$$L^{r(\cdot)}(\Omega) = \{u : \, u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{r(x)}\, dx < \infty\},$$

where $r \in C_+(\overline{\Omega})$. This space, endowed with the Luxemburg norm,

$$\|u\|_{L^{r(\cdot)}(\Omega)} = \inf \left\{ \mu > 0 : \quad \int_\Omega \left| \frac{u(x)}{\mu} \right|^{r(x)} dx \le 1 \right\},$$

is a separable and reflexive Banach space [15, Theorem 2.5, Corollary 2.7].

Furthermore, the Hölder-type inequality

$$\left| \int_\Omega u(x) v(x)\, dx \right| \le 2\, \|u\|_{L^{r(\cdot)}(\Omega)} \|v\|_{L^{r'(\cdot)}(\Omega)} \tag{4}$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ and $v \in L^{r'(\cdot)}(\Omega)$ (see [15, Theorem 2.1]), where we denoted by $L^{r'(\cdot)}(\Omega)$ the conjugate space of $L^{r(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $1/r(x) + 1/r'(x) = 1$ (see [15, Corollary 2.7]).

Starting from these variable exponent spaces, a theory of anisotropic spaces with variable exponents was developed (see [12]) and lately many problems were studies in this framework (see for example [5, 9]).

The anisotropic space with variable exponent is

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_M(\cdot)}(\Omega) : \quad \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega) \text{ for all } i \in \{1, \dots, N\}\}$$

and it is endowed with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|u\|_{L^{p_M(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)}.$$

The space $\left(W^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)}\right)$ is a reflexive Banach space (see [9, Theorems 2.1 and 2.2]).

Of course, these spaces have many other interesting properties, which were omitted for brevity, but they can be found in the previously mentioned references.

We recall a subspace of $W^{1,\vec{p}(\cdot)}(\Omega)$, that is,

$$V = \left\{u \in W^{1,\vec{p}(\cdot)}(\Omega) : \quad u\big|_{\partial\Omega} \equiv \text{constant}\right\}. \tag{5}$$

$(V, \|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)})$ is a reflexive Banach space (see [6, Theorem 5]) and it represents the space in which we search for weak solutions.

2. Main results

Taking into consideration condition (A4) we can introduce the notion of weak solution to our problem.

Definition 2.1. We define the weak solution for problem (1) as a function $u \in V$ satisfying:

$$\int_\Omega \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\partial_{x_i} v \, dx + \int_\Omega b(x)|u|^{p_M(x)-2} uv \, dx - \lambda \int_\Omega f(x,u)v \, dx = 0, \tag{6}$$

for all $v \in V$.

As in [6], the energy functional corresponding to (1) is defined as $I : V \to \mathbb{R}$,

$$I(u) = \int_\Omega \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) \, dx + \int_\Omega \frac{b(x)}{p_M(x)}|u|^{p_M(x)} \, dx - \lambda \int_\Omega F(x,u) \, dx. \tag{7}$$

Functional I is of class C^1 and its given by

$$\langle I'(u), v\rangle = \int_\Omega \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\partial_{x_i} v \, dx + \int_\Omega b(x)|u|^{p_M(x)-2} uv \, dx - \lambda \int_\Omega f(x,u)v \, dx,$$

for all u, $v \in V$.

2.1. The existence result.

Theorem 2.1. *Let $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \dots, N\}$ with $p_M^+ < \overline{p}^*(x)$ for all $x \in \Omega$. Assume that $b : \Omega \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, N\}$, satisfy conditions (B), (F0)-(F4), respectively (A0)-(A4). Then, problem* (1) *has a unique nontrivial weak solution in V for every $\lambda > 0$.*

We recall the existence result for problem (1) (see [6, Theorem 4]).

Theorem 2.2. *Let $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1,\dots,N\}$ with $p_M^+ < \overline{p}^*(x)$ for all $x \in \Omega$. Assume that $b : \Omega \to \mathbb{R}$ satisfies (B) and that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1,\dots,N\}$, are Carathéodory functions satisfying (F1)-(F3), respectively (A1)-(A4). Then, problem* (1) *has at least a nontrivial weak solution in V for every $\lambda > 0$.*

So, it remains to prove the uniqueness of the solution for problem (1). For this, we need the following lemma.

Lemma 2.3. *Let $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1,\dots,N\}$ with $p_M^+ < \overline{p}^*(x)$ for all $x \in \Omega$. Assume that $b : \Omega \to \mathbb{R}$ satisfies (B) and that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1,\dots,N\}$, are Carathéodory functions satisfying (F1)-(F4) respectively (A1)-(A4). Then, problem* (1) *has at most a nontrivial weak solution in V for every $\lambda > 0$.*

Proof. By proceeding as in [3], we suppose that there exist two nontrivial solutions to problem (1), that is, u_1 and u_2. Making in (6) the substitutions u with u_1 and v with $u_1 - u_2$, we obtain

$$\begin{aligned} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_1)\partial_{x_i}(u_1 - u_2)\, dx \quad &+ \quad \int_\Omega b(x)|u_1|^{p_M(x)-2} u_1(u_1 - u_2)\, dx \\ &- \quad \lambda \int_\Omega f(x, u_1)(u_1 - u_2)\, dx = 0. \end{aligned} \tag{8}$$

Now, we replace in (1) the solution u with u_2 and v with $u_2 - u_1$ and we obtain

$$\begin{aligned} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_2)\partial_{x_i}(u_2 - u_1)\, dx \quad &+ \quad \int_\Omega b(x)|u_2|^{p_M(x)-2} u_2(u_2 - u_1)\, dx \\ &- \quad \lambda \int_\Omega f(x, u_2)(u_2 - u_1)\, dx = 0. \end{aligned} \tag{9}$$

Combining relations (8) and (9) we infer

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N \left[a_i(x, \partial_{x_i} u_1) - a_i(x, \partial_{x_i} u_2)\right]\left(\partial_{x_i}(u_1 - u_2)\right)\, dx \\ &\quad + \quad \int_\Omega b(x)\left[|u_1|^{p_M(x)-2}u_1 - |u_2|^{p_M(x)-2}u_2\right](u_1 - u_2)\, dx \\ &\quad - \quad \lambda \int_\Omega \left[f(x, u_1) - f(x, u_2)\right](u_1 - u_2)\, dx = 0. \end{aligned}$$

The hypotheses (A3) and (F4) give us that all terms are positive unless $u_1 = u_2$. So, $u_1 = u_2$ and this shows that the solution of problem (1) is unique. □

Theorem 2.2 and Lemma 2.3 are sufficient to conclude that Theorem 2.1 takes place.

2.2. The multiplicity result. Adding to the hypotheses of Theorem 2.2 the symmetry conditions (A5) and (F5), we will present a multiplicity theorem based on the following symmetric mountain pass theorem.

Theorem 2.4. ([13, Theorem 11.5]) *Let X be a real infinite dimensional Banach space and $\Phi \in C^1(X;\mathbb{R})$ a functional satisfying the Palais-Smale condition. Assume that Φ satisfies:*

(i) *$\Phi(0) = 0$ and there are constants $\rho, \tau > 0$ such that*

$$\Phi_{|\partial B_\rho} \geq \tau,$$

(ii) Φ *is even, and*

(iii) *for all finite dimensional subspaces* $\widetilde{X} \subset X$ *there exists* $R = R(\widetilde{X}) > 0$ *such that*

$$\Phi(u) \leq 0 \textit{ for } u \in \widetilde{X} \setminus B_R(\widetilde{X}).$$

Then Φ possesses an unbounded sequence of critical values characterized by a minimax argument.

We present our multiplicity result.

Theorem 2.5. *Let $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \dots, N\}$ with $p_M^+ < \overline{p}^*(x)$ for all $x \in \Omega$. Assume that $b : \Omega \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, N\}$, satisfy conditions (B), (F0)-(F3), (F5) and (A0)-(A5). Then, problem* (1) *has infinitely many weak solutions in V for every $\lambda > 0$.*

The idea of the proof is to use Theorem 2.4 to obtain multiple weak solutions for problem (1). We recall here the following results concerning the functional I under hypotheses (A1)-(A4), (F1)-(F3).

Lemma 2.6. ([6, Lemma 1]) *The energy functional I introduced by* (7) *satisfies the Palais-Smale condition.*

Lemma 2.7. ([6, Lemma 2]) *There exist τ, $\rho > 0$ such that $I(u) \geq \tau$ for all $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ with $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \rho$.*

Moreover, by means of Lemma 2.6 it has been shown that I satisfies the Palais-smale condition, so it can be seen that $I(0) = 0$. In addition, the fact that A_i and F are even in the second variable implies that I is even. Hence, in order to get the multiplicity of solutions, we only need to prove the following.

Lemma 2.8. *For any finite dimensional $\overline{V} \subset V$ there exists $R = R(\widetilde{\overline{V}}) > 0$ such that*

$$I(u) \leq 0 \textit{ for all } u \in \overline{V} \setminus B_R(\overline{V}).$$

Proof. We adapt the techniques from [2] to our case. By (A1) and (A5), we have

$$0 \leq A_i(x, s) \leq \overline{c_i} \left| \int_0^{|s|} \left(d_i(x) + |t|^{p_i(x)-1} \right) dt \right| \leq \overline{c_i} \left(|d_i(x)||s| + \frac{|s|^{p_i(x)}}{p_i(x)} \right),$$

for all $x \in \Omega$, $s \in \mathbb{R}$. From the above we obtain

$$\begin{aligned} 0 \leq I(v) \leq \overline{C} \int_\Omega \sum_{i=1}^N |d_i(x)||\partial_{x_i} v| \, dx + \frac{\overline{C}}{p_m^-} \int_\Omega \sum_{i=1}^N |\partial_{x_i} v|^{p_i(x)} \, dx + \\ + \frac{\|b\|_{L^\infty(\Omega)}}{p_m^-} \int_\Omega |v|^{p_M(x)} \, dx - \lambda \int_\Omega F(x, v) \, dx \text{ for all } v \in V, \end{aligned} \tag{10}$$

where $\overline{C} = \max\{\overline{c_i} : i \in \{1, \dots, N\}\}$.

Let $\overline{V} \subset V$ be a finite dimensional subspace, $u \in \overline{V} \setminus \{0\}$ and $t > 1$. So, by (10)

$$\begin{aligned} I(tu) \quad \leq \quad & \overline{C} \int_\Omega \sum_{i=1}^N |d_i(x)||\partial_{x_i}(tu)| \, dx + \frac{\overline{C}}{p_m^-} \int_\Omega \sum_{i=1}^N |\partial_{x_i}(tu)|^{p_i(x)} \, dx \\ + \quad & \frac{\|b\|_{L^\infty(\Omega)}}{p_m^-} \int_\Omega |tu|^{p_M(x)} \, dx - -\lambda \int_\Omega F(x, tu) \, dx. \end{aligned} \tag{11}$$

By (F2), there exists $\widetilde{k} = \widetilde{k}(x) > 0$ such that

$$F(x, s) \geq \widetilde{k}(x)|s|^\gamma \text{ for all } x \in \Omega \text{ and all } s \in \mathbb{R} \text{ with } |s| > s_0.$$

By combining the previous two inequalities and by using the Hölder type inequality (4), we infer that, for all $t > 1$ and $u \in \overline{V} \setminus \{0\}$,

$$\begin{aligned} I(tu) &\leq \overline{C}t \sum_{i=1}^{N} \|d_i\|_{L^{p_i'(\cdot)}(\Omega)} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)} + \frac{\overline{C}t^{p_M^+}}{p_m^-} \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)}\, dx \\ &+ \frac{\|b\|_{L^\infty(\Omega)}}{p_m^-} \int_\Omega |u|^{p_M(x)}\, dx - \lambda t^\gamma \int_{\{x\in\Omega:|u(x)|>s_0\}} \widetilde{k}(x)|u|^\gamma\, dx \\ &- \lambda|\Omega| inf\{F(x,s) : x \in \Omega, |s| \leq s_0\}. \end{aligned} \tag{12}$$

For all $R > 0$,

$$\sup_{\|u\|_{\overrightarrow{p}(\cdot)}=R, u\in\overline{V}} I(u) = \sup_{\|tu\|_{\overrightarrow{p}(\cdot)}=R, tu\in\overline{V}} I(tu) = \sup_{\|tu\|_{\overrightarrow{p}(\cdot)}=R, u\in\overline{V}} I(tu). \tag{13}$$

Putting together (12) and (13) we get

$$\sup_{\|u\|_{\overrightarrow{p}(\cdot)}=R, u\in\overline{V}} I(u) \to -\infty \text{ as } R \to \infty.$$

Thus we can choose $R_0 > 0$ sufficiently large such that for all $R > R_0$ and for all $u \in \overline{V}$ with $\|u\|_{\overrightarrow{p}(\cdot)} = R$ we find that $I(u) \leq 0$. So,

$$I(u) \leq 0 \text{ for all } u \in \overline{V} \setminus B_{R_0}.$$

□

It is clear now that we can use the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz to deduce the existence of an unbounded sequence of weak solutions in V for problem (1).

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