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On (semi)topological residuated lattices

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Abstract. In this paper, we define the notions of semitopological and topological residuated lattices and derive here conditions that imply a residuated lattice to be a semitopological or topological residuated lattice. Also, we study the relationship between separation axioms T_0, T_1, T_2 and (semi)topological residuated lattices.

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1. Introduction

Algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. Because of this difference in nature, algebra and topology have a strong tendency to develop independency, not in direct contact with each other. However, in applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory, and others, topology and algebra come in contact most naturally. Many of the most important objects of mathematics represent a blend of algebraic and of topological structures. Topological function spaces and linear topological spaces in general, topological groups and topological fields, transformation groups, topological lattices are objects of this kind. In the 20th century many topologists and algebraists have contributed to topological algebra. Some outstanding mathematicians were involved, among them J. Dieudonné, L. S. Pontryagin, A. Weyl.

Residuated lattices have been introduced by M. Ward and R.P. Dilworth [10] as generalization of ideal lattices of rings with identity. Residuated lattices are a common structure among algebras associated with logical systems. The main examples of residuated lattices related to logic are MV-algebras and BL-algebras. In [4] Borzooei et.al introduced (semi)topological BL-algebras and in [5] and [6] they studied metrizability and separation axioms on (semi)topological BL-algebras. In section 3 of this note, we define semitopological and topological residuated lattices, and we state and prove some theorems that determine the relationships between them. It is quite clear that a topological residuated lattice is a semitopological residuated lattice, but the converse is not true. In this paper we find certain conditions under which the converse is true. In section 4 we deal with relations between T_i spaces and residuated lattices endowed with a topology. We bring a condition that T_1 spaces are equivalent to Hausdorff spaces on residuated lattices endowed with a topology.

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2. Preliminary

Recall that a set A with a family $\mathcal{U}$ of its subsets is called a *topological space*, denoted by $(A,\mathcal{U})$, if $A, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $U \in \mathcal{U}$, that is $A \setminus U$, is said to be a *closed set*. If B is a subset of A, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subfamily $\{U_\alpha\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$ there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is the union of members of $\{U_\alpha\}$. A subset P of A is said to be a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in A. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$. A *directed set* I is a partially ordered set such that, for any i and j of I, there is a $k \in I$ with $k \geq i$ and $k \geq j$. If I is a directed set, then the subset $\{x_i : i \in I\}$ of A is called a *net*. A net $\{x_i; i \in I\}$ *converges* to $x \in A$ if for each neighborhood U of x there exists a $j \in I$ such that for all $i \geq j$, $x_i \in U$. If $B \subseteq A$ and $x \in \overline{B}$, then there is a net in B that is converges to x.[See, [7]]

Definition 2.1. [4] Let $(A, *)$ be an algebra of type 2 and $\mathcal{U}$ be a topology on A. Then $(A, *, \mathcal{U})$ is called:
(i) *left (right) topological algebra* if for each $a \in A$, the map $l_a : A \hookrightarrow A$ ($r_a : A \hookrightarrow A$) is defined by $x \to a * x$ ($x \to x * a$) is continuous, or equivalently, for any $x \in A$, and any open neighborhood U of $a * x$ ($x * a$), there exists an open neighborhood V of x such that $a * V \subseteq U$ ($V * a \subseteq U$.) In this case we also call that the operation $*$ is continuous in the second(first) variable,
(ii) *semitopological algebra* if it is right and left topological algebra. In this case we also call that the operation $*$ is continuous in each variable separately,
(iii) *topological algebra* if the operation $*$ is continuous, or equivalently, if for any x, y in A and any open neighborhood W of $x * y$, there exist two open neighborhoods U and V of x and y, respectively, such that $U * V \subseteq W$.

Proposition 2.1. [4] *Let $(A, *)$ be a commutative algebra of type 2 and $\mathcal{U}$ be a topology on A. Then right and left topological algebras are equivalent. Moreover, $(A, *, \mathcal{U})$ is a semitopological algebra iff, it is right or left topological algebra.*

Definition 2.2. [4] Let A be a nonempty set, $\{*_i\}_{i\in I}$ be a family of operations of type 2 on A and $\mathcal{U}$ be a topology on A. Then:
(i) $(A, \{*_i\}_{i\in I}, \mathcal{U})$ is a right(left) topological algebra if for any $i \in I$, $(A, *_i, \mathcal{U})$ is a right (left) topological algebra,
(ii) $(A, \{*_i\}_{i\in I}, \mathcal{U})$ is a (semi)topological algebra if for all $i \in I$, $(A, *_i, \mathcal{U})$ is a (semi)topological algebra.

Definition 2.3. [12] A *residuated lattice* is an algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, $(L, \odot, 1)$ is a commutative monoid and for any $a, b, c \in A$, $c \leq a \to b$ iff, $a \odot c \leq b$.

Let L be a residuated lattice. We set $a' = a \to 0$ and denote $(a')'$ by a''. We call the map $p : L \to L$ by $p(a) = a'$, for any $a \in L$, the *negation map*. Also, for each $a \in L$, we define $a^0 = 1$ and $a^n = a^{n-1} \odot a$, for each natural numbers n.

Example 2.1. [12](i) Let $\odot$ and $\to$ on the real unit interval $I = [0, 1]$ be defined as follows:

$$x \odot y = \min\{x, y\} \ \&\ x \to y = \begin{cases} 1 & , x \le y, \\ y & , \text{otherwise.} \end{cases}$$

Then $\mathcal{I} = (I, \min, \max, \odot, \to, 0, 1)$ is a residuated lattice,
(ii) Let $\odot$ be the usual multiplication of real numbers on the unit interval $I = [0, 1]$ and $x \to y = 1$ iff, $x \le y$ and y/x otherwise. Then $\mathcal{I} = (I, \min, \max, \odot, \to, 0, 1)$ is a residuated lattice.
(iii) Let $L = \{0, a, b, c, 1\}$. Define $\odot$ and $\to$ as follows :

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Easily we can check that $(L, \odot, \to, 0, 1)$ is a residuated lattice, whose lattice $(L, \wedge, \vee, 0, 1)$ is given by the partial order $0 < a < c < 1$, $0 < b < c < 1$ and $x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$, $a \wedge b = 0$ and $a \vee b = 1$.

Example 2.2. [2] Let $\odot$ and $\to$ on the real unit interval $I = [0, 1]$ be defined as follows:

$$x \odot y = \begin{cases} 0 & , x + y \le 1/2, \\ x \wedge y & , \text{otherwise,} \end{cases} \ \&\ x \to y = \begin{cases} 1 & , x \le y, \\ max\{1/2 - x, y\} & , \text{otherwise.} \end{cases}$$

Then $\mathcal{I} = (I, \min, \max, \odot, \to, 0, 1)$ is a residuated lattice.

Proposition 2.2. [3] *Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice. The following properties hold.*
(R_1) $x \to (y \to z) = (x \odot y) \to z$,
(R_2) $x \le y$ *iff* $x \to y = 1$,
(R_3) $1 * x = x$, *where* $* \in \{\wedge, \odot, \to\}$,
(R_4) $x \odot 0 = 0$, $1' = 0$, $0' = 1$,
(R_5) $x \odot y \le x \wedge y \le x, y$, *and* $y \le (x \to y)$,
(R_6) $(x \to y) \odot x \le y$,
(R_7) $x \le y \to (x \odot y)$,
(R_8) $x \le y$ *implies* $x * z \le y * z$, *where* $* \in \{\wedge, \vee, \odot\}$,
(R_9) $x \le y$ *implies* $z \to x \le z \to y$ *and* $x \to z \ge y \to z$,
(R_{10}) $x \to y = x \to (x \wedge y)$,
(R_{11}) $x \le y$ *implies* $x \le z \to y$,
(R_{12}) $z \odot (x \wedge y) \le (z \odot x) \wedge (z \odot y)$,
(R_{13}) $x \to y \le (x \odot z) \to (y \odot z)$,
(R_{14}) $(x \to y) \odot (y \to z) \le x \to z$,
(R_{15}) $x \odot x' = 0$,
(R_{16}) $x \to y' = (x \odot y)'$,
(R_{17}) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(R_{18}) $(x \vee y) \to z = (x \to y) \wedge (x \to z)$,
(R_{19}) $x \to (y \wedge z) = (x \to y) \vee (x \to z)$,
(R_{20}) $(x \vee y)' = x' \wedge y'$, *and* $(x \wedge y)' \ge x' \vee y'$,
(R_{21}) *if* $x \vee y = 1$, *then* $x \to y = y$ *and* $x \odot y = x \vee y$,
(R_{22}) $x''' = x'$.

Definition 2.4. [11] Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice. A *filter* is a nonempty set $F \subseteq L$ such that for each $x, y \in L$:
(a) $x, y \in F$ implies $x \odot y \in F$,
(b) if $x \in F$ and $x \leq y$, then $y \in F$.

Lemma 2.3. [11] *Let L be a residuated lattice and $F \subseteq L$. Then:*
(i) *if $1 \in F$, then F is a filter if and only if $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$,*
(ii) *if F is a filter in L, then for each $x, y \in F$, $x \wedge y$, $x \vee y$ and $x \rightarrow y$ are in F.*

Let $(L, \wedge, \vee, \odot, \rightarrow)$ be a residuated lattice. Then $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$. If such element b exists it is called a *complement* of a. We denote $b = a^c$ and the set of all complemented elements in L by $B(L)$.[See, [12]]

Proposition 2.4. [12] *Let $(L, \wedge, \vee, \odot, \rightarrow)$ be a residuated lattice. Then for each $e, f \in B(A)$, the following properties hold.*
(i) *$e^2 = e$, $e^c = e'$ and $e'' = e$,*
(ii) *$e * f \in B(L)$, where $* \in \{\wedge, \vee, \odot, \rightarrow\}$,*
(iii) *$(e \vee f)^c = e^c \wedge f^c$ and $(e \wedge f)^c = e^c \vee f^c$,*
(iv) *$e \odot f = e \wedge f$.*

Notation. From now on, in this paper we let L be a residuated lattice and $\mathcal{U}$ be a topology on L, unless otherwise state.

3. (Semi)topological residuated lattices

In this section we define the notions of semitopological and topological residuated lattices and state and prove some theorems about them. Semitopological residuated lattice is a notation weaker than of topological residuated lattice. A topological residuated lattice is always a semitopological residuated lattice, but the converse is not true as shown by an example. We derive here conditions that imply a semitopological residuated lattice is a topological residuated lattice.

Definition 3.1. Let $(L, \{*_i\}, \mathcal{U})$, where $\{*_i\} \subseteq \{\wedge, \vee, \odot, \rightarrow\}$, be a (semi)topological algebra. Then $(L, \{*_i\}, \mathcal{U})$ is called a *(semi)topological residuated lattice.*

Example 3.1. Let $\mathcal{I}$ be the residuated lattice as Example 2.1(i), and $\mathcal{U}$ be a topology on $\mathcal{I}$ with the base $S = \{[a, b] \cap I : a, b \in \mathbb{R}\}$. We prove that $(\mathcal{I}, \mathcal{U})$ is a topological residuated lattice. For this, we must prove that the operations $\wedge, \vee, \odot$ and $\rightarrow$ are continuous. First, we show that $\wedge$ is continuous. Let $x \wedge y \in U \in \mathcal{U}$, where $x, y \in I$. W.O.L.G, let $x \leq y$. Then $x \wedge y = x \in U$. Since $x \in [0, x] \cap U \in \mathcal{U}$, $y \in [x, 1] \in \mathcal{U}$ and $([0, x] \cap U) \wedge [x, 1] \subseteq U$, the operation $\wedge$ is continuous. Now, we prove that $\vee$ is continuous. Let $x \vee y \in U \in \mathcal{U}$, and let $x \leq y$. Then $x \vee y = y \in U$. Now, $[0, y]$ and $[y, 1] \cap U$ are two open neighborhoods of x and y, respectively, such that $[0, y] \vee ([y, 1] \cap U) \subseteq U$. Hence $\vee$ is continuous. Since $\odot = \wedge$, the operation $\odot$ is continuous. Finally to complete the proof, we have to show that $\rightarrow$ is continuous. Let $x \rightarrow y \in U \in \mathcal{U}$, where $x, y \in I$. If $x \leq y$, then $[0, y]$ and $[y, 1]$ are two open neighborhoods of x and y, respectively, such that $[0, y] \rightarrow [y, 1] = \{1\} \subseteq U$. If $x > y$, then $x \rightarrow y = y \in U$. Hence $x \in [y, x] \in \mathcal{U}$, $y \in [0, y] \cap U \in \mathcal{U}$ and $[y, x] \rightarrow ([0, y] \cap U) \subseteq U$. Thus, we proved that $\rightarrow$ is continuous. Therefore, $(\mathcal{I}, \mathcal{U})$ is a topological residuated lattice.

In the following Example we introduce a semitopological residuated lattice which is not a topological residuated lattice.

Example 3.2. Let $\mathcal{I}$ be the residuated lattice as Example 2.2 and $\mathcal{U}$ be a topology on $\mathcal{I}$ with the base $S = \{[a,b] \cap I : a \neq b \in \mathbb{R}\}$. Clearly, $\{0\}$ and $\{1\}$ are in $\mathcal{U}$, and if $x \in I$, then $[a,x]$ and $[x,b]$ are two open neighborhoods of x. We prove that $(\mathcal{I},\mathcal{U})$ is a semitopological residuated lattice which is not a topological residuated lattice. For this, first we prove that the operations $\wedge, \vee, \odot$ and $\rightarrow$ are continuous in each variable separately. Let $x \leq y$ and $x \wedge y = x \in [a,b] \in \mathcal{U}$. If $x = y$ and $a \neq x$, then $[a,x]$ is an open neighborhood of x such that $[a,x] \wedge x = [a,x] \subseteq [a,b]$. If $x = y$ and $b \neq x$, then $[x,b]$ is an open neighborhood of x such that $[x,b] \wedge x = \{x\} \subseteq [a,b]$. Let $x \neq y$. If $y \leq b$, then $[x,y]$ is an open neighborhood of x such that $[x,y] \wedge y = [x,y] \subseteq [a,b]$, and if $y > b$, then $[a,b]$ is an open neighborhood of x such that $[a,b] \wedge y = [a,b]$. Thus, $\wedge$ is continuous in first variable. By Proposition 2.1, $\wedge$ is continuous in each variable separately. By the same argument as above, we can prove that $\vee$ is continuous in each variable separately. Now, we prove that $(\mathcal{I},\odot,\mathcal{U})$ is a semitopological residuated lattice. For this, we prove that $\odot$ is continuous in first variable. We consider the following cases:
Case1. let $x = y \in I$. Then $x \odot x = 0$ if $x \leq 1/4$ and x otherwise. If $x \odot x = 0 \in [0,b]$, then $[0,x]$ is an open neighborhood of x such that $[0,x] \odot x = \{0\} \subseteq [0,b]$. If $x \odot x = x \in [a,b]$, then $[x,1]$ is an open neighborhood of x such that $[x,1] \odot x = \{x\} \subseteq [a,b]$,
Case2. let x and y be in I and $x < y$. Then $x \odot y = 0$ if $x+y \leq 1/2$ and x if $x+y > 1/2$
(2-1) if $x + y \leq 1/2$, then $x \odot y = 0$. Let $x \odot y = 0 \in [0,b]$. Then $[0,x]$ is an open neighborhood of x such that $[0,x] \odot y = \{0\} \subseteq [0,b]$,
(2-2) if $x+y > 1/2$, then $x \odot y = x$. Now, $[a,x]$ and $[x,b]$ are two open neighborhoods of x. First, let $x \odot y = x \in [a,x]$. If $a \geq 1/2$, then $[a,x]$ is an open neighborhood of x such that $[a,x] \odot y = [a,x]$. If $a < 1/2$ and $x \neq 1/2$, then $[1/2,x]$ is an open neighborhood of x such that $[1/2,x] \odot y = [1/2,x] \subseteq [a,x]$. If $a < 1/2$ and $x = 1/2$, then $[a,1/2]$ is an open neighborhood of x such that $[a,1/2] \odot y = [a,1/2]$. Now, let $x \odot y = x \in [x,b]$. If $y < b$, then $[x,y]$ is an open neighborhood of x such that $[x,y] \odot y = [x,y] \subseteq [x,b]$. If $y \geq b$, then $[x,b]$ is an open neighborhood of x such that $[x,b] \odot y = [x,b]$.
Thus, in Cases (1) and (2), we can prove that $\odot$ is continuous in first variable. By Proposition 2.1, the operation $\odot$ is continuous in each variable separately. In continue, we prove that $(\mathcal{I},\rightarrow,\mathcal{U})$ is a semitopological residuated lattice. For this, let $x, y \in I$. Then $[a, x \rightarrow y]$ and $[x \rightarrow y, b]$ are two open neighborhoods of $x \rightarrow y$. We consider the following cases:
Case1. let $x \rightarrow y \in [a, x \rightarrow y]$. Then:
(1-1) if $x < y$, then $x \rightarrow y = 1 \in [a,1]$. Now, $[0,x]$ and $[x,y]$ are two open neighborhoods of x and y, respectively, such that $[0,x] \rightarrow y = \{1\} \subseteq [a,1]$ and $x \rightarrow [x,y] = \{1\} \subseteq [a,1]$. Let $x = y$. Then $x \rightarrow x = 1 \in [a,1]$. If $x \neq 0,1$, then $[0,x]$ and $[x,1]$ are two open neighborhoods of x such that $[0,x] \rightarrow x = \{1\} \subseteq [a,1]$ and $x \rightarrow [x,1] = \{1\} \subseteq [a,1]$. If $x = 0$ or $x = 1$, then $\{0\}$ and $\{1\}$ are two open neighborhoods of 0 and 1, respectively, $\{0\} \rightarrow \{0\} \subseteq [a,1]$ and $\{1\} \rightarrow \{1\} \subseteq [a,1]$,
(1-2) let $x > y$ and $x + y \leq 1/2$. Then $x \rightarrow y = 1/2 - x \in [a, 1/2 - x]$. $[0,y]$ is an open neighborhood of y such that $x \rightarrow [0,y] = \{1/2 - x\} \subseteq [a, 1/2 - x]$. This shows that $\rightarrow$ is continuous in second variable. we prove that $\rightarrow$ is continuous in first variable. If $y < a$, then $[x, 1/2 - a]$ is an open neighborhood of x such that $[x, 1/2 - a] \rightarrow y \subseteq [a, 1/2 - x]$, and if $y \geq a$, then $[x, 1/2 - y]$ is an open neighborhood of x such that $[x, 1/2 - y] \rightarrow y \subseteq [a, 1/2 - x]$. Hence $\rightarrow$ is continuous in first variable,
(1-3) let $x > y$ and $x + y > 1/2$. Then $x \rightarrow y = y \in [a,y]$. First, we prove that $\rightarrow$ is continuous in first variable. If $y < 1/2 - y$, then $[1/2 - y, x]$ is an open neighborhood

of x such that $[1/2 - y, x] \rightarrow y = \{y\} \subseteq [a, y]$, and if $1/2 - y \leq y$, then $[x + y/2, x]$ is an open neighborhood of x such that $[x + y/2, x] \rightarrow y = \{y\} \subseteq [a, y]$. Thus, the operation $\rightarrow$ is continuous in first variable. Now, we prove that $\rightarrow$ is continuous in second variable. If $1/2 - x \leq a$, then $[a + y/2, y]$ is an open neighborhood of y such that $x \rightarrow [a + y/2, y] = [a + y/2, y] \subseteq [a, y]$, and if $a < 1/2 - x$ and $1/2 - x < c < y$, then $[c, y]$ is an open neighborhood of y such that $x \rightarrow [c, y] \subseteq [c, y] \subseteq [a, y]$. Therefore, $\rightarrow$ is continuous in second variable,
Case2. let $x \rightarrow y \in [x \rightarrow y, b]$. Then:
(2-1) if $x \leq y$, then by the same argument from *(1-1)* of Case1, we can prove that $\rightarrow$ is continuous in each variable separately,
(2-2) let $x < y$ and $x + y \leq 1/2$. Then $x \rightarrow y = 1/2 - x \in [1/2 - x, b]$. If $y < 1/2 - b$, then $[1/2 - b, x]$ is an open neighborhood of x such that $[1/2 - b, x] \rightarrow y \subseteq [1/2 - x, b]$, and if $y \geq 1/2 - b$, then $[x + y/2, x]$ is an open neighborhood of x such that $[x + y/2, x] \rightarrow y \subseteq [1/2 - x, b]$. Hence $\rightarrow$ is continuous in first variable. On the other hand, $[0, y]$ is an open neighborhood of y such that $x \rightarrow [0, y] = \{1/2 - x\} \subseteq [1/2 - x, b]$. This proves that $\rightarrow$ is continuous in second variable,
(2-3) let $x > y$ and $x + y \geq 1/2$. Then $x \rightarrow y = y \in [y, b]$. Then by the same argument of $(2 - 2)$ of Case2, we can prove that $\rightarrow$ is continuous in first variable. If $x \leq b$, then $[y, x + y/2]$ is an open neighborhood of y such that $x \rightarrow [y, y + x/2] \subseteq [y, y + x/2] \subseteq [y, b]$, and if $x > b$, then $[y, b]$ is an open neighborhood of y such that $x \rightarrow [y, b] \subseteq [y, b]$. Hence $\rightarrow$ is continuous in second variable. Thus, we can prove that $(\mathcal{I}, \mathcal{U})$ is a semitopological residuated lattice.
Now, we prove that $(\mathcal{I}, \mathcal{U})$ is not a topological residuated lattice. For this, let $1/4 \odot 1/4 = 1/2 \in [3/8, 1/2]$. Let $1/4 \in U \in \mathcal{U}$. Then there is a $\epsilon > 0$ such that $[1/4, 1/4 + \epsilon] \subseteq U$ or $[1/4 - \epsilon, 1/4] \subseteq U$. It is easy to prove that $[1/4, 1/4 + \epsilon] \odot [1/4, 1/4 + \epsilon]$ and $[1/4 - \epsilon, 1/4] \odot [1/4 - \epsilon, 1/4]$ and $[1/4 - \epsilon, 1/4] \odot [1/4, 1/4 + \epsilon]$ are not subsets of $[3/8, 1/2]$. Therefore, the operation $\odot$ is not continuous in $(1/4, 1/4)$, and so $(\mathcal{I}, \mathcal{U})$ is not a topological residuated lattice.

Example 3.3. Let $\mathcal{I}$ be the residuated lattice in Example 2.1(i), and $\mathcal{U}$ be the subspace topology induced from $\mathbb{R}$. Then for each $a, b \in I$, the intervals $(a, b), [0, a)$ and $(b, 1]$ form a base of $\mathcal{U}$. We prove that the operations $\wedge, \vee, \odot$ are continuous in each variable separately but the operation $\rightarrow$ is not. First, we prove that $(L, \wedge, \mathcal{U})$ is a semitopological residuated lattice. For this, let $x, y \in I$, $x \leq y$ and $x \wedge y \in U \in \mathcal{U}$. If $x = y$ and $x \wedge y = 0$, then U is an open neighborhood of x such that $U \wedge y = \{0\} \subseteq U$. If $x \neq y$ and $x \wedge y = 0$, then there is an $a < y$ such that $x \wedge y \in [0, a) \subseteq U$. Now $[0, a)$ is an open neighborhood of x such that $[0, a) \wedge y = [0, a) \subseteq U$.
If $x \wedge y = 1$, then by (R_5), $x = y = 1$. Hence U is an open neighborhood of x such that $U \wedge y = \{1\} \subseteq U$.
Now, let $x \wedge y \neq 0, 1$. Then there are $a, b \in I$ such that $x \wedge y \in (a, b) \subseteq U$. If $x = y$, then (a, b) is an open neighborhood of x such that $(a, b) \wedge y \subseteq (a, b)$. Let $x \neq y$. Then if $y < b$, then (a, y) is an open neighborhood of x such that $(a, y) \wedge y \subseteq (a, b)$, and if $y > b$, then (a, b) is an open neighborhood of x such that $(a, b) \wedge y \subseteq (a, b)$. Thus, we can prove that the operation $\wedge$ is continuous in first variable. By Proposition 2.1, $(L, \wedge, \mathcal{U})$ is a semitopological residuated lattice. By the same argument as above, we can prove that $\vee$ is continuous in each variable separately. Since $\odot = \wedge$, the operation $\odot$ is continuous in each variable separately. But $\rightarrow$ is not continuous in first variable, because let $0 \rightarrow 0 = 1 \in (1/2, 1]$, and $[0, b)$ be an open neighborhood of 0. Then for each $x \in (0, b)$, $x \rightarrow 0 = 0 \notin (1/2, 1]$.

Recall, a function $f : X \hookrightarrow Y$ between topological spaces is *homeomorphism* if

$f^{-1} : Y \hookrightarrow X$ exists and f and f^{-1} are continuous.[See, [7]]

Proposition 3.1. *Let $(L,\mathcal{U})$ be a semitopological residuated lattice. Then:*
(i) if $1 \in U \in \mathcal{U}$ and $x \in L$, then there is an open neighborhood V of 1 such that $x * V \subseteq U$, where $* \in \{\vee, \rightarrow\}$,
(ii) if $1 \in U \in \mathcal{U}$, then there is an open neighborhood V of 0 such that $V' \subseteq U$,
(iii) if $x \in U \in \mathcal{U}$, there is an open neighborhood V of 1 such that $V * x \subseteq U$, where $* \in \{\wedge, \odot, \rightarrow\}$,
(iv) if $x \in U \in \mathcal{U}$, there is an open neighborhood V of x such that $V * x \subseteq U$, where $* \in \{\wedge, \vee\}$,
(v) the negation map $p : L \hookrightarrow L$ by $p(x) = x'$ is homeomorphism iff, is one to one iff, is onto.

Proof. (i) Let $* \in \{\vee, \rightarrow\}$, $x \in L$ and $1 \in U \in \mathcal{U}$. By (R_2) and (R_8), $x * 1 = 1 \in U$. Since $*$ is continuous in second variable, there is an open neighborhood V of 1 such that $x * V \subseteq U$.
(ii) Let $1 \in U \in \mathcal{U}$. By (R_4), $0 \rightarrow 0 = 1 \in U$. Since the operation $\rightarrow$ is continuous in first variable, there is an open neighborhood V of 0 such that $V' = V \rightarrow 0 \subseteq U$.
(iii) Let $* \in \{\wedge, \odot, \rightarrow\}$ and $x \in U \in \mathcal{U}$. By (R_3), $1 * x = x$. Since $*$ is continuous in first variable, there is an open neighborhood V of 1 such that $V * x \subseteq U$.
(iv) The proof is similar to the proof of (iii).
(v) If p is homeomorphism, it is clear that P is an one to one map. Let p be one to one. We prove that p is onto. For this, let $y \in L$. By (R_{21}), $y' = y'''$. Since p is one to one, $y = y'' = p(y')$. Hence p is onto. Now, let p be onto and let y be an arbitrary element of L. Since p is onto, there is a $x \in L$ such that $y = p(x)$. Now, by (R_{21}), $y'' = x''' = x' = y$. Hence for each $x \in L$, $p(p(x)) = x'' = x$, which implies that $p^{-1} = p$. Since p is continuous, p is homeomorphism. □

Proposition 3.2. *Let $(L,\mathcal{U})$ be a topological residuated lattice. Then,*
(i) if $1 \in U \in \mathcal{U}$, then there is an open neighborhood V of 1 such that $V * V \subseteq U$, where $* \in \{\wedge, \vee, \odot, \rightarrow\}$,
(ii) if $1 \in U \in \mathcal{U}$ and $x \in L$, then there is an open neighborhood V of x such that $V \rightarrow V \subseteq U$,
(iii) if $1 \in U \in \mathcal{U}$, then there is an open neighborhood V of 0 such that $V' \subseteq U$,
(iv) if $x \in U \in \mathcal{U}$, there are two open neighborhoods V and W of 1 and x, respectively, such that $V * W \subseteq U$, where $* \in \{\wedge, \odot, \rightarrow\}$,
(v) if $x \in U \in \mathcal{U}$, there is an open neighborhood V of x such that $V * V \subseteq U$, where $* \in \{\wedge, \vee\}$.

Proof. The proof is similar to the proof of Proposition 3.1. □

Proposition 3.3. *Let F be a Filter in a topological residuated lattice $(L, \odot, \rightarrow, \mathcal{U})$. If F is an open neighborhood of 1, then $\overline{F}$, closure F in L, is a filter in L. Moreover, if $\{1\}$ is an open set in L, then for each filter F in L, $\overline{F}$ is a filter.*

Proof. Since $\odot$ is continuous, we get that $\overline{F} \odot \overline{F} \subseteq \overline{F \odot F}$. Let $x \leq y$ and $x \in \overline{F}$. There is a net $\{x_i : i \in I\}$ in F such that is converges to x. Since $\rightarrow$ is continuous, the net $\{x_i \rightarrow y : i \in I\}$ is converges to $x \rightarrow y = 1$. Since F is an open neighborhood of 1, there is a $j \in I$ such that $x_j \rightarrow y$ is in F. Hence $y \in F \subseteq \overline{F}$. Therefore, $\overline{F}$ is a filter. The proof of other case is clear. □

Theorem 3.4. *Let L be a residuated lattice. Then there is a topology $\mathcal{U}$ on L such that $(L, \wedge, \vee, \odot, \mathcal{U})$ is a topological residuated lattice and $(L, \rightarrow, \mathcal{U})$ is a left topological residuated lattice.*

Proof. Let $a \in L$, $L_a = \{x \in L : a \leq x\}$ and $\mathbf{B} = \{L_a : a \in L\}$. It is easy to prove that $L \subseteq \bigcup_{a \in L} L_a$, and for each $a, b, c \in L$ if $c \in L_a \cap L_b$, then $L_c \subseteq L_a \cap L_b$. Hence $\mathbf{B}$ is a base for a topology $\mathcal{U}$ on L. Let $a, b \in L$ and $* \in \{\wedge, \vee, \odot\}$. Then by (R_8) and (R_9), we can prove that $L_a * L_b \subseteq L_{a*b}$ and $a \to L_b \subseteq L_{a \to b}$. These relations prove that $(L, \wedge, \vee, \odot, \mathcal{U})$ is a topological residuated lattice and $(L, \to, \mathcal{U})$ is a left topological residuated lattice. □

Example 3.4. In Example 2.1(i), for each $a \in I$, $L_a = [a, 1]$. By Theorem 3.4, $(L, \wedge, \vee, \odot, \mathcal{U})$ is a topological residuated lattice and $(L, \to, \mathcal{U})$ is a left topological residuated lattice. We show that $(L, \to, \mathcal{U})$ is not a right topological residuated lattice. For this, let $1 \neq y \in L$ and $x \to y \in [x \to y, 1]$. Then it is easy to prove that $[x, 1] \to y \nsubseteq [x \to y, 1]$. This proves that $\to$ is not continuous in first variable.

Proposition 3.5. *Let $(L, \wedge, \mathcal{U})$ be a topological residuated lattice, and the negation map $p : L \hookrightarrow L$ by $p(x) = x'$ be continuous. Then if $\mathcal{U}_B$ is the induced topology from L on $B(L)$, then the operations $\wedge, \vee, \odot, \to$ are continuous on $(B(L), \mathcal{U}_B)$.*

Proof. By Proposition 2.4, $\wedge, \vee, \odot, \to$ are operations on $B(L)$, and the negation map p is also a continuous map of $B(L)$ onto $B(L)$. Since $\wedge$ is continuous on L, by [[4], Proposition 3.9], the operation $\wedge$ is continuous on $B(L)$. On the other hand, since $\wedge = \odot$ on $B(L)$, we conclude that $\odot$ is continuous on $B(L)$. Now, we prove that $\vee$ is continuous in $B(L)$. Let $x, y \in B(L)$, and U be an open set in L such that $x \vee y \in B(L) \cap U$. By Proposition 2.4(i), $(x \vee y)'' = x \vee y \in B(L) \cap U$. By (R_{20}), $(x' \wedge y')' \in U$. Since the negation map p is continuous in L, there is an open neighborhood V of $x' \wedge y'$ in L such that $V' \subseteq U$. Since $\wedge$ is continuous in L, there are two open neighborhoods W_1 and W_2 of x' and y',respectively, in L such that $W_1 \wedge W_2 \subseteq V$. Since the negation map p is continuous in L, there are two open neighborhoods U_1 and U_2 of x and y, respectively, in L such that $U_1' \subseteq W_1$ and $U_2' \subseteq W_2$. Now, $B(L) \cap U_1$ and $B(L) \cap U_2$ are two open neighborhoods of x and y, respectively, in $B(L)$. We prove that $(B(L) \cap U_1) \vee ((B(L) \cap U_2) \subseteq B(L)) \cap U$. Let $z_1 \in B(L) \cap U_1$ and $z_2 \in B(L) \cap U_2$. Then $z_1' \in W_1$ and $z_2' \in W_2$. Hence $z_1' \wedge Z_2' \in W_1 \wedge W_2 \subseteq V$. Thus, $(z_1' \wedge z_2')' \in U$. Since $z_1 \vee z_2 = (z_1 \vee z_2)'' = (z_1' \wedge z_2')'$, it follows that $z_1 \vee z_2 \in B(L) \cap U$. Therefore, $\vee$ is continuous in $B(L)$. Finally, to complete the proof, we have to show that $\to$ is continuous in $B(L)$. Let $x, y \in B(L)$ and $U \in \mathcal{U}$ and $x \to y \in B(L) \cap U$. By Proposition 2.4(i), $x \to y'' \in B(L) \cap U$ and by (R_{16}), $(x \odot y')' \in B(L) \cap U$. Since the negation map p is continuous on $B(L)$, there is an open neighborhood $B(L) \cap V$ in $(B(L)$, of $x \odot y'$ such that $(B(L) \cap V)' \subseteq B(L) \cap U$. Since $\odot$ is continuous in $B(L)$, there are two open neighborhoods $B(L) \cap W_1$ and $B(L) \cap W_2$ of x and y', respectively, such that $(B(L) \cap W_1) \odot (B(L) \cap W_2) \subseteq B(L) \cap V$. Since the negation map p on $B(L)$ is continuous, there is an open neighborhood $B(L) \cap U_1$ of y in $B(L)$ such that $(B(L) \cap U_1)' \subseteq B(L) \cap W_2$. We show that $(B(L) \cap W_1) \to (B(L) \cap U_1) \subseteq B(L) \cap U$. For this, let $z_1 \in B(L) \cap W_1$ and $z_2 \in B(L) \cap U_1$. Hence $z_2' \in B(L) \cap W_2$, and so $(z_1 \odot z_2')'$ is in $((B(L) \cap W_1) \odot B(L) \cap W_2)' \subseteq B(L) \cap U$. Since $z_1 \to z_2 = z_1 \to z_2'' = (z_1 \odot z_2')'$, we conclude that $z_1 \to z_2 \in B(L) \cap U$. □

In Proposition 3.5, if $(L, \wedge, \mathcal{U})$ is a semitopological residuated lattice, then by the same argument,we can prove that $\wedge, \vee, \odot, \to$ are continuous in each variable separately in $B(L)$.

Example 3.5. Let L be the residuated lattice in Example 2.1(iii). Then:
(i) it is easy to prove that the set $\{\{0\}, \{a\}, \{b\}, \{c, 1\}\}$ is a base for a nontrivial topology $\mathcal{U}$ on L such that $(L, \wedge, \mathcal{U})$ is a topological residuated lattice and the negation

map p is continuous. By Proposition 3.5, the operations $\wedge, \vee, \odot, \rightarrow$ are continuous on $B(L) = \{0, a, b, 1\}$,
(ii) the set $\mathcal{U} = \{\{a\}, \{a, b\}, L, \phi\}$ is a topology on L such that $(L, \wedge, \mathcal{U})$ is a semitopological residuated lattice and the negation map p is continuous. Hence the operations $\wedge, \vee, \odot, \rightarrow$ are continuous in each variable separately in $B(L) = \{0, a, b, 1\}$.

Theorem 3.6. *Let $(L, \wedge, \odot, \mathcal{U})$ be a (semi)topological residuated lattice, and the negation map $p : L \hookrightarrow L$ by $p(x) = x'$ be continuous. If for each $a \in L$, $a'' = a$, then $(L, \mathcal{U})$ is a (semi)topological residuated lattice.*

Proof. Suppose $(L, \wedge, \odot, \mathcal{U})$ is a topological residuated lattice. We prove that $(L, \mathcal{U})$ is a topological residuated lattice. The proof of other case is similar. First, we show that $\vee$ is continuous. For this, let $x, y \in L$, $U \in \mathcal{U}$ and $x \vee y \in U$. Then by (R_{20}), $(x' \wedge y')' = (x \vee y)'' \in U$. Since the negation map p is continuous, there is an open neighborhood V of $x' \wedge y'$ such that $V' \subseteq U$. Since $\wedge$ is continuous, there are two open neighborhoods V_1 and V_2 of x' and y', respectively, such that $V_1 \wedge V_2 \subseteq V$. Since the negation map p is continuous, there are two open neighborhoods W_1 and W_2 of x and y such that $W_1' \subseteq V_1$ and $W_2' \subseteq V_2$. Now, W_1 and W_2 are two open neighborhoods of x and y, respectively, such that $W_1 \vee W_2 \subseteq U$ because if $z_1 \in W_1$ and $z_2 \in W_2$, then by (R_{20}),

$$z_1 \vee z_2 = (z_1 \vee z_2)'' = (z_1' \wedge z_2')' \in (W_1' \wedge W_2')' \subseteq (V_1 \wedge V_2)' \subseteq V' \subseteq U.$$

Now, we prove that $\rightarrow$ is continuous. For this, let $x, y \in L$ and $x \rightarrow y \in U \in \mathcal{U}$. Then by (R_{16}), $(x \odot y')' = x \rightarrow y'' = x \rightarrow y \in U$. Since the negation map p is continuous, there is an open neighborhood V of $x \odot y'$ such that $V' \subseteq U$. Since $\odot$ is continuous, there are two open neighborhoods W_1 and W_2 of x and y', respectively, such that $W_1 \odot W_2 \subseteq V$. Since the negation map p is continuous, there is an open neighborhood W_3 of y such that $W_3' \subseteq W_2$. Now, W_1 and W_3 are two open neighborhoods of x and y, respectively, such that $W_1 \rightarrow W_3 \subseteq U$ because if $z_1 \in W_1$ and $z_2 \in W_3$, then by (R_{16}),

$$z_1 \rightarrow z_2 = z_1 \rightarrow z_2'' = (z_1 \odot z_2')' \in (W_1 \odot W_3')' \subseteq (W_1 \odot W_2)' \subseteq V' \subseteq U.$$

□

Theorem 3.7. *Let L be a residuated lattice. Then there is a nontrivial topology $\mathcal{U}$ on L such that the negation map $p : L \hookrightarrow L$ by $p(x) = x'$ is continuous in L. Moreover, if the negation map p is an open map in $(L, \mathcal{U})$ and $(L, \wedge, \rightarrow, \mathcal{U})$ is a semitopological residuated lattice, then $(L, \mathcal{U})$ is a topological residuated lattice.*

Proof. Let $\mathcal{U} = \{U \subseteq L : x \in U \Leftrightarrow x' \in U'\} \cup \{\phi\}$. It is clear that L and the empty set ϕ are in $\mathcal{U}$. Let $\{U_i : i \in I\}$ be a family of members of $\mathcal{U}$. If $x' \in (\cup_{i \in I} U_i)'$, then there is a $i \in I$ and $z \in U_i$ such that $x' = z'$. Since $z' \in U_i'$, we get that $x' \in U_i'$. Hence $x \in U_i \subseteq \cup_{i \in I} U_i$. Thus, we can prove that $\cup_{i \in I} U_i$ is in $\mathcal{U}$. Now, if $x' \in (\cap_{i \in I} U_i)'$, then there is a $z \in \cap_{i \in I} U_i$ such that $x' = z'$. Since for each $i \in I$, $z' \in U_i'$, and $U_i \in \mathcal{U}$, we get that $x \in U_i$. Hence $x \in \cap_{i \in I} U_i$. Thus, $\cap_{i \in I} U_i$ is in $\mathcal{U}$. This proves that $\mathcal{U}$ is a topology on L. Since $\phi \neq \{x \in L : x' = 0\} \in \mathcal{U}$ and $0 \notin \{x \in L : x' = 0\}$, we get that $\mathcal{U}$ is a nontrivial topology on L. Now, we prove that the negation map p is continuous. For this, let $U \in \mathcal{U}$ and $x' \in (p^{-1}(U))'$. Then there is a $z \in p^{-1}(U)$ such that $x' = z'$. Since $z' \in U$, we get that $x \in p^{-1}(U)$. Hence $p^{-1}(U) \in \mathcal{U}$ and so p is continuous.
Let the negation map p be an open map and $(L, \wedge, \rightarrow, \mathcal{U})$ be a semitopological residuated lattice. First, we prove that $(L, \mathcal{U})$ is a semitopological residuated lattice. For this, we have to show that $\vee$ and $\odot$ are continuous in each variable separately. Let

$x, y \in L$ and $x \vee y \in U \in \mathcal{U}$. Then by (R_{20}) and hypothesis $x' \wedge y' \in U' \in \mathcal{U}$. Since $\wedge$ is continuous in first variable, there is an open neighborhood V of x' such that $V \wedge y' \subseteq U$. Since the negation map p is continuous, there is an open neighborhood W of x such that $W' \subseteq V$. Now, W is an open neighborhood of x such that $W \vee y \subseteq U$. Hence $\vee$ is continuous in first variable. By Proposition 2.1, $\vee$ is continuous in each variable separately. In continue we will prove that $\odot$ is continuous in each variable separately. Let $x, y \in L$ and $x \odot y \in U \in \mathcal{U}$. Then by (R_{16}) and hypothesis, $x \to y' \in U' \in \mathcal{U}$. Since $\to$ is continuous in first variable, there is an open neighborhood V of x such that $V \to y' \subseteq U$. Thus, V is an open neighborhood of x such that $V \odot y \subseteq U$. Hence $\odot$ is continuous in first variable. By Proposition 2.1, $\odot$ is continuous in each variable separately. Finally, to complete the proof, we have to show that $(L, \mathcal{U})$ is a topological residuated lattice. Let $* \in \{\wedge, \vee, \odot, \to\}$ and $x, y \in L$ and $x * y \in U \in \mathcal{U}$. Since $*$ is continuous in first variable, there is an open neighborhood V of x such that $V * y \subseteq U$. Since $*$ is continuous in second variable, for each $z \in V$, there is an open neighborhood W_z of y such that $z * W_z \subseteq U$. Let $W = \cap_{z \in V} W_z$. Since arbitrary intersection of members of $\mathcal{U}$ is in $\mathcal{U}$, we get that $W \in \mathcal{U}$. Thus, V and W are two open neighborhoods of x and y, respectively, such that $V * W \subseteq U$. Therefore, $*$ is continuous and so $(L, \mathcal{U})$ is a topological residuated lattice. □

If F is a filter in a (semi)topological residuated lattice L, then by [[4], Proposition 3.9] F, endowed with the topology induced from L, is a (semi)topological algebra. In this case we say that F is a (semi)topological filter in (semi)toplogical residuated lattice L.

Theorem 3.8. *Let $L^* = L \backslash \{0\}$, be a filter in a residuated lattice L. Let $\mathcal{U}$ be a topology on L such that $\{0\}$ is an open and closed set in L. Then $(L, \mathcal{U})$ is a semitopological residuated lattice iff, there is a chain $\mathcal{F}$ of proper open semitopological filters in L such that $L^* = \cup \mathcal{F}$.*

Proof. Let p be the negation map. Since L^* is a filter in L, we get that for each $x \in L^*$, $p(x) = 0$ and $p(0) = 1$. We prove that p is continuous. Let U be an open set in L. If $0 \in U$, then $p^{-1}(U) = L^*$ which is an open set in L. If $1 \in U$ and $0 \notin U$, then $p^{-1}(U) = \{0\}$ which is an open set in L. If $0, 1$, both are not in U, then $p^{-1}(U) = \phi$ which is an open set in L. Thus, the negation map p is continuous. Now, let $(L, \mathcal{U})$ be a semitopological residuated lattice. Then $\mathcal{F} = \{L^*\}$ is a chain of proper open semitopological filters in L such that $L^* = \cup \mathcal{F}$.
Conversely, Let $\mathcal{F}$ be a chain of proper open semitopological filters in L such that $L^* = \cup \mathcal{F}$. We prove that $(L, \mathcal{U})$ is a semitopological residuated lattice. First, we show that $\wedge$ is continuous in each variable separately. Let $x, y \in L$ and $x \wedge y \in U \in \mathcal{U}$. If $x = 0$, then $\{0\}$ is an open neighborhood of x such that $\{0\} \wedge y \subseteq U$. If $y = 0$, then L is an open neighborhood of x such that $L \wedge y \subseteq U$. Let x and y are nonzero. Then there is a $F \in \mathcal{F}$ such that $x, y \in F$. Since $x \wedge y \in F \cap U$ and $\wedge$ is continuous in F, there is an open neighborhood V of x in L such that $(F \cap V) \wedge y \subseteq U$. Now, since F is an open set in L, we get that $F \cap V$ is an open neighborhood of x such that $(F \cap V) \wedge y \subseteq U$. Hence $\wedge$ is continuous in first variable. By Proposition 2.1, $\wedge$ is continuous in each variable separately. By the same argument we can prove that $\vee$ and $\odot$ are continuous in each variable separately. Finally, to complete the proof, we have to show that $\to$ is continuous in each variable separately. For this, let $x, y \in L$ and $x \to y \in U \in \mathcal{U}$. If $x = 0$, then $\{0\}$ and L are two open neighborhoods of x and y, respectively, such that $\{0\} \to y \subseteq U$ and $x \to L \subseteq U$. If $y = 0$, then $x \to y = x'$. Since the negation map p is continuous, there is an open neighborhood of x such that

$V' \subseteq U$. Now, V and $\{0\}$ are two open neighborhoods of x and y, respectively, such that $V \to y \subseteq U$ and $x \to \{0\} \subseteq U$. Let x and y both are nonzero. Then there is a $F \in \mathcal{F}$ such that $x, y \in F$. Since $x \to y \in F \cap U$ and $\to$ is continuous in each variable in F, there are two open neighborhoods V and W in L of x and y, respectively, such that $(F \cap V) \to y \subseteq U$ and $x \to (F \cap W) \subseteq U$. Now, since F is an open set in L, $F \cap V$ and $F \cap W$ are two open neighborhoods of x and y in L, respectively, such that $(F \cap V) \to y \subseteq U$ and $x \to (F \cap W) \subseteq U$. Therefore, $\to$ is continuous in each variable separately in L. □

4. Hausdorff (semi)topological residuated lattice

In this section we study the relationship between separation axioms T_0, T_1 and T_2 and (semi)topological residuated lattice. We bring some conditions under which a (semi)topological residuated lattice becomes a T_0 or T_1 or Hausdorff space.

Definition 4.1. Let L be a residuated lattice, $\mathcal{U}$ be a topology on L, and f be a function from L into L. We call that L is a *f-open* if f is an open map.

Notation. Let L be a residuated lattice. From now on in this paper, for each $a \in L$, we let m_a, j_a, t_a, l_a and r_a be the functions from L into L by $m_a(x) = a \wedge x$, $j_a(x) = a \vee x$, $t_a(x) = a \odot x$, $l_a(x) = a \to x$ and $r_a(x) = x \to a$.
It is easy to see that $(L, \mathcal{U})$ is a semitopological residuated lattice iff, for each $a \in L$, the mappings m_a, j_a, t_a, l_a and r_a are continuous.

Proposition 4.1. *Let L be a residuated lattice, and $p : L \hookrightarrow L$ be the negation map. Then:*
(i) there is a nontrivial topology $\mathcal{U}$ on L such that L is a p-open,
(ii) if $f \in \{m, j, t, l\}$, then there is a nontrivial topology $\mathcal{U}$ on L such that for each $a \in L$, $f_a : L \hookrightarrow L$ is continuous and L is f_a-open.

Proof. (i) Let $\mathcal{U} = \{U \subseteq L : U' \subseteq U\}$. It ie easy to prove that $\mathcal{U}$ is a topology on L. Since $\{0\} \notin \mathcal{U}$ and $\{0, 1\} \in \mathcal{U}$, it follows that topology is nontrivial. Since for each $U \in \mathcal{U}$, $U' \subseteq U$, we get that $U'' \subseteq U'$. Hence U' is an open set in L.
(ii) Let $F \neq \{1\}$ be a proper filter in L, $* \in \{\wedge, \vee, \odot, \to\}$ and $\mathcal{U}_* = \{U \subseteq L : \forall x \in U,\ F * x \subseteq U\}$. It is easy to prove that $\mathcal{U}_*$ is a topology on L. If $* \in \{\wedge, \vee, \odot\}$, since $*$ is associative, it is easy to see that for each $a \in L$, $F * a \in \mathcal{U}_*$. If $* = \to$, then by (R_1), for each $a \in L$ and $y \in F \to a$, $F \to y \subseteq F \to (F \to a) = (F \odot F) \to a \subseteq F \to a$ which implies that $F \to a \in \mathcal{U}_*$. Therefore, $\mathcal{U}_*$ is a nontrivial topology on L. Let $* \in \{\wedge, \vee, \odot\}$ and $f \in \{m, j, t\}$. If $x, y \in L$ and $x * y \in U \in \mathcal{U}_*$, then $F * x$ is an open neighborhood of x such that $(F * x) * y \subseteq U$. Hence $(L, *, \mathcal{U}_*)$ is a semitopological residuated lattice and so for each $a \in L$, f_a is continuous. Now, we prove that L is f-open. For this, let $a \in L$ and $U \in \mathcal{U}_*$. If $y \in a * U$, then there is a $x \in U$ such that $y = a * x$. Since $F * x \subseteq U$, it follows that $F * y = F * a * x \subseteq a * U$. Hence $a * U$ is an open set in L. This proves that L is f_a-open.

Let $* = \to$. We show that for each $a \in L$, l_a is continuous and L is l_a-open. Let $a \in L$ and $U \in \mathcal{U}_*$. First, we prove that l_a is continuous. Let $a \to x \in U$. Then $F \to x$ is an open neighborhood of x such that by (R_1), $a \to (F \to x) = F \to (a \to x) \subseteq U$. Hence l_a is continuous. Now, we prove that L is l_a-open. For this, we have to prove that $a \to U$ is an open set in L. Let $y \in a \to U$. Then there is a $x \in U$ such that $y = a \to x$. Now, $F \to x$ is an open neighborhood of x such that by (R_1), $F \to y = F \to (a \to x) = a \to (F \to x) \subseteq a \to U$. Therefore, L is l_a-open. □

Proposition 4.2. *Let $\mathcal{U}$ be a topology on a residuated lattice, and let $p : L \hookrightarrow L$ by $p(x) = x'$ be the negation map. If L is p-open and for each $A \in L$, $a'' = a$, then:*
(i) L is t-open iff, L is r-open,
(ii) if L is j-open, then L is m-open.

Proof. (i) Let L is t-open, $U \in \mathcal{U}$ and $a \in L$. Since p and t_a are open and by (R_{16}), $U \to a = U \to a'' = (U \odot a')'$, we get that $U \to a$ is an open set in L which implies that L is r-open. Conversely, Let L be r-open, $U \in \mathcal{U}$ and $a \in L$. Since p and r_a are open and by (R_{16}), $U \odot a = (U \odot a)'' = (U \to a')'$, we get that $U \odot a$ is an open set in L which implies that L is t-open.
(ii) By (R_{20}), the proof is similar (i). □

Let A be a topological space. Recall that A is a
(i) T_0-space if for each $x \neq y \in A$, there is at least one in an open neighborhood excluding the other,
(ii) T_1-space if for each $x \neq y \in A$, each has an open neighborhood not containing the other,
(iii) T_2-space if for each $x \neq y \in A$, there two disjoint open neighborhoods U, V of x and y, respectively.
A T_2-space is also known as a *Hausdorff space.*[See, [7]]

Proposition 4.3. *Let $\mathcal{U}$ be a topology on a residuated lattice L. If L is t-open or r-open, then L is a T_0-space.*

Proof. First, let for each $a \in L$, the mapping t_a is open, and $x \neq y \in L$. If U is an open neighborhood of 1, then $U \odot x$ and $U \odot y$ are two open neighborhoods of x and y, respectively. But $x \notin U \odot y$ or $y \notin U \odot x$ because if $x \in U \odot y$ and $y \in U \odot x$, then by (R_5), $x = y$ which is a contradiction. Hence L is a T_0-space. Let for each $a \in L$, the mapping r_a be open and let $x \neq y \in L$. Then $U \to x$ and $U \to y$ are two open neighborhoods of x and y, respectively. We show that $x \notin U \to y$ or $y \notin U \to x$. Let $x \in U \to y$ and $y \in U \to x$. Then by (R_5), $x = y$ which is a contradiction. Hence L is a T_0-space. □

Theorem 4.4. *Let L be a residuated lattice. Then there is a nontrivial topology $\mathcal{U}$ on L such that L is a T_0-space.*

Proof. Let $F \neq \{1\}$ be a proper filter in L. By Proposition 4.1(ii), $\mathcal{U} = \{U \subseteq L : \forall x \in U, F \odot x \subseteq U\}$ is a nontrivial topology on L such that L is t-open. Now, by Proposition 4.3, L is a T_0-space. □

Proposition 4.5. *Let $(L, \to, \mathcal{U})$ be a left topological residuated lattice. If for each $x \in L \setminus \{1\}$, there is an open neighborhood U of 1 such that $x \notin U$, then L is a T_0-space.*

Proof. Let $x \neq y \in L$. Then $x \to y \neq 1$ or $y \to x \neq 1$. W.O.L.G, let $x \to y \neq 1$. Then there exists an open neighborhood U of $x \to y$ such that $1 \notin U$. Since $(L, \to, \mathcal{U})$ is a left topological residuated lattice, there exists an open neighborhood V of x such that $V \to y \subseteq U$. We prove that $y \notin V$. If $y \in V$, then by (R_2), $1 = y \to y \in U$, which is a contradiction. Hence $(L, \mathcal{U})$ is a T_0-space. □

Theorem 4.6. *Let L be a residuated lattice such that for each $a \neq 1$, there is a $b \in L$ such that $a < b < 1$. Then there is a nontrivial topology $\mathcal{U}$ on L such that L is a T_0-space.*

Proof. Let for each $a \in L$, $L_a = \{x : a \leq x\}$. Then by Theorem 3.4, $\mathbf{B} = \{L_a : a \in L\}$ is a base of a nontrivial topology $\mathcal{U}$ on L such that $(L, \rightarrow, \mathcal{U})$ is a left topological residuated lattice. Let $1 \neq a \in L$. Then there is a $b \in L$ such that $a < b < 1$. Now, L_b is an open neighborhood of 1 such that $a \notin L_b$. By Proposition 4.5, L is a T_0-space. □

Theorem 4.7. *Let L be a residuated lattice. Then, there is a nontrivial topology $\mathcal{U}$ on L such that L is a T_1-space.*

Proof. Let $F \neq \{1\}$ be a proper filter in L. By Proposition 4.1(ii), $\mathcal{U} = \{U \subseteq L : \forall x \in U, F \rightarrow x \subseteq U\}$ is a nontrivial topology on L. For each $x \in L$, $F \rightarrow x$ is an open set because if $y \in F \rightarrow x$, then there is a $z \in F$ such that $y = z \rightarrow x$ and by (R_1), $F \rightarrow y = F \rightarrow (z \rightarrow x) = (F \odot z) \rightarrow x \subseteq F \rightarrow x$. Now, if $x \neq y \in L$, then $F \rightarrow x$ and $F \rightarrow y$ are two open neighborhoods of x and y, respectively, which by (R_5), $x \notin F \rightarrow y$ and $y \notin F \rightarrow x$. □

Proposition 4.8. *Let $(L, \odot, \mathcal{U})$ be a semitopological residuated lattice. If L is t-open and r-open, then L is T_1-space iff, for each $1 \neq x \in L$, there is an open neighborhood U of 1 such that $x \notin U$.*

Proof. If L is a T_1-space, then the proof is clear. Conversely, Let for each $x \neq 1$ there exists an open neighborhood U of 1 such that $x \notin U$. We prove that L is a T_1-space. Let $x, y \in L$ and $x \neq y$. We consider the following cases:
<u>Case 1.</u> Let $x = 1$. Then $y \neq 1$. Hence there is an open neighborhood U of $x = 1$ such that $y \notin U$. Since L is t-open, $y \odot U$ is an open neighborhood of y. But $1 \notin y \odot U$ because if $1 \in y \odot U$, then there is a $z \in U$ such that $1 = y \odot z$. By (R_5), $1 = y$ which is a contradiction.
<u>Case 2.</u> Let $x, y \neq 1$ and $x < y$. Then there is an open neighborhood U of 1 such that $y \notin U$. Since L is t-open and r-open, $U \odot x$ and $U \rightarrow y$ are two open neighborhoods of x and y, respectively. But $y \notin U \odot x$ and $x \notin U \rightarrow y$ because if $y \in U \odot x$ or $x \in U \rightarrow y$, then by (R_5), $x \leq y$ which is a contradiction. If $y < x$, then the proof is similar.
<u>Case 3.</u> Let $x, y \neq 1$ and $x \not< y$ and $y \not< x$. Then there is an open neighborhood U of 1 such that $y \notin U$. Since L is t-open, $x \odot U$ and $y \odot U$ are two open neighborhoods of x and y, respectively. Now, by (R_5), we get that $x \notin y \odot U$ and $y \notin x \odot U$. □

Proposition 4.9. *Let $(L, \rightarrow, \mathcal{U})$ be a semitopological residuated lattice. Then $(L, \mathcal{U})$ is a T_1-space if and only if for any $x \neq 1$ there are neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$.*

Proof. If L is a T_1-space, then the proof is clear. Conversely, let for any $x \neq 1$ there are two open neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$. We prove that L is a T_1-space. Let $x, y \in L$ and $x \neq y$. Then $x \rightarrow y \neq 1$ or $y \rightarrow x \neq 1$. W.O.L.G, let $x \rightarrow y \neq 1$. Let U be an open neighborhood of $x \rightarrow y$ such that $1 \notin U$. Since $\rightarrow$ is continuous in each variable separately, there are two open neighborhoods V and W of x and y, respectively, such that $V \rightarrow y \subseteq U$ and $x \rightarrow W \subseteq U$. But $x \notin W$ and $y \notin V$ because if $y \in V$ or $x \in W$, then $1 = y \rightarrow y \in U$ or $1 = x \rightarrow x \in U$ which both are contradictions. Hence $(L, \mathcal{U})$ is a T_1-space. □

Proposition 4.10. *Let $(L, \rightarrow, \mathcal{U})$ be a topological residuated lattice. Then $(L, \mathcal{U})$ is a Hausdorff space iff, for each $x \neq 1$ there exist two open neighborhoods U and V of x and 1, respectively, such that $U \cap V = \phi$.*

Proof. If L is a T_2-space, then the proof is clear. Conversely, let for each $x \neq 1$, there exist two open neighborhoods U and V of x and 1, respectively, such that $U \cap V = \phi$. Let $x, y \in L$ and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. W.O.L.G, let $x \to y \neq 1$ and U and V be two disjoint open neighborhoods of $x \to y$ and 1. Since $(L, \to, \mathcal{U})$ is a topological residuated lattice, there are two open neighborhoods W_1 and W_2 of x and y, respectively, such that $W_1 \to W_2 \subseteq U$. W_1 and W_2 are disjoint because if $z \in W_1 \cap W_2$, then $1 = z \to z \in U$, which implies that $1 \in U \cap V$, a contradiction. □

Proposition 4.11. *Let L be an ordered residuated lattice i.e, for each $x, y \in L$, $x \leq y$ or $y \leq x$. Then there is a nontrivial topology $\mathcal{U}$ on L such that $(L, \mathcal{U})$ is a Hausdorff space.*

Proof. Let for each $a, b \in L$, $[a, b] = \{x \in L : a \leq x \leq b\}$. Then it is easy to prove that $\mathbf{B} = \{[a, b] : a, b \in L\}$ is a base of a nontrivial topology $\mathcal{U}$ on L. Let $x, y \in L$ and $x < y$. Then $[0, x]$ and $[y, 1]$ are two disjoint open neighborhoods of x and y, respectively. Hence $(L, \mathcal{U})$ is a Hausdorff space. □

Recall that a topological space $(A, \mathcal{U})$ is *regular* if for each $x \in U \in \mathcal{U}$, there is an open set V in A such that $x \in V \subseteq \overline{V} \subseteq U$, where $\overline{V}$ is closure of V in A.[See, [7]]

Theorem 4.12. *Let $(L, \to, \mathcal{U})$ be a regular topological residuated lattice. Then the following statements are equivalent:*
(i) $(L, \mathcal{U})$ is a Hausdorff space,
(ii) $(L, \mathcal{U})$ is a T_1 space,
(iii) $\bigcap_{U \in \mathcal{U}} U = 1$, where $\mathcal{U}$ is a fundamental system of neighborhoods of 1.

Proof. $(i \Rightarrow ii)$ The proof is clear.
$(ii \Rightarrow iii)$ The proof is clear by Proposition 4.9.
$(iii \Rightarrow i)$ Let $\bigcap_{U \in \mathcal{U}} U = 1$, $x \neq y \in L$. Then $x \to y \neq 1$ or $y \to x \neq 1$. W.O.L.G, let $x \to y \neq 1$. Then there exists an open neighborhood U of 1 such that $x \to y \notin U$. Since $(A, \mathcal{U})$ is a regular space, there is a $V \in \mathcal{U}$ such that $1 \in V \subseteq \overline{V} \subseteq U$. Since $L \setminus \overline{V}$ is an open neighborhood of $x \to y$, and $(L, \to, \mathcal{U})$ is a topological residuated lattice, there exist two open neighborhoods W_1 and W_2 of x and y, respectively, such that $W_1 \to W_2 \subseteq A \setminus \overline{V}$. W_1 and W_2 are disjoint because if $z \in W_1 \cap W_2$, then $1 = z \to z \in W_1 \to W_2 \subseteq L \setminus \overline{V}$, which is a contradiction. □

Proposition 4.13. *Let $(L, \odot, \mathcal{U})$ be a Hausdorff semitopological residuated lattice. Then for each $a \in L$, $J_a = \{x \in L : a \odot x = a\}$ is a closed filter in L.*

Proof. Let $a \in L$ and $x, y \in J_a$. Then $a \odot (x \odot y) = (a \odot x) \odot y = a \odot y = a$. If $x \leq y$ and $x \in L$, then by (R_5) and (R_8), $a = a \odot x \leq a \odot y \leq a$, which implies that $a \odot y = a$. Hence J_a is a filter in L. Since L is Hausdorff and t_a is continuous, we get that $J_a = t_a^{-1}(a)$ is closed in L. □

Proposition 4.14. *Let $(L, \odot, \mathcal{U})$ be a Hausdorff compact semitopological residuated lattice. If $\{1\} \in \mathcal{U}$ and for each $1 \neq a \in L$, there is a $z \in L \setminus \{1\}$ such that $a \odot z = a$, then there is a filter $J \neq \{1\}$ in L such that for each $a \in J \setminus \{1\}$, a is a maximal idempotent.*

Proof. Let $\mathcal{F} = \{F \subseteq L : 1 \neq F \textit{ is a closed filter}\}$ and $\mathcal{A} = \{F^* : F \in \mathcal{F}\}$, where $F^* = F \setminus \{1\}$. Since for each $1 \neq a \in L$, by hypothesis and by Proposition 4.8, $J_a = \{x \in L : a \odot x = a\}$ is in $\mathcal{F}$, we get thate $\mathcal{F}$ and $\mathcal{A}$ are nonempty. Since for each $F \in \mathcal{F}$, $F^* = F \cap L \setminus \{1\}$, we get that each of members of $\mathcal{A}$ are closed in L. Let $S = \{F_i^* : i \in I\}$ be a chain in partial order set $(\mathcal{A}, \subseteq)$. Since L is compact, and finite intersection of members S is nonempty, we get that $\cap_{i\in I}F_i^*$ is a nonempty closed set in L. On the other hand, since $\cap_{i\in I}F_i^* = (\cap_{i\in I}F_i)^*$, we conclude that $\cap_{i\in I}F_i \neq \{1\}$. Hence $\cap_{i\in I}F_i^*$ is a lower bound of S in $\mathcal{A}$. By zorn's Lemma, $\mathcal{A}$ has a minimal, say J^*. Thus, $J \neq \{1\}$ is a filter in L. We prove that all of members of J are idempotent. Let $a \in J$. It is easy to prove that $J_a = \{x \in L : a \odot x = a\}$ is a closed filter in L which is contained in J. Hence J_a^* is a member of $\mathcal{A}$ which is contained in J^*. Since J^* is minimal, $J_a = J$. This follows that $a \odot a = a$. To complete the proof, we have to prove that all of members of J are maximal idempotent. Let $1 \neq a \in J$ and b be an idempotent in L such that $a \leq b$. Since J is filter, b is in J. Since b is idempotent and J^* is minimal in $\mathcal{A}$, it is easy to prove that $< b >= J$. Thus, $b = b \odot a \leq a$ which implies that $a = b$. □

5. Conclusion

In this paper we introduced (semi)topological residuated lattices and studied separation axioms T_0, T_1 and T_2 on them. Next researches can study normality, regularity, metrizability and uniformity on (semi)topological residuated lattices.

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