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Existence and uniqueness of solutions of a nonlocal problem involving the $p(x)$-Laplacian

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Abstract. The object of this paper is to study a nonlocal problem involving the $p(x)$-Laplacian where nonlinearities f do not necessarily satisfy the classical conditions, such as Ambrosetti-Rabinowitz condition, but are limited by functions that do satisfy some specific conditions. By using the direct variational approach and the theory of the variable exponent Sobolev spaces, the existence and uniqueness of solutions is obtained.

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1. Introduction

We study the existence and uniqueness of solutions of the following nonlocal problem

$$\begin{cases} -M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\Delta_{p(x)}u = f(x,u) + h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{$\mathbf{P}$}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, $p \in C\left(\overline{\Omega}\right)$ such that $2 \leq p(x) < N$ for any $x \in \overline{\Omega}$, M and f are continuous functions which obey some specific conditions, and $h \in L^{p(x)/p(x)-1}(\Omega)$.

Problem ($\mathbf{P}$) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [15]. To be more precise, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, P_0, h, E, L are constants, which extends the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. For $p(x)$-Kirchhoff-type equations see, for example, [3, 7, 8, 10].

The importance of problem ($\mathbf{P}$) arises mainly from the existence of the $p(x)$-Laplacian $\triangle_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$. Obviously, when $p(x) = 2$, $\triangle_2 = \triangle$ is the usual Laplace operator. However, in case $p(x) \neq 2$ the situation is very crucial, as for example, one encounters the lack of the homogenity, and a result of this, some classical theories, such as the theory of Sobolev spaces, is not applicable. For the papers involving the $p(x)$-Laplacian operator we refer the readers to [4, 13, 17, 19] and references therein. Moreover, the nonlinear problems involving the $p(x)$-Laplacian extremely attractive because they can be used to model dynamical

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phenomena which arise from the study of electrorheological fluids or elastic mechanics. Problems with variable exponent growth conditions also appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium. The detailed application backgrounds of the $p(x)$-Laplacian can be found in [2, 5, 18, 21] and references therein.

One of the most widely used results for solving problem (**P**) is the mountain pass theorem [23]. In order to apply this theorem, it is necessary that the Euler-Lagrange functional associated to the problem has the Palais-Smale property. One way to ensure this is to assume that f satisfies some Ambrosetti-Rabinowitz-type condition [1]. In order to use any of the techniques above, it is necessary that the nonlinearity f has subcritical growth.

The object of this paper is to study problem (**P**) for nonlinearities f which do not necessarily satisfy the classical conditions, such as Ambrosetti-Rabinowitz condition, but are limited by functions that do satisfy some specific conditions.

2. Abstract framework and preliminaries

We state some basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain (for details, see [9, 11, 12, 16]).

Set
$$C_+\left(\overline{\Omega}\right) = \left\{p; p \in C\left(\overline{\Omega}\right), \inf p\left(x\right) > 1, \forall x \in \overline{\Omega}\right\}.$$
Let $p \in C_+\left(\overline{\Omega}\right)$ and denote
$$p^- := \inf_{x\in\overline{\Omega}} p\left(x\right) \leq p\left(x\right) \leq p^+ := \sup_{x\in\overline{\Omega}} p\left(x\right) < \infty.$$
For any $p \in C_+\left(\overline{\Omega}\right)$, we define the variable exponent Lebesgue space by
$$L^{p(x)}\left(\Omega\right) = \left\{u \mid \text{the map } u : \Omega \to \mathbb{R} \text{ is measurable; } \int_\Omega |u\left(x\right)|^{p(x)} dx < \infty\right\},$$
then $L^{p(x)}(\Omega)$ endowed with the norm
$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u\left(x\right)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$
becomes a Banach space.

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by
$$\rho\left(u\right) = \int_\Omega |u\left(x\right)|^{\,p(x)} dx, \quad \forall u \in L^{p(x)}\left(\Omega\right).$$

Proposition 2.1. [11, 16] *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, ...)$, we have*
(i) $|u|_{p(x)} < 1\,(= 1; > 1) \Leftrightarrow \rho\left(u\right) < 1\,(= 1; > 1)$;
(ii) $|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \leq \rho\left(u\right) \leq |u|_{p(x)}^{p^+}$,
$|u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \leq \rho\left(u\right) \leq |u|_{p(x)}^{p^-}$;
(iii) $\lim\limits_{n\to\infty} |u_n|_{p(x)} = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho(u_n) = 0$; $\lim\limits_{n\to\infty} |u_n|_{p(x)} = \infty \Leftrightarrow \lim\limits_{n\to\infty} \rho(u_n) = \infty$.

Proposition 2.2. [11, 16] *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, ...)$, then the following statements are equivalent:*
(i) $\lim\limits_{n\to\infty} |u_n - u|_{p(x)} = 0$;
(ii) $\lim\limits_{n\to\infty} \rho(u_n - u) = 0$;
(iii) $u_n \to u$ *in measure in* Ω *and* $\lim\limits_{n\to\infty} \rho(u_n) = \rho(u)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

for all $u \in W^{1,p(x)}(\Omega)$. The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)},$$

since Poincaré inequality holds, i.e., there exists a positive constant $c > 0$ such that

$$|u|_{p(x)} \leq c|\nabla u|_{p(x)},$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Proposition 2.3. [16] *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv|\, dx \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right) |u|_{p(x)} |v|_{p'(x)}.$$

Proposition 2.4. [11, 16] *Let $r \in C_+(\overline{\Omega})$. If $r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ +\infty & \text{if} \quad p(x) \geq N. \end{cases}$$

3. Main results

Definition 3.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a *weak solution* of (**P**) if any $\varphi \in W_0^{1,p(x)}(\Omega)$,

$$M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx - \int_\Omega f(x,u)\, \varphi dx - \int_\Omega h\varphi dx = 0.$$

The energy functional corresponding to problem (**P**) is defined as $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$,

$$J(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) - \int_\Omega F(x,u)\, dx - \int_\Omega hudx,$$

where $F(x,t) = \int_0^t f(x,s)\, ds$ and $\widehat{M}(t) = \int_0^t M(s)\, ds$ for $x \in \Omega$ and $t \in \mathbb{R}$. It is well known that weak solutions of (**P**) correspond to critical points of the functional J.

The main result of the present paper is:

Theorem 3.1. *Assume that the following conditions hold:*
(M_0) $M : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and satisfies the condition*

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1},$$

for all $t > 0$, *where* m_1,m_2 *and* α *real numbers such that* $0 < m_1 \leq m_2$ *and* $\alpha \geq 1$;
(f_0) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function, and assume that there exist* $a, b > 0$ *such that*

$$|f(x,t)| \leq a + b\,|t|^{q(x)-1}$$

for all $x \in \overline{\Omega}$ *and all* $t \in \mathbb{R}$, *where* $q \in C_+\left(\overline{\Omega}\right)$ *such that* $q(x) < p^*(x)$;
(f_1) $f(x,t)\,t \leq 0$ *and* $(f(x,t) - f(x,s))(t-s) \leq 0$ *for all* $x \in \overline{\Omega}$ *and for all* $t, s \in \mathbb{R}$.
Then problem $(\mathbf{P})$ *has exactly one solution.*

Remark 3.1. As an example of function f satisfying the assumptions of Theorem 3.1, one can take $f(x,t) = -|t|^{\beta(x)-2}t$ with $\beta(x) \in [1, p^*(x))$.

First, we give the following well-known Propositions which are necessary through the present paper (see, e.g., [6, 14, 23]).

Proposition 3.2. *Let* X *be a Banach space and let* $I : X \to \mathbb{R}$ *be a differentiable functional. Assume that for all* $u, v \in X$,

$$\langle I'(u) - I'(v), u - v\rangle \geq 0.$$

Then I *is convex. If the strict inequality holds when* $u \neq v$, *then* I *is strictly convex.*

Proposition 3.3. *Let* X *be a Banach space and let* $I : X \to \mathbb{R}$ *be strictly convex and differentiable functional. Then* I *has at most one critical point in* X.

To obtain the result of Theorem 3.1, we need to show that the following two lemmas hold.

Lemma 3.4. (i) *The functional* J *is well-defined on* $W_0^{1,p(x)}(\Omega)$.
(ii) J *is a continuously Gâteaux differentiable functional, i.e.* J *is of class* $C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, *whose derivative is*

$$\langle J'(u), \varphi\rangle = M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi dx - \int_\Omega f(x,u)\varphi dx - \int_\Omega h\varphi dx,$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$.

Proof. (i) From (f_0), (f_1), (M_0), continuous embeddings and Proposition 2.3, we have

$$\begin{aligned} J(u) &\leq m_2 \int_0^{\frac{1}{p^-}\rho(\nabla u)} s^{\alpha-1} ds + a\int_\Omega |u|\,dx + \frac{b}{q^-}\int_\Omega |u|^{q(x)}\,dx - |h|_{p'(x)}\,|u|_{p(x)} \\ &\leq \frac{m_2}{\alpha\,(p^-)^\alpha}\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^\alpha + a\int_\Omega |u|\,dx + \frac{b}{q^-}\int_\Omega |u|^{q(x)}\,dx - |h|_{p'(x)}\,|u|_{p(x)} \\ &\leq \frac{m_2}{\alpha\,(p^-)^\alpha}\,\|u\|^{\alpha p^+} + c_1\,\|u\|^{q^+} + c_2\,\|u\|, \end{aligned}$$

this implies that J is well-defined on $W_0^{1,p(x)}(\Omega)$.
(ii) For simplicity, we denote by $K : W_0^{1,p(x)} \to \mathbb{R}$,

$$K(u) := \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right),$$

and

$$G(u) := \int_\Omega F(x,u)\,dx + \int_\Omega hudx.$$

Then, we write

$$J(u) = K(u) - G(u).$$

First of all, since M is a continuous function and satisfies growth condition (M_0), it is easy to see that the composition functional $K(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)$ is well-defined and of class $C^1(W_0^{1,p(x)}, \mathbb{R})$ and its derivative $K' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is

$$\langle K'(u), v\rangle = M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx.$$

Therefore, showing that G is continuously Gâteaux differentiable is equivalent to saying that J is continuously Gâteaux differentiable. First, we will show that G is Gâteaux differentiable. Therefore we have to prove that for fixed $u, v \in W_0^{1,p(x)}(\Omega)$,

$$\lim_{t\to 0}\frac{G(u+tv)-G(u)}{t} = \int_\Omega f(x,u)\,vdx + \int_\Omega hvdx.$$

After elementary calculations, one can easily show that, for almost every $x \in \Omega$,

$$\begin{aligned}\lim_{t\to 0}&\frac{F(x,u(x)+tv(x)) + h(u(x)+tv(x)) - F(x,u(x)) - hu(x)}{t}\\ &= (f(x,u(x)) + h(x))\,v(x).\end{aligned}$$

Then, by the Lagrange theorem, there exists a real number θ such that $|\theta| \leq |t|$ and

$$\begin{aligned}&\left|\frac{F(x,u(x)+tv(x)) + h(u(x)+tv(x)) - F(x,u(x)) - hu(x)}{t}\right|\\ &= |(f(x,u(x)+\theta v(x)) + h(x))\,v(x)|\\ &\leq \left(a + b|u(x)+\theta v(x)|^{q(x)-1}\right)|v(x)| + |h(x)v(x)|\\ &\leq |u(x)|^{q(x)-1}|v(x)| + |v(x)|^{q(x)} + |v(x)| + |h(x)v(x)|. \quad (1)\end{aligned}$$

By Proposition 2.3, we get

$$\int_\Omega |u|^{q(x)-1}|v|\,dx \leq \left||u|^{q(x)-1}\right|_{q'(x)}|v|_{q(x)},$$

and

$$\int_\Omega |hv|\,dx \leq |h|_{p'(x)}|v|_{p(x)}.$$

From the above inequalities, one concludes that the expression (1) is in $L^1(\Omega)$. Therefore by the dominated convergence theorem we have

$$\lim_{t\to 0}\frac{G(u+tv)-G(u)}{t} = \int_\Omega f(x,u)\,vdx + \int_\Omega hvdx.$$

Since the right-hand side, as a function of v, is a continuous linear functional on $W_0^{1,p(x)}(\Omega)$, it is the Gâteaux differential of G.

Now, we will prove that $G' : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ is continuous. Assume $u_k \to u$ in $W_0^{1,p(x)}(\Omega)$. Up to a subsequence, we may assume that $u_k \to u$ in $L^{q(x)}(\Omega)$

and $u_k(x) \to u(x)$ a.e. in Ω as $k \to \infty$. Then, using Proposition 2.3 and (f_0) we have

$$\begin{aligned} |\langle G'(u_k) - G'(u), v\rangle| &\leq \int_\Omega |f(x,u_k) - f(x,u)|\, |v|\, dx \\ &\leq \int_\Omega \left| |u_k|^{q(x)-1} - |u|^{q(x)-1} \right| |v|\, dx \\ &\leq \int_\Omega |u_k|^{q(x)-1} |v|\, dx + \int_\Omega |u|^{q(x)-1} |v|\, dx. \end{aligned}$$

Since $u_k \to u$ in $L^{q(x)}(\Omega)$, there exists $\varphi \in L^{q(x)}(\Omega)$ such that $|u_k(x)| \leq \varphi(x)$ a.e. in Ω and for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned} |\langle G'(u_k) - G'(u), v\rangle| &\leq \int_\Omega |\varphi|^{q(x)-1} |v|\, dx + \int_\Omega |u|^{q(x)-1} |v|\, dx \\ &\leq \left(\left| |\varphi|^{q(x)-1} \right|_{q'(x)} + \left| |u|^{q(x)-1} \right|_{q'(x)} \right) |v|_{q(x)} \in L^1(\Omega). \end{aligned}$$

On the other hand, taking into account that $u_k(x) \to u(x)$ a.e. in Ω as $k \to \infty$, we also get

$$\lim_{k\to\infty} |f(x, u_k(x)) - f(x, u(x))| = 0.$$

If we consider the above inequalities and apply the dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \int_\Omega |f(x,u_k) - f(x,u)|\, dx = 0,$$

which implies

$$\lim_{k\to\infty} \sup \|G'(u_k) - G'(u)\| = 0.$$

So we deduce that Gâteaux differential of G is continuous, i.e., G is of class $C^1\left(W_0^{1,p(x)}(\Omega), \mathbb{R}\right)$. □

Lemma 3.5. (i) *The functional J is coercive.*
(ii) *The functional J is strictly convex.*

Proof. (i) From (f_1), it is clear that $F(x,t) \leq 0$ for all $t \in \mathbb{R}$. Moreover considering (M_0), it follows

$$\begin{aligned} J(u) &= \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) - \int_\Omega F(x,u)\, dx - \int_\Omega hu dx \\ &\geq \frac{m_1}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^-} - c\|u\|, \end{aligned}$$

this implies that J is coercive.
(ii) For all $u, v \in W_0^{1,p(x)}(\Omega)$ for $u \neq v$, from (f_1) and (M_0) we have

$$\begin{aligned} \langle I'(u) - I'(v), u - v\rangle &= M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v \right) \\ &\quad \times (\nabla u - \nabla v)\, dx - \int_\Omega (f(x,u) - f(x,v))(u - v)\, dx \\ &\geq \frac{m_1}{(p^+)^{\alpha-1}} \min\left\{ \|u\|^{p^-(\alpha-1)}, \|u\|^{p^+(\alpha-1)} \right\} \\ &\quad \times \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v \right) (\nabla u - \nabla v)\, dx. \end{aligned}$$

Next, we apply the following well-known inequality (see, e.g., [20, 22]), for any $\xi, \eta \in \mathbb{R}^N$

$$\left(|\xi|^{r(x)-2}\xi - |\eta|^{r(x)-2}\eta\right)(\xi - \eta) \geq 2^{2-r^+}|\xi - \eta|^{r(x)},\ r(x) \geq 2.$$

Therefore, one easily concludes that

$$\begin{aligned} \langle I'(u) - I'(v), u - v\rangle &\geq c_1 2^{2-p^+}\int_\Omega |\nabla(u-v)|^{p(x)}\,dx \\ &\geq c_2 2^{2-p^+}\min\left\{\|u-v\|^{p^-}, \|u-v\|^{p^+}\right\} > 0. \end{aligned}$$

By Proposition 3.1, we conclude that J is strictly convex. □

Proof. [Proof of Theorem 3.1] The functional J is continuous and convex, and thus (sequentially) weakly lower semi-continuous. Further, since it is coercive, it has a global minimum point, which is a critical point. Moreover, since J is strictly convex and differentiable, by Proposition 3.2, J must have only one critical point. □

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