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Semi-slant warped product submanifolds of a trans-Sasakian manifold

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Abstract. In this paper we study semi-slant warped product submanifolds of a trans-Sasakian manifold. A characterization for warped product submanifolds in terms of warping function and shape operator is obtained and finally we proved an inequality for squared norm of second fundamental form.

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1. Introduction

In the Gray Hervella classification of almost Hermitian manifolds [1], there appears a class $\mathcal{W}_4$, of Hermitian manifolds which are closely related to a locally conformal Kaehler manifolds. An almost contact metric structure on a manifold $\bar{M}$ is called a trans-Sasakian structure if the product manifold $\bar{M} \times R$ belongs to $\mathcal{W}_4$ [9]. The class $C_6 \oplus C_5$ [6] coincides with the class of trans-Sasakian structure of type (α, β) . We note that the trans-Sasakian structure of type $(0, 0)$ are cosymplectic [5], trans-Sasakian structure of type $(0, \beta)$ are $\beta-$Kenmotsu and trans-Sasakian structure of type $(\alpha, 0)$ are $\alpha-$Sasakian [6].

The notion of semi-slant submanifolds of almost Hermitian manifolds were introduced by N. Papaghuic [17], after that J. L. Cabrerizo et al. [12] defined and study semi-slant submanifolds in the setting of Sasakian manifolds.

R. L. Bishop and B. O. Neil [18] introduced the notion of warped product manifolds. These manifolds are generalization of Riemannian product manifolds and occur naturally. Many important physical applications of warped product manifolds have been discovered (c.f., [9], [17]). Due to wide applications of warped product submanifolds, this becomes a fascinating and interesting topic for research, and many articles are available in literature. CR-warped prod- uct was introduced by Chen [3], they studied warped products CR-submanifolds in the setting of Kaehler manifolds. In the available literature, many geometers have studied warped products in the setting of almost contact metric manifolds (c.f., [7], [15], [22]). Hesigawa and Mihai [8] obtained the inequality for squared norm of the second fundamental form in term of the warping function for contact CR-warped product in Sasakian manifolds. Recently, Falleh R. Al-Solamy and Meraj Ali Khan [7] study the semi-slant warped product submanifolds in the setting of Kenmotsu manifolds.

In this paper we study nontrivial warped product submanifolds of the type $N_T \times_f N_\theta$ where N_T and N_θ are the invariant and proper slant submanifolds of trans-Sasakian manifolds one can also study the warped product of the type $N_\theta \times_f N_T$. From here there emerges the natural problem of finding the geometric behavior of shape

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operator and warping function. We also obtain some interesting results exploring geometric properties of second fundamental form and finally we calculate an estimate for squared norm of second fundamental form.

2. Preliminaries

A $2n+1$ dimensional C^∞ manifold $\bar{M}$ is said to have an *almost contact structure* if there exist on M a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η satisfying the following properties

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold $\bar{M}$ satisfying the following conditions

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

where X, Y are vector fields on $\bar{M}$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be *normal* if the almost complex structure J on the product manifold $\bar{M} \times R$ given by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

has no torsion i.e., J is integrable where f is a $C^\infty-$ function on $\bar{M}\times R$, or alternately we can say the tensor $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of.

An almost contact metric manifold is said to be *trans-Sasakian manifold* [10] if

$$(\bar{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}. \tag{2.3}$$

for all $X, Y \in T\bar{M}$, where α, β are smooth functions on $\bar{M}$ and $\bar{\nabla}$ is the Levi-Civita connection of g and in this case we say that the trans-Sasakian structure is of the type (α, β).

If $\alpha = 0$ then $\bar{M}$ is $\beta-$Kenmotsu manifold and if $\beta = 0$ then $\bar{M}$ is $\alpha-$Sasakian manifold. Moreover, if $\alpha = 1$ and $\beta = 0$ then $\bar{M}$ is a Sasakian manifold and if $\alpha = 0$ and $\beta = 1$ then $\bar{M}$ is a Kenmotsu manifold. From (2.3), it follows that

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.4}$$

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are the induced connection on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.6}$$

for each $X, Y \in TM$ and $N \in T^\perp M$, where h and A_N are the second fundamental form and the shape operator respectively for the immersion of M into $\bar{M}$ and they are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M.

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly, for any $N \in T^{\perp}M$, we write

$$\phi N = tN + fN, \tag{2.9}$$

where tN is the tangential component and fN is the normal component of ϕN. The covariant derivatives of the tensor field P and F are defined as

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y. \tag{2.11}$$

From equations (2.3), (2.5), (2.6), (2.8) and (2.9) we have

$$(\bar{\nabla}_X P)Y = A_{FY}X + th(X,Y) + \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(PX,Y)\xi - \eta(Y)PX\} \tag{2.12}$$

$$(\bar{\nabla}_X F)Y = fh(X,Y) - h(X,PY) - \eta(Y)FX. \tag{2.13}$$

Definition 2.1. [2] A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be *slant submanifold* if for any $x \in M$ and $X \in T_xM - \langle\xi\rangle$ the angle between X and ϕX is constant. The constant angle $\theta \in [0, \pi/2]$ is then called *slant angle* of M in $\bar{M}$. If $\theta = 0$ the submanifold is *invariant* submanifold, if $\theta = \pi/2$ then it is *anti-invariant* submanifold, if $\theta \neq 0, \pi/2$ then it is *proper slant* submanifold.

For slant submanifolds of contact manifolds J. L. Cabrerizo et al. [13] proved the following Lemma

Lemma 2.1. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$, such that $\xi \in TM$ then M is slant submanifold if and only if there exist a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(I - \eta \otimes \xi), \tag{2.14}$$

where $\lambda = -\cos^2\theta$.

Thus, one has the following consequences of above formulae

$$g(PX, PY) = \cos^2\theta[g(X,Y) - \eta(X)\eta(Y)], \tag{2.15}$$

$$g(FX, FY) = \sin^2\theta[g(X,Y) - \eta(X)\eta(Y)]. \tag{2.16}$$

A submanifold M of $\bar{M}$ is said to be *semi-slant submanifold* of an almost contact manifold $\bar{M}$ if there exist two orthogonal complementary distributions D_T and D_θ on M such that

(i) $TM = D_T \oplus D_\theta \oplus \langle\xi\rangle$,
(ii) The distribution D_T is invariant i.e., $\phi D_T \subseteq D_T$,
(iii) The distribution D_θ is slant with slant angle $\theta \neq 0$.

It is straight forward to see that semi-invariant submanifolds and slant submanifolds are semi-slant submanifolds with $\theta = \pi/2$ and $D_T = \{0\}$ respectively.

If μ is invariant subspace under ϕ of the normal bundle $T^{\perp}M$, then in the case of semi-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \mu \oplus FD_\theta. \tag{2.17}$$

A semi-slant submanifold M is called a *semi-slant product* if the distributions D_T and D_θ are parallel on M. In this case M is foliated by the leaves of these distributions.

As a generalization of the product manifolds and in particular of a semi-slant product submanifold, one can consider warped product of manifolds which are defined as

Definition 2.2. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with Riemannian metric g_B and g_F respectively and f be a positive differentiable function on B. The *warped product* of B and F is the Riemannian manifold $(B \times F, g)$, where

$$g = g_B + f^2 g_F.$$

For a warped product manifold $N_1 \times_f N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers respectively. In other words, D_1 is obtained by the tangent vectors of N_1 via the horizontal lift and D_2 is obtained by the tangent vectors of N_2 via vertical lift. In the case of semi-slant warped product submanifolds D_1 and D_2 are replaced by D_T and D_θ respectively.

The warped product manifold $(B \times F, g)$ is denoted by $B \times_f F$. If X is the tangent vector field to $M = B \times_f F$ at (p, q) then

$$\|X\|^2 = \|d\pi_1 X\|^2 + f^2(p)\|d\pi_2 X\|^2.$$

R. L. Bishop and B. O'Neill [13] proved the following

Theorem 2.2. *Let $M = B \times_f F$ be warped product manifolds. If $X, Y \in TB$ and $V, W \in TF$ then*
(i) $\nabla_X Y \in TB$,
(ii) $\nabla_X V = \nabla_V X = (\frac{Xf}{f})V$,
(iii) $\nabla_V W = \frac{-g(V,W)}{f}\nabla f$.
∇f is the gradient of f and is defined as

$$g(\nabla f, X) = Xf, \tag{2.18}$$

for all $X \in TM$.

Corollary 2.3. *On a warped product manifold $M = N_1 \times_f N_2$, the following statements hold*
(i) *N_1 is totally geodesic in M,*
(ii) *N_2 is totally umbilical in M.*

Throughout, we denote by N_T and N_θ an invariant and a slant submanifold respectively of an almost contact metric manifold $\bar{M}$.

Siraj Uddin et al. [21] proved the following Corollary

Corollary 2.4. *Let $\bar{M}$ be a trans-Sasakian manifold and N_1 and N_2 be any Riemannian submanifolds of $\bar{M}$, then there do not exist a warped product submanifold $M = N_1 \times_f N_2$ of $\bar{M}$ such that ξ is tangential to N_2.*

Thus, we assume that the structure vector field ξ is tangential to N_1 of a warped product submanifold $N_1 \times_f N_2$ of $\bar{M}$.

In this paper we will consider the warped product of the type $N_T \times_f N_\theta$ and is called semi-slant warped product submanifolds.

3. Semi-slant warped product submanifolds

Throughout this section we will study the warped product of the type $N_T \times_f N_\theta$, for these submanifolds by Theorem 2.2 we have

$$\nabla_X Z = \nabla_Z X = X ln f Z, \tag{3.1}$$

for any $X \in TN_T$ and $Z \in TN_\theta$.

We start the section exploring some important relation of second fundamental form

Lemma 3.1. *On a semi-slant warped product submanifold* $M = N_T \times_f N_\theta$ *of a trans-Sasakian manifold* $\bar{M}$*, we have*
(i) $g(h(X,Y), FZ) = 0$,
(ii) $g(h(X,Z), FZ) = -\alpha\eta(X)\|Z\|^2 - PXlnf\|Z\|^2$,
(iii) $g(h(X,Z), FPZ) = \cos^2\theta(Xlnf - \beta\eta(X))$,
(iv) $g(h(X,PZ), FZ) = \beta\eta(X) - Xlnf$,
(v) $\xi lnf = \beta$,
for any $X, Y \in TN_T$ *and* $Z \in TN_\theta$.

Proof. For any $X, Y \in TN_T$ and by equation (2.12) we get

$$\nabla_X PY - P\nabla_X Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)PX) + th(X,Y),$$

taking inner product with $Z \in TN_\theta$ in above equation and using equation (3.1), we get part (i) of Lemma.
From equations (2.3) and (3.1) we have

$$PXlnZ - XlnfPZ = -\alpha\eta(X)Z - \beta\eta(X)PZ + th(X,Z)$$

taking inner product with $Z \in TN_\theta$, we get

$$g(h(X,Z), FZ) = -\alpha\eta(X)\|Z\|^2 - PXlnf\|Z\|^2. \quad (3.2)$$

(3.2) is the part (ii) of Lemma.
Taking inner product with $PZ \in TN_\theta$ in (3.2), we find

$$g(h(X,Z), FPZ) = \cos^2\theta(Xlnf - \beta\eta(X)), \quad (3.3)$$

replacing Z by PZ, and using equation (2.14), the above equation gives

$$g(h(X,PZ), FZ) = \beta\eta(X) - Xlnf. \quad (3.4)$$

(3.3) and (3.4) prove the (iii) and (iv) parts of Lemma respectively.
Part (v) of Lemma follows from equations (2.4) and (3.1). □

Note 3.1. From part (v) of Lemma 3.1 and equation (2.18), it is easy to see that

$$\nabla lnf = \beta\xi,$$

where ∇lnf denotes the gradient of lnf and above equation can also be written as

$$\sum_{i=0}^{p} \frac{\partial lnf}{\partial x_i} = \beta\xi, \quad (3.5)$$

where $(p+1)$ is the dimension of N_θ, the equation (3.5) is the first order partial differential equation and has a unique solution, it confirms the existence of warped product of the type $N_\theta \times_f N_T$.

Corollary 3.2. *Let* $M = N_T \times_f N_\theta$ *be a semi-slant warped product submanifolds of a trans-Sasakian manifold, then*

$$g(h(PX,Z), FZ) = Xlnf\|Z\|^2, \quad (3.6)$$

for any $X \in TN_T$ *and* $Z \in TN_\theta$.

The Corollary follows immediately from equation (3.3). Now we have the following characterization for semi-slant warped product

Theorem 3.3. *Let us consider a semi-slant submanifold M with involutive distributions $D_T \oplus \langle\xi\rangle$ and D_θ of a trans-Sasakian manifold $\bar{M}$. Then M is a semi-slant warped product submanifold of $\bar{M}$ if and only if*

$$A_{FZ}X = (X\mu - \beta\eta(X))PZ - (PX\mu + \alpha\eta(X))Z,$$

for any $X \in D_T \oplus \langle\xi\rangle$ and $Z \in TN_\theta$ and μ is a $C^\infty-$function on M satisfying $Z\mu = 0$ for each $Z \in D_\theta$.

Proof. Let M be a semi-slant warped product of type $N_T \times_f N_\theta$, then for any $X \in TN_T$ and $Z \in TN_\theta$, from equations (2.10), (2.12) and (3.1) we have
from equations (3.6) and (3.7), we get

$$A_{FZ}X = -\alpha\eta(X)Z - \beta\eta(X)PZ - PXlnfZ + XlnfPZ, \tag{3.7}$$

taking inner product with $W \in TN_\theta$

$$g(A_{FZ}X, W) = (Xlnf - \beta(X))g(PZ, W) - (PXlnf + \alpha\eta(X))g(Z, W). \tag{3.8}$$

Also from part (i) of Lemma 3.1

$$g(A_{FZ}X, Y) = 0. \tag{3.9}$$

It is easy to observe from equations (3.8) and (3.9), that

$$A_{FZ}X = (X\mu - \beta\eta(X))PZ - (PX\mu + \alpha\eta(X))Z.$$

Conversely, let M be a semi-slant submanifold of $\bar{M}$ satisfying the hypothesis of theorem, then for any $X, Y \in D_T \oplus \langle\xi\rangle$ and $Z \in D_\theta$, then

$$g(h(X, Y), FZ) = 0,$$

this mean $h(X, Y) \in \mu$, then from equation (2.13),

$$-F\nabla_X Y = -h(X, PY) + fh(X, Y).$$

Since $h(X, Y) \in \mu$, $F\nabla_X Y = 0$, i.e., $\nabla_X Y \in D_T \oplus \langle\xi\rangle$, hence each leaf of $D_T \oplus \langle\xi\rangle$ is totally geodesic in $\bar{M}$.

Further, suppose N_θ be a leaf of D_θ and h_θ be second fundamental form of the immersion of N_θ in M, then for any $X \in D_T$ and $Z \in D_\theta$. For any $X \in D_T$ and $Z \in D_\theta$ we have

$$g(h_\theta(Z, Z), \phi X) = g(\nabla_Z Z, \phi X),$$

using equations (2.8), (2.5) and (2.6) the above equation yields

$$g(h_\theta(Z, Z), \phi X) = g(\nabla_Z PZ, X) + g(A_{FZ}Z, X),$$

applying the hypothesis, we get

$$g(h_\theta(Z, Z), \phi X) = -PXlnfg(Z, Z).$$

Replacing X by PX, the above equation gives

$$h_\theta(Z, Z) = \nabla\mu g(Z, Z).$$

From above equation it is easy to derive

$$h_\theta(Z, W) = \nabla\mu g(Z, W),$$

i.e., N_θ is totally umbilical and as $Z\mu = 0$, for all $Z \in D_\theta$, $\nabla\mu$ is defined on N_T, this mean that mean curvature vector of N_θ is parallel i.e., the leaves of D_θ are extrinsic spheres in M. Hence by virtue of result of [20] which says that if the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of nontrivial vector subbundles such that E_1 is spherical and its orthogonal complement E_0 is auto parallel, then the, manifold M is locally isometric to a warped product $M_0 \times_f M_1$, we can say M is locally semi-slant warped product submanifold $N_T \times_f N_\theta$, where the warping function $f = e^\mu$.

Let us denote by $D_T \oplus \langle\xi\rangle$ and D_θ the tangent bundles on N_T and N_θ respectively and let $\{X_0 = \xi, X_1, \dots X_p, X_{p+1} = \phi X_1, \dots X_{2p} = \phi X_p\}$ and $\{Z_1, \dots Z_q,\ Z_{q+1} = PZ_1, \dots Z_{2q} = PZ_q\}$ be local orthonormal frames of vector fields on N_T and N_θ respectively with $2p+1$ and $2q$ being real dimension. Then the second fundamental form can be written as

$$\|h\|^2 = \sum_{i,j=1}^{2p+1} g(h(X_i, X_j), h(X_i, X_j)) + \sum_{i=1}^{2p+1}\sum_{r=1}^{2q}(g(h(X_i, Z_r), h(X_i, Z_r)) + \sum_{r,s=1}^{2q} g(h(Z_r, Z_s), h(Z_r, Z_s)). \tag{3.10}$$

□

Now, on a semi-slant warped product submanifold of a trans-Sasakian manifold, we prove

Theorem 3.4. *Let $M = N_T \times_f N_\theta$ be a semi-slant warped product submanifold of a trans-Sasakian manifold $\bar{M}$ with N_T and N_θ invariant and slant submanifolds respectively of $\bar{M}$. If $\beta\eta(X) \geq 2Xlnf$ for all $X \in TN_T$, then the squared norm of the second fundamental form h satisfies*

$$\|h\|^2 \geq q\csc^2\theta(\beta^2 + 2\|\nabla lnf\|^2)(\cos^4\theta + 3), \tag{3.11}$$

where ∇lnf is the gradient of lnf and $2q$ is the dimension N_θ.

Proof. In view of the decomposition (2.17), we may write

$$h(U, V) = h_{FD_\theta}(U, V) + h_\mu(U, V), \tag{3.12}$$

for each $U, V \in TM$, where $h_{FD_\theta}(U, V) \in FD_\theta$ and $h_\mu(U, V) \in \mu$ with

$$h_{FD_\theta}(U, V) = \sum_{r=1}^{2q} h^r(U, V)FZ_r \tag{3.13}$$

$$h^r(U, V) = \csc^2\theta g(h(U, V), FZ_r) \tag{3.14}$$

for each $U, V \in TM$. In view of above formulae we have

$$g(h_{FD_\theta}(PX_i, Z_r), h_{FD_\theta}(PX_i, Z_r)) = h^r(PX_i, Z_r)X_i lnf + \sin^2\theta \sum_{s\neq r}(h^s(PX_i, Z_r))^2.$$

In view of equations (3.14) and (3.5), we get

$$g(h_{FD_\theta}(PX_i, Z_r), h_{FD_\theta}(PX_i, Z_r)) = \csc^2\theta(X_i lnf)^2 + \sin^2\theta \sum_{s\neq r}(h^s(PX_i, Z_r))^2. \tag{3.15}$$

Summing over $i = 1, \dots 2p$ and $r = 1, \dots 2q$ and using part (v) of Lemma 3.1, we have

$$\sum_{i=1}^{2p}\sum_{r=1}^{2q} g(h_{FD_\theta}(PX_i, Z_r), h_{FD_\theta}(PX_i, Z_r)) = 2q\csc^2\theta\beta^2 + 4q\csc^2\theta\|\nabla lnf\|^2 + \sin^2\theta \sum_{i=1}^{2p}\sum_{r,s=1, r\neq s}^{2q}(h^s(PX_i, Z_r))^2. \tag{3.16}$$

Since we have choose the orthonormal frame of vector fields on D_θ as $\{Z_1, \dots Z_q, Z_{q+1} = PZ_1, \dots Z_{2q} = PZ_q\}$, then the third term in the right hand side of equation (3.16) is written as

$$\csc^2\theta \sum_{i=1}^{2p} [\sum_{r=1}^{q} \{(g(h(PX_i, Z_r), FPZ_r))^2 + (g(h(PX_i, PZ_r), FZ_r))^2\}$$

$$+ \sum_{r=1}^{q} \sum_{s=1, s\neq r}^{q} \{(g(h(PX_i, Z_r), FPZ_s))^2 + (g(h(PX_i, PZ_r), FZ_s))^2\}].$$

From equations (3.4) and (3.5), the first two terms of above equation can be written as

$$\csc^2\theta \sum_{i=1}^{2p+1} [q \cos^4\theta (X_i lnf - \beta\eta(X_i))^2 + q(\beta\eta(X_i) - X_i lnf))^2].$$

In account of to hypothesis $\beta\eta(X_i) \geq 2X_i lnf$ and the fact that $\beta = \xi lnf$ the above expression is greater than equal to the following term

$$q \csc^2\theta [(\cos^4\theta + 1)(\beta^2 + 2\|\nabla lnf\|^2)]. \tag{3.17}$$

Using above inequality into (3.16), we have

$$g(h_{FD_\theta}(PX_i, Z_r), h_{FD_\theta}(PX_i, Z_r)) \geq q \csc^2\theta (\beta^2 + 2\|\nabla lnf\|^2)(\cos^4\theta + 3). \tag{3.18}$$

By Similar calculation and using part (i) of Lemma 3.1, it is easy to see that

$$g(h_{FD_\theta}(X_I, Y_j), h_{FD_\theta}(X_i, Y_j)) = 0 \tag{3.19}$$

The inequality (3.11) follows from (3.10),(3.17), (3.18) and (3.19). □

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