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The Ţăndăreanu theory of generalized Boolean functions

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Abstract. This survey paper is devoted to a class of functions with arguments and values in an arbitrary Boolean algebra, introduced and studied by Nicolae Ţăndăreanu. It includes strictly the class of Boolean functions and it is a proper subclass of the class of all the functions that can be defined over the Boolean algebra.

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In memory of Nicolae Ţăndăreanu (1947-2013)

Introduction

Nowadays the term "Boolean function" has two meanings. One of them is a function with arguments and values in the Boolean algebra $\{0,1\}$; these functions, more properly called truth functions or switching functions, are largely used in numerous applications. The other meaning is much more general: it designates the algebraic functions over an arbitrary Boolean algebra, that is, those functions with arguments and values in an arbitrary Boolean algebra $(B,\vee,\cdot,',0,1)$ that are obtained from constants and variables by superpositions of the basic operations $\vee,\cdot,'$. The works reported in this survey paper refer to the general meaning.

It is well known that Boolean functions are characterized by the fact that they can be represented in the canonical disjunctive form

(CDF) $$f(x_1,\ldots,x_n)=\bigvee\nolimits_{a_1,\ldots,a_n\in\{0,1\}} f(a_1,\ldots,a_n)x_1^{a_1}\ldots x_n^{a_n}\ ,$$

where $\bigvee$ designates iterated disjunction $\vee$ and the notation x^a with $a\in\{0,1\}$ is defined by $x^1=x$, $x^0=x'$. It can be said that the properties of Boolean functions follow from (CDF).

Ţăndăreanu noted that (CDF) is obtained by using the following properties:

$$1^1=1,\ 0^0=0'=1\ ,x\vee x'=1\ ,xx'=0\ .$$

This has led him to the idea of defining a class of functions resembling Boolean functions, with the difference that the functions x^0 and x^1 are replaced by a family of functions $g(a,x)$, where a runs in $\{0,1\}$ or in a bigger set A, such that

$$g(0,0)=g(1,1)=1\ ,\bigvee_{a\in A} g(a,x)=1\ ,g(a,x)g(b,x)=0 \text{ if } a\neq b\ .$$

Between 1981–1985, Nicolae Ţăndăreanu elaborated his theory of generalized Boolean functions, which is the subject-matter of the present survey paper. Our presentation differs from the original works in two major respects. First, we have replaced the term "generalized Boolean function" by "Ţăndăreanu function", yielding a corresponding slight change of notation. Besides, with the exception of Theorem 1, we have dropped

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the condition $B \neq \{0,1\}$, which was not only unnecessary, but prevented Ţăndăreanu functions from being a generalization of Boolean functions, and we have replaced the hypoyheses $A \subset B$ by $A \subseteq B$. Except Theorem 1, we have succeeded in recapturing all of the original results in this slightly modified form.

The first two Sections of this paper, 1 The functions $\mathcal{G}(A,B)$, 2 The Ţăndăreanu functions, present the fundamentals of the theory, introduced in [4]. Sections 3 $TFn(A_1 \cap A_2)$ and $TFn(A_1 \cup A_2)$, 4 $TF1(A)$, 5 $TF1(\{0,1\})$, 6 $TFn(\{0,1\})$ are devoted to the main classes of Ţăndăreanu functions. Papers [8], [9] and the seemingly unpublished paper [10] are presented in Section 7 Other results. Finally we offer a few Conclusions.

In the present paper $(B, \vee, \cdot, ', 0, 1)$ is an arbitrary Boolean algebra. For every n, the functions $f : B^n \longrightarrow B$ form a Boolean algebra $(B^{B^n}, \vee, \cdot, ', 0, 1)$, where the operations are defined pointwise: $(f \vee g)(X) = f(X) \vee g(X)$, $(fg)(X) = f(X)g(X)$, $f'(X) = (f(X))'$, $0(X) = 0$, $1(X) = 1$. Let BFn denote the Boolean algebra of all Boolean functions $f : B^n \longrightarrow B$, which is a Boolean subalgebra of B^{B^n}.

1. The functions $\mathcal{G}(A,B)$

The starting point is the following definition [4].

Definition 1. Let A be a finite set satisfying $\{0,1\} \subseteq A \subseteq B$.[1] We denote by $\mathcal{G}(A,B)$ the set of those functions $g : A \times B \longrightarrow B$ which satisfy the following conditions:

$$g(0,0) = g(1,1) = 1 \ , \tag{1}$$

$$\bigvee_{a \in A} g(a,x) = 1 \ \ (\forall\, x \in B) \ , \tag{2}$$

$$g(a,x)g(b,x) = 0 \ \ (\forall\, a,b \in A,\ a \neq b) \ \ (\forall\, x \in B) \ . \tag{3}$$

It is easy to see that conditions (2), (3) are equivalent to

$$g(a,x) = \prod_{b \in A \setminus \{a\}} g'(b,x) \ \ (\forall\, a \in A) \ \ (\forall\, x \in B) \ . \tag{4}$$

Here is an example proving that $\mathcal{G}(A,B) \neq \varnothing$.

Example 1. Define $g : A \times B \longrightarrow B$ by $g(a,x) = x^a$ for $a \in \{0,1\}$ and $x \in B$, else $g(a,x) = 0$. Then $g \in \mathcal{G}(A,B)$. Indeed, we have already noted that $g(0,0) = 0^0 = 1$, $g(1,1) = 1^1 = 1$, then $\bigvee_{a \in A} g(a,x) = g(0,x) \vee g(1,x) = x \vee x' = 1$ and (3) is satisfied for $a \notin \{0,1\}$ or $b \notin \{0,1\}$, while $g(0,x)g(1,x) = x'x = 0$.

The following result will be used in the sequel.

Lemma 1. *If $\{0,1\} \subseteq A \subset A_1 \subseteq B$, where A_1 is a finite set, then $\mathcal{G}(A_1,B) \setminus \mathcal{G}(A,B) \neq \varnothing$.*

Proof. Take $a_0 \in A_1$ and define $g : A_1 \times B \longrightarrow B$ by

$$g(a,a) = 1 \ (\forall\, a \in A_1),\ g(a_0,x) = 1 \ (\forall\, x \in B \setminus A_1),\ \text{else } g(a,x) = 0 \ .$$

The first two prescriptions imply $g(0,0) = g(1,1) = 1$. Besides, $\bigvee_{a \in A_1} g(a,x) \geq g(a_0,x) = 1$, no matter whether $x \in A_1$ or $x \in B \setminus A_1$. If $a,b \in A_1$ and $a \neq b$, take $x \in B$. If $x \in A_1$, since we cannot have both $x = a$ and $x = b$, it follows that $g(a,x)g(b,x) = 0$. If $x \in B \setminus A_1$, since we cannot have both $a = a_0$ and $b = a_0$, it follows that $g(a,x)g(b,x) = 0$ again. We have thus proved that $g \in \mathcal{G}(A_1,B)$. On the other hand, if $a_0 \notin A$ then $\bigvee_{a \in A} g(a,a_0) = 0$, therefore $g \notin \mathcal{G}(A,B)$. □

[1]The restriction $A \neq B$ required by Ţăndăreanu is not really necessary.

The meaning of the next Proposition is that if $A \subset A_1$ then $\mathcal{G}(A,B)$ can be embedded into $\mathcal{G}(A_1,B)$ by sending each $g \in \mathcal{G}(A,B)$ to its unique extension to $\mathcal{G}(A_1,B)$.

Proposition 1. *If $\{0,1\} \subseteq A \subset A_1 \subseteq B$, where A_1 is a finite set, then every function $g \in \mathcal{G}(A,B)$ has a unique extension to a function $g_1 \in \mathcal{G}(A_1,B)$.*

Proof. Given $g \in \mathcal{G}(A,B)$, we define $g_1 : A_1 \times B \longrightarrow B$ by $g_1(a,x) = g(a,x)$ if $a \in A$ and $g_1(a,x) = 0$ if $a \in A_1 \setminus A$. It is easy to see that $g \in \mathcal{G}(A_1,B)$. For uniqueness we prove that if $h \in \mathcal{G}(A_1,B)$ is an extension of g, then $h(a,x) = 0$ for all $a \in A_1 \setminus A$ and all $x \in B$. Indeed, for every $a \in A_1 \setminus A$ we have $A \subseteq A_1 \setminus \{a\}$, hence for all $x \in B$, using (4) and (2) we get

$$h(a,x) = \prod_{b \in A_1 \setminus \{a\}} h'(b,x) \leq \prod_{b \in A} h'(b,x) = \prod_{b \in A} g'(b,x) = 0 \ .$$

□

An important specialization of $\mathcal{G}(A,B)$ is the case when $A = \{0,1\}$.

Remark 1. $\mathcal{G}(\{0,1\},B)$ consists of those functions $g : \{0,1\} \times B \longrightarrow B$ that satisfy (1) and

$$g(0,x) = g'(1,x) \ \ (\forall\, x \in B) \ , \tag{5}$$

because (5) is obtained from (4) by taking $A = \{0,1\}$.

An alternative description of $\mathcal{G}(\{0,1\})$ is a parametric representation.

Proposition 2. *A map $g : \{0,1\} \times B \longrightarrow B$ belongs to $\mathcal{G}(\{0,1\},B)$ if and only if it is of the form*

$$g(1,x) = h(x), \ \ g(0,x) = h'(x) \ \ (\forall\, x \in B) \ , \tag{6}$$

where $h : B \longrightarrow B$ is a function which satisfies $h(a) = a$ $(\forall\, a \in \{0,1\})$.

Proof. The representation (6) implies (1) and (5). Conversely, if $g \in \mathcal{G}(\{0,1\},B)$ then it follows from (1) and (5) that the function h defined by $h(x) = g(1,x)$ for all $x \in B$ satisfies $h(1) = 1,\ g(0,x) = g'(1,x) = h'(x)$ and $h(0) = g(1,0) = g'(0,0) = 0$. □

An even more particular case yields a singleton.

Proposition 3. [2] *For every finite Boolean algebra B, $\mathcal{G}(B,B)$ contains the Kronecker function $\delta : B \times B \longrightarrow B$, where $\delta(x,x) = 1$ and $\delta(x,y) = 0$ if $x \neq y$.*

Proof. Immediate from the fact that the matrix $\{\delta(x,y)\}_{x,y \in B}$ is the unit matrix of order card$B$. □

Proposition 4. *$\mathcal{G}(\{0,1\},\{0,1\}) = \{\delta\}$, where $\delta : \{0,1\}^2 \longrightarrow \{0,1\}$ is the Kroneker delta.*

Proof. Immediate from Proposition 3 or from Remark 1. □

Exercise 1. There are 4 functions in $\mathcal{G}(\{0,1\}^2,\{0,1\}^2)$.

Remark 2. The restriction to $\{0,1\}^2$ of every $g \in \mathcal{G}(A,B)$ is the Kronecker $\delta \in \mathcal{G}(\{0,1\},\{0,1\})$ because of (1) and (5), which imply $g(0,1) = g'(1,1) = 0$ and $1 = g(0,0) = g'(1,0)$.

Liu [1] has counted the cardinality of the set $\mathcal{G}(B) = \bigcup_{\{0,1\} \subseteq A \subset B} \mathcal{G}(A,B)$ for a finite Boolean algebra B.

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[2] Propositions 3, 4 and 5 are due to the present author.

2. Ţăndăreanu functions

Definition 2. [4][3] For every $g \in \mathcal{G}(A,B)$ and every natural number n, let $TFn(g)$ denote the set of those functions $f : B^n \longrightarrow B$ which satisfy the identity

$$f(x_1,\ldots,x_n) = \bigvee_{a_1,\ldots,a_n \in A} f(a_1,\ldots,a_n)g(a_1,x_1)\ldots g(a_n,x_n)\ . \tag{7}$$

The functions belonging to $TFn(A) = \bigcup_{g\in\mathcal{G}(A,B)} TFn(g)$ will be called *Ţăndăreanu functions* or *T-functions* for short. In particular if B is finite and $A = B$ (case excluded by Ţăndăreanu), we will refer to the functions $f \in TFn(B)$ as *improper T-functions.*

Remark 3. Every Boolean function satisfies (7) with the function g from Example 1, therefore $BFn \subseteq TFn(A) \subseteq B^{B^n}$ for every A and every n.

Remark 4. In view of orthonormality, the computations with $g(a_1,x_1)\ldots g(a_n,x_n)$ obey the same rules as computations with the minterms $x_1^{a_1}\ldots x_n^{a_n}$, where $a_1,\ldots,a_n \in \{0,1\}$; see e.g. [3], Theorem 1.5.

Remark 5. It follows from Remark 4 that every $f \in TFn(A)$ satisfies

$$\prod_{a_1,\ldots,a_n\in A} f(a_1,\ldots,a_n) \le f(x_1,\ldots,x_n) \le \bigvee_{a_1,\ldots,a_n\in A} f(a_1,\ldots,a_n)\ ,$$

because

$$\prod_{a_1,\ldots,a_n} f(a_1,\ldots,a_n) = \bigvee_{b_1,\ldots,b_n} (\prod_{a_1,\ldots,a_n} f(a_1,\ldots,a_n))g(b_1,x_1)\ldots g(b_n,x_n).$$

Remark 6. It also follows from Remark 4 that the set $TFn(A)$ and each set $TFn(g)$, endowed with the pointwise defined operations (see introduction to this paper) are Boolean algebras which contain all constant functions. The proof is similar to the proof that BFn is a Boolean algebra (see e.g. [3], beginning of §1.4 and Theorem 1.17). The inclusions in Remark 3 become "is a subalgebra of".

In most cases the inclusions from Remark 3 are in fact strict inclusions.

Theorem 1. *If $B \ne \{0,1\}$ and A is a finite set satisfying $\{0,1\} \subseteq A \subset B$ then $BFn \subset TFn(A) \subset B^{B^n}$ for every n.*

Proof. The function $\varphi : B^n \longrightarrow B$ defined by $\varphi(X) = 0$ if $X \in A^n$, else $\varphi(X) = 1$ does not satisfy property (7), which defines T-functions. It remains to provide a T-function which is not Boolean.

Let g be the function from Lemma 1 for $A_1 := A$ and define $f : B^n \longrightarrow B$ as follows: for every $x_2,\ldots,x_n \in B$,

$$f(x_1,\ldots,x_n) = \begin{cases} x_1, & \text{if } x_1 \in A,\\ a_0, & \text{if } x_1 \in B\setminus A,\end{cases}$$

and let $h(x_1,\ldots,x_n)$ denote the right side of (7). Then

$$h(x_1,\ldots,x_n) = \bigvee_{a_1\in A} g(a_1,x_1) \bigvee_{a_2,\ldots a_n\in A} f(a_1,\ldots,a_n)g(a_2,x_2)\ldots g(a_n,x_n)\ .$$

Note that $g(x_1,x_1) = 1$ if $x_1 \in A$, and $g(a_0,x_1) = 1$ if $x_1 \in B\setminus A$, while $g(a,x) = 0$ in the other cases. Therefore, taking also into account Remark 4, we see that if $x_1 \in A$ then

$$h(x_1,\ldots,x_n) = \bigvee_{a_2,\ldots,a_n\in A} f(x_1,a_2,\ldots,a_n)g(a_2,x_2)\ldots g(a_n,x_n) = x_1\ ,$$

[3]Except the names T-function, improper T-function and the notation TF.

while if $x_1 \in B \setminus A$ then

$$h(x_1,\ldots,x_n) = \bigvee_{a_2,\ldots,a_n \in A} f(a_0,a_2,\ldots,a_n) = a_0 = f(x_1,\ldots,x_n) \; .$$

We have thus proved that $f(x_1,\ldots,x_n) = h(x_1,\ldots,x_n)$, that is, f satisfies (7). In other words, $f \in TFn(A)$.

On the other hand, if $x_1 \in B \setminus A$ then

$$\bigvee_{a_1,\ldots,a_n \in \{0,1\}} f(a_1,\ldots,a_n) x_1^{a_1} \ldots x_n^{a_n} = \bigvee_{a_1,\ldots,a_n \in \{0,1\}} a_1 x_1^{a_1} \ldots x_n^{a_n}$$

$$= \bigvee_{a_2,\ldots,a_n \in \{0,1\}} x_1 x_2^{a_2} \ldots x_n^{a_n} = x_1 \neq a_0 = f(x_1,\ldots,x_n) \; ,$$

therefore f is not a Boolean function. □

Theorem 1 is the single place in this survey in which we have used conditions $B \neq \{0,1\}$ and $A \neq B$; they are essential for this theorem. Indeed, if $B = \{0,1\}$ then $BFn = TFn(A) = \{0,1\}^{\{0,1\}^n}$. Condition $A = B$ makes sense only if B is finite, in which case we obtain the following result.

Proposition 5. *In a finite Boolean algebra B every function $f : B^n \longrightarrow B$ is an improper T-function.*

Proof. It follows from Proposition 3 that $f \in TFn(B)$, because for every $x_1,\ldots,x_n \in B$,

$$\bigvee_{a_1,\ldots,a_n \in B} f(a_1,\ldots,a_n)\delta(a_1,x_1)\ldots\delta(a_n,x_n) = f(x_1,\ldots,x_n) \; .$$

□

3. $TFn(A_1 \cap A_2)$ and $TFn(A_1 \cup A_2)$

First we prove that $TFn(A)$ is invariant to the introduction of fictitious variables.

Lemma 2. *If $f \in TFn(g)$ and $h : B^{n+p} \longrightarrow B$ is defined by $h(X,Y) = f(X)$ for $X \in B^n$ and $Y \in B^p$ then $h \in TF(n+p)(g)$.*

Proof. We have $h(x_1,\ldots,x_n,y_1,\ldots,y_p) =$

$$= \bigvee_{a_1,\ldots,a_n \in A} f(a_1,\ldots,a_n) g(a_1,x_1)\ldots g(a_n,x_n) \bigvee_{b_1,\ldots,b_p \in A} g(b_1,y_1)\ldots g(b_p,y_p),$$

which is the expansion (7) for h. □

Theorem 2. *If $\{0,1\} \subseteq A \subset A_1 \subseteq B$, where A_1 is a finite set, then $TFn(A) \subseteq TFn(A_1)$.*

Proof. The function $g \in \mathcal{G}(A,B)$ from (7) has the extension $g_1 \in \mathcal{G}(A_1,B)$ described in Proposition 1. Since $g_1(a_i,x)$ is 0 if $a_i \in A_1 \setminus A$ and is $g(a_i,x)$ if $a_i \in A$, for all $i = 1,\ldots,n$, it follows that

$$\bigvee_{a_1,\ldots,a_n \in A_1} f(a_1,\ldots,a_n) g_1(a_1,x_1)\ldots g_1(a_n,x_n)$$

$$= \bigvee_{a_1,\ldots,a_n \in A} f(a_1,\ldots,a_n) g(a_1,x_1)\ldots g(a_n.x_n) = f(x_1,\ldots,x_n),$$

showing that $f \in TFn(A_1)$. □

It follows from Theorem 2 that for every $TFn(A_1)$ and $TFn(A_2)$,

$$TFn(A_1 \cap A_2) \subseteq TFn(A_1) \cap TFn(A_2) \text{ and } TFn(A_1) \cup TFn(A_2) \subseteq TFn(A_1 \cup A_2)\ .$$

Of course, if $A_1 \subseteq A_2$ or $A_2, \subseteq A_1$) then the above inclusions are fulfilled as equalities. The next two theorems show that except this trivial case, the inclusions are strict.

Theorem 3. *If A_1, A_2 are finite sets such that $\{0,1\} \subseteq A_1, A_2 \subseteq B$, $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$ then $TFn(A_1 \cap A_2) \subset TFn(A_1) \cap TFn(A_2)$.*

Proof. We must show that $TFn(A_1) \cap TFn(A_2) \setminus TFn(A_1 \cap A_2) \neq \varnothing$.

Note first that the hypotheses imply $A_1, A_2 \subset B$.

Take $y_1 \in A_1 \setminus A_2$ and $y_2 \in A_2 \setminus A_1$ and define $g_j : A_j \times B \longrightarrow B$ $(j = 1,2)$ like in Lemma 1, that is,

$$g_j(x,x) = 1\ (\forall\, x \in A_j),\ g_j(y_j,x) = 1\ (\forall\, x \in B \setminus A_j),\ \text{else } g_j(y,x) = 0\ \ (j = 1,2)\ .$$

It follows by Lemma 1 that $g_j \in \mathcal{G}(A_j, B)$ $(j = 1,2)$.

Define $f : B^n \longrightarrow B$ by $f(x_1,\dots,x_n) = 0$ if $x_1,\dots,x_n \in A_1 \cap A_2$, else $f(x_1,\dots,x_n) = 1$. Define also

$$h_j(x_1,\dots,x_n) = \bigvee_{a_1,\dots,a_n \in A_j} f(a_1,\dots,a_n) g_j(a_1,x_1)\dots g_j(a_n,x_n)\ \ (j = 1,2)\ .$$

We will prove that $f \in TFn(A_1) \cap TFn(A_2)$ by checking that $f = h_1 = h_2$.

If $x_1,\dots,x_n \in A_1 \cap A_2$ then $g_j(a_1,x_1)\dots g_j(a_n,x_n) = 0$ except the case $a_i = x_i$ $(i = 1,\dots,n)$, so that $h_j(x_1,\dots,x_n) = f(x_1,\dots,x_n)$ $(j = 1,2)$.

If there is some $x_i \in A_1 \setminus A_2$ then $g_1(x_i,x_i) = 1$ and $g_2(y_2,x_i) = 1$. Setting $z_1 = x_i$ and $z_2 = y_2$, we have $g_j(z_j,x_i) = 1$, $g_j(a_i,x_i) = 0$ for $a_i \neq z_j$ and $z_j \notin A_1 \cap A_2$ $(j = 1,2)$. Further, for $t \in \{1,\dots,n\} \setminus \{i\}$ set $b_t = x_t$ if $x_t \in A_j$ and $b_t = y_j$ if $x_t \in B \setminus A_j$. It follows that

$$h_j(x_1,\dots,x_n) = \bigvee_{a_1,\dots,a_{i-1},a_{i+1},\dots,a_n \in A_j} f(a_1,\dots,a_{i-1},z_j,a_{i+1},\dots,a_n) \prod_{t \neq i} g_j(a_t,x_t)$$

$$= \bigvee_{a_1,\dots,a_{i-1},a_{i+1},\dots,a_n \in A_j} \prod_{t \neq i} g_j(a_t,x_t) \geq \prod_{t \neq i} g_j(b_t,x_t) = 1 = f(x_1,\dots,x_n)\ .$$

If there is some $x_i \in A_2 \setminus A_1$, interchange 1 and 2 in the above proof.

If $x_1,\dots,x_n \in B \setminus (A_1 \cup A_2)$ then $g_j(y_j,x_i) = 1$ $(i = 1,\dots,n)$ $(j = 1,2)$, hence $g_j(a_i,x_i) = 0$ for $a_i \neq y_j$, therefore $h_j(x_1,\dots,x_n) = f(y_j,\dots,y_j) = 1 = f(x_1,\dots,x_n)$.

We have thus proved that $f \in TFn(A_1) \cap TFn(A_2)$. Yet $f \notin TFn(A_1 \cap A_2)$, otherwise there would exist $\gamma \in \mathcal{G}(A_1 \cap A_2, B)$ such that

$$f(x_1,\dots,x_n) = \bigvee_{a_1,\dots,a_n \in A_1 \cap A_2} f(a_1,\dots,a_n) \gamma(a_1,x_1)\dots\gamma(x_n,x_n) = 0,$$

a contradiction. □

Theorem 4. *If A_1, A_2 are finite sets such that $\{0,1\} \subseteq A_1, A_2 \subseteq B$, $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$, then $TFn(A_1) \cup TFn(A_2) \subset TFn(A_1 \cup A_2)$.*

Proof. We must show that $TFn(A_1 \cup A_2) \setminus (TFn(A_1) \cup TFn(A_2)) \neq \varnothing$.

Take $a \in A_1 \setminus A_2$ and $b \in A_2 \setminus A_1$; note that $a \neq b$ and $a, b \notin \{0,1\}$. Define the function with fictitious variables $f : B^n \longrightarrow B$ by

$$f(x_1,\dots,x_n) = a \text{ or } a' \text{ or } 0\ \text{ according as }\ x_1 = a \text{ or } x_1 = b \text{ or } x_1 \notin \{a,b\}\ .$$

Let $g \in \mathcal{G}(A_1 \cup A_2, B)$ be the function obtained from Lemma 1 by taking $A_1 := A_1 \cup A_2$, $a_0 := 0$ and disregarding A, that is,

$$g(x,x) = 1 \ (\forall\, x \in A_1 \cup A_2),\ g(0,x) = 1 \ (\forall\, x \in B \setminus (A_1 \cup A_2)) \text{ else } g(y,x) = 0\ .$$

It follows that

$$\begin{aligned} f(x_1,\ldots,x_n) &= ag(a,x_1) \vee a'g(b,x_1) \vee \bigvee_{a_1 \in (A_1\cup A_2)\setminus\{0,1\}} 0 \cdot g(a_1,x_1) \\ &= \bigvee_{a_1 \in A_1 \cup A_2} f(a_1,\ldots,a_n) g(a_1,x_1) \bigvee_{a_2,\ldots,a_n \in A_1 \cup A_2} g(a_2,x_2)\ldots g(a_n,x_n)\ , \end{aligned}$$

which is the expansion (7) for f, thus proving that $f \in TFn(A_1 \cup A_2)$.

Finally, supposing that $f \in TFn(A_1)$, Remark 5 would imply that

$$a' = f(b,x_2,\ldots,x_n) \le \bigvee_{a_1,\ldots,a_n \in A_1} f(a_1,\ldots,a_n) = a,$$

which is possible only for $a = 1$, a contradiction. Thus $f \notin TFn(A_1)$ and $f \notin TFn(A_2)$ by a similar proof, therefore $f \notin TFn(A_1) \cup TFn(A_2)$. ☐

4. The class TF1(A)

As usual, A is a finite set satisfying $\{0,1\} \subseteq A \subseteq B$.

Proposition 6. *A function $F : B \longrightarrow B$ belongs to $TF1(A)$ if and only if it satisfies identically*

$$\prod_{a\in A} f(a) \le f(x) \le \bigvee_{a\in A} f(a)\ , \tag{8}$$

or equivalently, if and only if (8) *holds for $x \in B \setminus \{0,1\}$.*

Proof. The equivalence of the two variants follows from the fact that every function $f : B \longrightarrow B$ satisfies (8) for $x \in \{0,1\}$.

The condition is necessary by Remark 5.

Conversely, if condition (8) holds, then we can find a function $g \in \mathcal{G}(A,B)$ such that $f(x) = \bigvee_{a\in A} f(a)g(a,x)$. Indeed, we must take $g(0,0) = g(1,1) = 1$. Besides, for an arbitrary but fixed element $x \in B$, set $g(a,x) = y_a$ $(\forall\, a \in A)$. Then $(y_a)_{a\in A}$ must be an orthonormal system satisfying the equation $\bigvee_{a\in A} f(a)y_a = f(x)$. In view of Theorem 4.8 from [3], a necessary and sufficient condition for the existence of an orthonormal solution to the above equation is (8). ☐

Theorems 4.7 and 4.8 from [3] provide also an actual construction of the solution.

In [6] it is also proved that the following strengthening of (8) is necessary and sufficient in order that the function $g \in \mathcal{G}(A,B)$ associated with f be unique:

$$\begin{aligned} &\prod_{a\in A} f(a) \vee \bigvee_{a,b\in A, a\neq b} f'(a)f'(b) \le f(x) \le \\ &\le \left(\bigvee_{a\in A} f(a)\right) \prod_{a,b\in A, a\neq b} (f'(a) \vee f'(b))\ . \end{aligned} \tag{9}$$

If B is a finite Boolean algebra then condition (8) is satisfied for $A = B$, therefore every $f : B \longrightarrow B$ belongs to $TF1(B)$. This is the particular case $n := 1$ of Proposition 5.

Open question. For a finite Boolean algebra B, is it possible that a function $f : B \longrightarrow B$ not satisfying (9) belong to $TF1(A)$ for a conveniently chosen set $A \subset B$?

Paper [5] provides an example in which this is not possible: $B = \{0,1,b,b'\}$ and the function f defined by $f(0) = f(1) = 1$, $f(b) = b$, $f(b') = b'$.

5. The class $TF1(\{0,1\})$

In this section we work with the set $H = \{f : B \longrightarrow B \mid h(0) = 0,\ h(1) = 1\}$.

Remark 7. It follows from Proposition 6 that $H \subseteq TF1(\{0,1\})$.

It follows from Proposition 2 that we obtain a map $u : H \longrightarrow \mathcal{G}(\{0,1\}, B)$ by setting

$$u(h)(a, x) = h^a(x) \ \ (a \in \{0,1\}) \ .$$

In other words, $u(h) = g \Longleftrightarrow g(a, x) = h^a(x)$ $(a \in \{0,1\})$ and if $g \in \mathcal{G}(\{0,1\}, B)$ then $u(h) = g \Longleftrightarrow g(1, x) = h(x)$, again by Proposition 2.

Since $g(1,1) = 1$ and $g(1,0) = 0$, it follows that u is a surjection. If $u(h_1) = u(h_2) = g$ then $h_1(x) = g(1, x) = h_2(x)$. We have thus proved:

Proposition 7. *The map $u : H \longrightarrow \mathcal{G}(\{0,1\}, B)$ is a bijection and if $g \in \mathcal{G}(\{0,1\}, B)$ then $h = u^{-1}(g) \Longleftrightarrow h(x) = g(1, x)$.*

Corollary 1. *If $g \in \mathcal{G}(\{0,1\}, B)$ then $u^{-1}(g) \in TF1(g)$.*

Proof. If $u^{-1}(g) = h$ then $h(x) = g(1, x) = h(0)g(0, x) \vee h(1)g(1, x)$. □

Proposition 8. *For every $g_1, g_2 \in \mathcal{G}(\{0,1\}, B)$ there is a Boolean-algebra isomorphism $\varphi : TF1(g_1) \longrightarrow TF1(g_2)$ such that $\varphi(u^{-1}(g_1)) = u^{-1}(g_2)$ and $\varphi(c) = c$ for every constant function c.*

Proof. For $f(x) = f(0)g_1(0, x) \vee f(1)g_1(1, x)$ set $(\varphi f)(x) = f(0)g_2(0, x) \vee f(1)g_2(1, x)$. It is easy to check that φ is a bijection, $\varphi(f_1 \vee f_2) = \varphi(f_1) \vee \varphi(f_2)$ and $\varphi(f') = \varphi'(f)$ by Proposition 2.. If $u^{-1}(g_1) = h_1$ then $h_1(x) = h_1(0)g_1(0, x) \vee h_1(1)g_1(1, x)$, hence $\varphi(h_1)(x) = h_1(0)g_2(0, x) \vee h_1(1)g_2(1, x) = g_2(1, x) = u^{-1}(g_2)(x)$. The last claim is obvious. □

Corollary 2. *For every $g \in \mathcal{G}(\{0,1\}, B)$ there is an isomorphism $\varphi : TF1(g) \longrightarrow BF1$ such that $\varphi(u^{-1}(g)) = i$ (the identity function) and $\varphi(c) = c$ for every constant function c.*

Proof. Apply Proposition 8 with $g_1 := g$ and $g_2(a, x) := x^a$; we have $u^{-1}(g_2) = i$ by Proposition 7. □

The next Proposition generalizes Theorem 2 from [2].

Proposition 9. *For every $g \in \mathcal{G}(\{0,1\}, B)$, a function $f : B \longrightarrow B$ is in $TF1(g)$ if and only if it can be written in the form*

$$f(x) = f_0 h(0) \vee f_1 h(x) \vee f_2 h(x') \vee f_3 h(1) \ , \tag{10}$$

where $h = u^{-1}(g)$ and (f_0, f_1, f_2, f_3) is an orthonormal system in B.

When this is the case, $f_0 = f'(0)f'(1)$, $f_1 = f'(0)f(1)$, $f_2 = f(0)f'(1)$, $f_3 = f(0)f(1)$ and $f(0) = f_2 \vee f_3$, $f(1) = f_1 \vee f_3$.

Proof. If f satisfies (10) where $h = u^{-1}(g)$ then, since $u^{-1}(g) \in TF1(g)$ by Corollary 1, it follows that $f \in TF1(g)$ by Remark 6.

To prove the converse we need some preparation. Let $g_2 \in \mathcal{G}(\{0,1\}, B)$ be defined by $g_2(1, x) = x$, hence $g_2(0, x) = x'$ (cf. Remark 1). It follows that $TF1(g_2) = BF1$, so that Proposition 1 implies the existence of an isomorphism $\varphi : TF1(g) \longrightarrow BF1$ such that $\varphi(u^{-1}(g)) = u^{-1}(g_2)$ and φ preserves the constants. Further, the identity i on B belongs to H and $g_2(1, x) = i(x)$, hence $u(i) = g_2$ by the definition of u. Therefore $\varphi^{-1}(i) = \varphi^{-1}(u^{-1}(g_2)) = u^{-1}(g)$.

Now suppose $f \in TF1(g)$ and set $\varphi(f) = \ell$. Since $\ell \in BF1$, Theorem 2 from [2] implies the existence of an orthonormal quadruple $(f_0, f_1, f_2, f_3) \in B^4$ such that $\ell(x) = f_0 0 \vee f_1 x \vee f_2 x' \vee f_3 1$. This identity in B yields $\ell = f_0 0 \vee f_1 i \vee f_2 i' \vee f_3 1$ in $BF1$. By applying φ^{-1} it follows that $f = f_0 0 \vee f_1 \varphi^{-1}(i) \vee f_2 \varphi^{-1}(i') \vee f_3 1$, which is (10) because $\varphi^{-1}(i) = h$.

It follows from (10) that $f(0) = f_2 \vee f_3$ and $f(1) = f_1 \vee f_3$. Finally $f'(0)f'(1)$ and the other three products are easily computed using orthonormality. ☐

6. The class $TFn(\{0,1\})$

Recall that $H = \{f : B \longrightarrow B \mid h(0) = 0,\ h(1) = 1\}$.

Proposition 10. *A function $f : B^n \longrightarrow B$ belongs to $TFn(\{0,1\})$ if and only if it is of the form*

$$f(x_1, \ldots, x_n) = p(h(x_1), \ldots, h(x_n)) \tag{11}$$

for some $p \in BFn$ and some $h \in H$.

Proof. In view of Proposition 2, the expansion (7) of $f \in TFn(\{0,1\})$ can be written in the form

$$f(x_1, \ldots, x_n) = \bigvee_{a_1,\ldots,a_n \in \{0,1\}} f(a_1, \ldots, a_n) h^{a_1}(x_1) \ldots h^{a_n}(x_n)\ ,$$

which is of the form (11) with $p \in BFn$ defined by

$$p(x_1, \ldots, x_n) = \bigvee\nolimits_{a_1,\ldots,a_n \in \{0,1\}} f(a_1, \ldots, a_n) x_1^{a_1} \ldots x_n^{a_n}\ . \tag{12}$$

Conversely, the expansion (11) can be written in the form

$$f(x_1, \ldots, x_n) = \bigvee_{a_1,\ldots,a_n \in \{0,1\}} p(a_1, \ldots, a_n) h^{a_1}(x_1) \ldots h^{a_n}(x_n)\ ,$$

which implies that for every $b_1, \ldots, b_n \in \{0,1\}$, $f(b_1, \ldots, b_n) = p(b_1, \ldots, b_n)$, therefore, using again Proposition 2, the latter identity becomes (7), thus proving that $f \in TFn(\{0,1\})$.

☐

The main result is the following.

Proposition 11. *A function $f : B^n \longrightarrow B$ belongs to $TFn\{0,1\})$ if and only if it is of the form*

$$f(x_1, \ldots, x_n) = p(k(x_1), \ldots, k(x_n)) \tag{13}$$

for some $p \in BFn$ and some $k \in TF1(\{0,1\})$.

Proof. In view of Proposition 10, every $f \in TFn(\{0,1\})$ satisfies (13) with p defined by (12) and $k := h$, which belongs to $TF1(\{0,1\})$ by Remark 7.

Conversely, suppose (13) holds. In view of Proposition 10, k is of the form $k(x) = \pi(h(x))$, where $\pi(x) = \pi(0)x' \vee \pi(1)x$. Therefore $k(x) = \pi(0)h'(x) \vee \pi(1)h(x)$, hence for $a \in \{0,1\})$ we have

$$k^a(x) = \pi^a(0)h'(x) \vee \pi^a(1)h(x) = \bigvee_{b \in \{0.1\}} \pi^a(b)h^b(x)\ .$$

In the following the indices of iterated disjunctions run in $\{0,1\}$. We have

$$f(x_1, \ldots, x_n) = \bigvee_{a_1,\ldots,a_n} p(a_1, \ldots, a_n) k^{a_1}(x_1) \ldots k^{a_n}(x_n)$$

$$
\begin{aligned}
&= \bigvee_{a_1,\dots,a_n} p(a_1,\dots,a_n)(\bigvee_{b_1} \pi^{a_1}(b_1)h^{b_1}(x_1))\dots(\bigvee_{b_n} \pi^{a_n}(b_n)h^{b_n}(x_n)) \\
&= \bigvee_{a_1,\dots,a_n} p(a_1,\dots,a_n) \bigvee_{b_1,\dots,b_n} \pi^{a_1}(b_1)\dots\pi^{a_n}(b_n)h^{b_1}(x_1)\dots h^{b_n}(x_n)) \\
&= \bigvee_{b_1,\dots,b_n} (\bigvee_{a_1,\dots,a_n} p(a_1,\dots,a_n)\pi^{a_1}(b_1)\dots\pi^{a_n}(b_n))h^{b_1}(x_1)\dots h^{b_n}(x_n)) \ .
\end{aligned}
$$

Thus we have found 2^n constants $c_{b_1\dots b_n}$ such that

$$
f(x_1,\dots,x_n) = \bigvee_{b_1,\dots,b_n} c_{b_1\dots b_n}h^{b_1}(x_1)\dots h^{b_n}(x_n) = q(h(x_1),\dots,h(x_n)) \ ,
$$

where $q(x_1,\dots,x_n) = \bigvee_{b_1,\dots,b_n} x_1^{b_1}\dots x_n^{b_n}$, showing that $q \in BFn$. Therefore $f \in TFn(\{0,1\})$ by Proposition 10. $\square$

We have noticed in Remark 3 that $BFn = TFn(\bar{g})$, where $\bar{g} \in \mathcal{G}(A,B)$ is defined by $\bar{g}(1,x) = x,\ \bar{g}(0,x) = x'$, else $\bar{g}(a,x) = 0$. Note that $\bar{g}$ is the unique extension to $\mathcal{G}(A,B)$ of the function $g_2 \in \mathcal{G}(\{0,1\},B)$ used in the proof of Proposition 9 (cf. Proposition 1).

Now we prove that if $\bar{g}$ is replaced by g_2 or, more generally, by an arbitrary function from $\mathcal{G}(\{0,1\},B)$, the equality is recaptured as an isomorphism.

Theorem 5. *For any function $g \in \mathcal{G}(\{0,1\},B)$, the Boolean algebras BFn and $TFn(g)$ are isomorphic.*

Proof. We map every function $f \in BFn$ to the function $p \in TFn(g)$ defined by

$$
p(x_1,\dots,x_n) = \bigvee_{a_1,\dots,a_n\in\{0,1\}} f(a_1,\dots,a_n)g(a_1,x_1)\dots g(a_n,x_n) \ .
$$

The map is surjective because every $q \in TFn(g)$ is of the form

$$
q(x_1,\dots,x_n) = \bigvee_{a_1,\dots,a_n\in\{0,1\}} c_{a_1\dots a_n}g(a_1,x_1)\dots g(a_n,x_n) \ ,
$$

for some 2^n constants $c_{a_1\dots a_n} \in B$, so that q is the image p of the function $f \in BFn$ defined by

$$
f(x_1,\dots,x_n) = \bigvee_{a_1,\dots,a_n\in\{0,1\}} c_{a_1\dots a_n}x_1^{a_1}\dots x_n^{a_n} \ .
$$

To prove injectivity, suppose that $f_1, f_2 \in BFn$ have the same image p. By equating the above representations of p for $f := f_1$ and $f := f_2$ and then taking $x_1 := b_1,\dots,x_n := b_n$, where $b_1,\dots,b_n$ are arbitrary in $\{0,1\}$, we get $f_1(b_1,\dots,b_n) = f_2(b_1,\dots,b_n)$, therefore $f_1 = f_2$.

Thus the map is a bijection and it is a homomorphism by Remark 4. $\square$

7. Other results

At first glance the isotony of a function with arguments and values in a Boolean algebra is a property which has nothing to do with the property of being a Boolean function or a T-function. Yet in paper [8] Ţăndăreanu remarks that in every Boolean algebra B *the isotone functions* $f : B \longrightarrow B$ *belong to* $TF1(\{0,1\})$ by Proposition 6 because $f(0)f(1) \leq f(0) \leq f(x) \leq f(1) \leq f(0) \vee f(1)$.

A function $f : B \longrightarrow B$ is isotone if and only if it satisfies conditions $f(0) \leq f(1)$ and $f(x) = f(0)h'(x) \vee f(1)h(x)$ for some isotone function $h \in H$.

Indeed, these conditions imply easily that if $x \leq y$ then $f(x)f'(y) = 0$. Conversely, it is also easy to check that an isotone function f satisfies these conditions with the isotone function $h \in H$ defined by $h(0) = 0$, $h(1) = 1$, else $h(x) = f(x)$.

The characterizations of isotone functions yield characterizations of antitone functions, because a function f is antitone if and only if f' is isotone.

In [9] the partial derivative of a function $f \in TFn\{0,1\}$ is defined by the very formula used for Boolean functions, that is, $\partial f/\partial x_i : B^{n-1} \longrightarrow B$ is given by

$$(\partial f/\partial x_i)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) + \\ + f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) \,, \tag{14}$$

where $x + y = xy' \vee x'y$, and it is proved that if $f \in TFn(g)$ then $\partial f/\partial x_i \in TF(n-1)(g)$. Besides, f does not depend on the variable x_i if and only if $\partial f/\partial x_i = 0$ identically, and several computation rules are provided, everything like for Boolean functions.

The seemingly unpublished paper [10] deals with the following problem. Given a finite set A such that $\{0,1\} \subseteq A \subset B$ and a positive integer n, characterize the subsets $F \subseteq B^{B^n}$ of the form $TFn(g)$ for some $g \in \mathcal{G}(A,B)$. We present below [10] decomposed into elementary steps.

To solve the problem, Ţăndăreanu uses as parameter a family (f_a) of functions $f_a : B \longrightarrow B$ ($\forall\, a \in A$). With this family one associates the function $g : A \times B \longrightarrow B$ defined by $g(a,x) = f_a(x)$ and the family of functions $h_{ai} : B^n \longrightarrow B$ ($a \in A$, $i \in \{1,\ldots,n\}$) defined by $h_{ai}(x_1,\ldots,x_n) = f_a(x_i)$. This notation is fixed in the following.

Lemma 3. *$(f_a)_{a\in A}$ is orthonormal and satisfies $f_a(a) = 1$ ($\forall\, a \in A$) if and only if $g \in \mathcal{G}(A,B)$ and $g(a,a) = 1$ ($\forall\, a \in A$).*

Proof. Immediate. □

Lemma 4. *If $(f_a)_{a\in A}$ is an orthonormal system such that $f_a(a) = 1$ ($\forall\, a \in A$) then the following hold:*

(i) *$g \in \mathcal{G}(A,B)$ and $\{f_a\}_{a\in A} \subseteq TF1(g)$;*

(ii) *$\{h_{ai}\}_{a\in A, i\in\{1,\ldots,n\}} \subseteq TFn(g)$;*

(iii) *the sublattice of $(B^{B^n}, \vee, \cdot)$ generated by the functions h_{ai} ($a \in A$, $i \in \{1,\ldots,n\}$) and the constant functions of n variables is $TFn(g)$.*

Proof. (i) $g \in \mathcal{G}(A,B)$ by Lemma 3, and $f_a(x) = g(a,x) = \bigvee_{b\in A} \delta(a,b)g(b,x)$.

(ii) By (i) and Lemma 2.

(iii) Let F be the sublattice.

If $f \in F$ then f has an expansion of the form $f = \bigvee u_{i_1\ldots i_t} h_{a_1 i_1} \ldots h_{a_t i_t}$, where $u_{i_1\ldots i_t}$ are constant functions. From this, (ii) and Remark 6 it follows that $f \in TFn(g)$. Therefore $F \subseteq TFn(g)$.

If $f \in TFn(g)$ then using the fact that $g(a,x_i) = f_a(x_i) = h_{ai}(x_1,\ldots,x_n)$, we get

$$f(x_1,\ldots,x_n) = \bigvee_{a_1,\ldots,a_n \in A} f(a_1,\ldots,a_n) h_{a1}(x_1,\ldots,x_n) \ldots h_{an}(x_1,\ldots,x_n) \,,$$

hence $f \in F$. Therefore $TFn(g) \subseteq F$. □

Theorem 6. *If A is a finite set such that $\{0,1\} \subseteq A \subset B$ and F is a subset of B^{B^n}, then the following conditions are equivalent:*

(α) *there is $g \in \mathcal{G}(A,B)$ such that $g(a,a) = 1$ ($\forall\, a \in A$) and $F = TFn(g)$;*

(β) *there is an orthonormal system* $\{f_a\}_{a\in A} \subset B^B$ *satisfying* $f_a(a) = 1$ ($\forall\, a \in A$) *such that* F *is the sublattice of* $(B^{B^n}, \vee, \cdot)$ *generated by the functions* h_{ai} $(a, \in A,\ i \in \{1, \dots, b\})$ *and the constant functions of* n *variables.*

Proof. $(\alpha) \Longrightarrow (\beta)$: It follows from Lemma 3 that the hypotheses of Lemma 4 are satisfied. According to Lemma 4(iii), the subalgebra mentioned in (β) is $TFn(g)$, therefore $F = TFn(g)$.

$(\beta) \Longrightarrow (\alpha)$: The function g associated with $(f_a)_{a\in A}$ belongs to $\mathcal{G}(A, B)$ by Lemma 3 and the sublattice mentioned in (β) is $TFn(g)$ by Lemma 4(iii), therefore $F = TFn(g)$. □

Conclusions. The Ţăndăreanu functions are a proper generalization of Boolean functions. There are two kinds of T-functions, $TFn(g)$ and $TFn(A)$, the latter being set-theoretical unions of the former. The sets $TFn(g)$ are essentially Boolean algebras, regarded from another points of view. The algebras $TFn(A)$ are more general than Boolean algebras. It is our conviction that the theory of Ţăndăreanu functions offers numerous prospects of continuation, which should be exploited.

The most natural problem would be the extension of the wide theory of Boolean functions and equations (cf. [3]) to Ţăndăreanu functions. It is likely that this would work well for $TFn(g)$ algebras, whereas the fact that the functions of a $TFn(A)$ algebra belong to various $TFn(g)$'s seems to raise serious difficulties.

Like Boolean functions, Ţăndăreanu functions are defined over Boolean algebras. Another line of research might consist in working with functions defined over a more general algebraic structure. Post algebras might be an appropriate framework for this kind of generalization, because they share many features of Boolean algebras; cf. my monograph Lattice Functions and Equations, Springer Verlag, London 2001.

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