Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 40(2), 2013, Pages 211–225
ISSN: 1223-6934

Entropy solution for strongly nonlinear elliptic problems with lower order terms and L^1-data

Mostafa El Moumni

Abstract. We give an existence result for strongly nonlinear elliptic equations of the type

$$-\mathrm{div}\Big(a(x,u,\nabla u)+\Phi(u)\Big)+g(x,u,\nabla u)+H(x,\nabla u)=f \text{ in } \Omega,$$

where the right hand side f belongs to $L^1(\Omega)$, $-\mathrm{div}(a(x,u,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u and $\Phi \in C^0(\mathbb{R},\mathbb{R}^N)$. The critical growth condition on g is with respect to ∇u and no growth condition with respect to u, while the function $H(x,\nabla u)$ grows as $|\nabla u|^{p-1}$.

2010 Mathematics Subject Classification. 35B45, 35J66.
Key words and phrases. A priori estimates, Nonlinear elliptic equation.

1. Introduction

In the present paper, we show the existence of an entropy solution for strongly nonlinear elliptic problem of the type

$$-\mathrm{div}\Big(a(x,u,\nabla u)+\Phi(u)\Big)+g(x,u,\nabla u)+H(x,\nabla u)=f \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$. The operator $-\mathrm{div}(a(x,u,\nabla u))$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, which is coercive and grows like $|\nabla u|^{p-1}$ with respect to ∇u, $p' := \frac{p}{p-1}$. Furthermore, the functions g and H are two the Carathéodory functions with suitable assumptions (see Assumption H(2)). The function Φ is just assumed to be continuous on $\mathbb{R}$. Main difficulties in this work arise from the fact that we consider data f which only belong to $L^1(\Omega)$.

Many physicals models lead to elliptic and parabolic problems. For instance, in [14] the authors study the modeling of an electronically device. The derived elliptic system coupled the temperature (denoted u) and the electronically potential (denoted φ). The temperature equation is considered as an elliptic equation where the second member $f = |\nabla \varphi|^2$ belongs to $L^1(\Omega)$. In [15] a Fokker-Planck equation arising in populations dynamics is studied. Models of turbulent flows in oceanography and climatology also lead to such kind of problems (see [16] and the references therein).

In [17] the author studies the Navier-Stokes equations completed by an equation for the temperature ($u = T$). Note that for compressible flows the divergence of the velocity does not vanish, and the temperature equation can be considered with linear terms having the form $b(x)\nabla u$. These linear terms introduce new difficulties in the sense that the compactness results, do not apply directly which needs further technical investigations.

Received September 16, 2013.

We are interested in existence results for entropy solutions to (1.1). For instance, in the variational case (i.e. when $f \in W^{-1,p'}(\Omega)$), existence result can be found in [6] while if $f \in L^1(\Omega)$ initiated basic works were given in [12, 9, 23], also an existence result for (1.1) was proved in [8] (see the references therein). Related topics can be found in [21, 22].

When H is not necessarily the null function and $\Phi \equiv 0$, existence result for problem (1.1) was proved first in [13] in the case where g does not depend on the gradient and then in [20] using, in both works, the rearrangement techniques. For different approach used in the setting of Orlicz Sobolev space the reader can refer to [3, 4]. See also [5] for related topics.

The main features of (1.1) are both the fact that the operator has two lower order terms, which produce a lack of coercivity and the right-hand side which is a measure. The operator has no lower order terms (i.e. $H \equiv g \equiv 0$), in this case the difficulties in studying problem (1.1) are due only to the right-hand side belongs to $L^1(\Omega)$ and the functions $a(x, u, \nabla u)$ and $\Phi(u)$ do not belong to $(L^1_{loc}(\Omega))^N$ in general. In the present paper we consider operators where both the two lower order terms $\Phi(u)$ and $H(x, \nabla u)$ appear without any coerciveness assumption on the operator. Simple examples (the Laplace operator in a ball, i.e. $p = 2$, $\Phi \equiv H \equiv g \equiv 0$, and second member the Dirac mass in the center) show that, in general, the solution of (1.1) does not belong to the space $W^{1,1}_{loc}(\Omega)$. Thus it is necessary to change the classical framework of Sobolev spaces in order to prove existence results. In the present paper we consider operators where both the lower order terms $H(x, \nabla u)$ appear without any coerciveness assumption on the operator.

Our aim in this paper is to investigate the existence of entropy solutions to strongly nonlinear elliptic equations (1.1), in the case where the right-hand side belongs to $L^1(\Omega)$, $\Phi \not\equiv 0$. The function $g(x, s, \xi)$ is assumed to have exactly the natural growth (i.e. of order p), but no growth assumption is imposed with respect to s to the function g which only satisfies the sign condition and the coercivity condition. The function $H(x, \xi)$, which induces a convection term, is assumed only to grow at most as $|\xi|^{p-1}$.

Now we state a slight modification of Gronwall's Lemma (see [2]).

Lemma 1.1. *Given the function $\lambda, \gamma, \varphi, \rho$ defined on $[a, +\infty[$, suppose that $a \geq 0$, $\lambda \geq 0$, $\gamma \geq 0$ and that $\lambda\gamma, \lambda\varphi$ and $\lambda\rho$ belong to $L^1(a, +\infty)$. If for a.e. $t \geq 0$, we have*

$$\varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \lambda(\tau)\varphi(\tau)d\tau.$$

Then, for a.e. $t \geq 0$,

$$\varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \rho(\tau)\lambda(\tau) \left(\int_t^{\tau} \lambda(r)\gamma(r)dr \right) d\tau.$$

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

2. Main results

Let us now give the precise hypotheses on the problem (1.1), we assume that the following assumptions:

Assumption H(1). Ω is a bounded open set of $\mathbb{R}^N$ ($N \geq 1$), let $a : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, such that

$$|a(x, s, \xi)| \leq \beta[k(x) + |s|^{p-1} + |\xi|^{p-1}], \tag{2.1}$$

for a.e. $(x) \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, some positive function $k(x) \in L^{p'}(\Omega)$ and $\beta > 0$.

$$\Big[a(x, s, \xi) - a(x, s, \eta)\Big] \cdot (\xi - \eta) > 0 \text{ for all } (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ with } \xi \neq \eta, \tag{2.2}$$

$$a(x, s, \xi) \cdot \xi \geq \alpha|\xi|^p, \tag{2.3}$$

where α is a strictly positive constant.

Assumption H(2). Furthermore, let $g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $H(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be two Carathéodory functions which satisfy, for almost every $x \in \Omega$ and for all $s \in \mathbb{R},\ \xi \in \mathbb{R}^N$, the following conditions

$$|g(x, s, \xi)| \leq L_1(|s|)(L_2(x) + |\xi|^p), \tag{2.4}$$

$$g(x, s, \xi)s \geq 0, \tag{2.5}$$

where $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function, while $L_2(x)$ is positive and belongs to $L^1(\Omega)$.

$$\exists \quad \delta > 0,\ \nu' > 0 : \text{ for } |s| \geq \delta,\ |g(x, s, \xi)| \geq \nu'|\xi|^p, \tag{2.6}$$

$$|H(x, \xi)| \leq b(x)|\xi|^{p-1}, \tag{2.7}$$

where $b(x)$ is positive and belongs to $L^r(\Omega)$ with $r >$ $\max(N, p)$.

$$\Phi \in C^0(\mathbb{R}, \mathbb{R}^N), \tag{2.8}$$

we point out that no growth hypothesis is assumed on the function Φ. This implies that for a function $u \in W_0^{1,p}(\Omega)$, the term div $\Phi(u)$ may be meaningless, even as a distribution.

Assumption H(3). As far as the right-hand side of (1.1) is concerned, we assume that

$$f \in L^1(\Omega). \tag{2.9}$$

We shall use the following definitions of entropy solutions solutions for problem (1.1) in the following sense:

Definition 2.1. An *entropy solution* of (1.1) is a function $u : \Omega \to \overline{\mathbb{R}}$ such that $T_k(u) \in W_0^{1,p}(\Omega)$, and $\forall\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$

$$\begin{aligned}\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - \varphi)\, dx + \int_\Omega \Phi(u) \cdot \nabla T_k(u - \varphi)\, dx \\ + \int_\Omega g(x, u, \nabla u)T_k(u - \varphi)\, dx + \int_\Omega H(x, \nabla u)T_k(u - \varphi)\, dx = \int_\Omega fT_k(u - \varphi)\, dx.\end{aligned} \tag{2.10}$$

Existence result. Our main results are collected in the following theorems:

Theorem 2.2. *Assume that* (2.1)–(2.9) *hold true. Then the problem* (1.1) *has at least one solution* u *in the sense of definition 2.1.*

Proof. The proof of Theorem 2.2 is done in five steps.

Step 1: Approximate problem and a priori estimates. For $n > 0$, let us define the following approximation of Φ, g, H, and f. First, let Φ_n be a Lipschitz continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$, such that Φ_n uniformly converges to Φ on any compact subset of $\mathbb{R}^N$ as n tends to $+\infty$. Set

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|} \text{ and } H_n(x,\xi) = \frac{H(x,\xi)}{1+\frac{1}{n}|H(x,\xi)|}. \tag{2.11}$$

Note that $g_n(x,s,\xi)$ and $H_n(x,\xi)$ are satisfying the following conditions

$$|g_n(x,s,\xi)| \le n \text{ and } |H_n(x,\xi)| \le n.$$

Let f_n is a regular functions such that f_n strongly converges to f in $L^1(\Omega)$ and $||f_n||_{L^1} \le c_1$ for some constant c_1.

Let us now consider the approximate problem

$$-\text{div}\Big(a(x,u_n,\nabla u_n) + \Phi_n(u_n)\Big) + g_n(x,u_n,\nabla u_n) + H_n(x,\nabla u_n) = f_n \text{ in } \Omega. \tag{2.12}$$

From the Leray-Lions existence theorem (cf. Theorem 2.1 and Remark 2.1 in chapter 2 of [19]), there exists at least one weak solution $u_n \in W_0^{1,\,p}(\Omega)$ of the approximate problem (2.12).

Now, we prove the solution u_n of problem (2.12) is bounded in $W_0^{1,\,p}(\Omega)$, we prove the following

Lemma 2.3. *Let $u_n \in W_0^{1,\,p}(\Omega)$ be a weak solution of (2.12). Then, the following estimate holds,*

$$||u_n||_{W_0^{1,\,p}(\Omega)} \le D, \tag{2.13}$$

where D depends only on Ω, N, p, p', f and $||b||_{L^r(\Omega)}$.

Proof. To get (2.13), we divide the integral $\displaystyle\int_\Omega |\nabla u_n|^p\,dx$ in two parts and we prove the following estimates: for all $k \ge 0$

$$\int_{\{|u_n|\le k\}} |\nabla u_n|^p\,dx \le M_1 k, \tag{2.14}$$

and

$$\int_{\{|u_n|>k\}} |\nabla u_n|^p\,dx \quad \le \quad M_2, \tag{2.15}$$

where M_1 and M_2 are positive constants. In what follows we will denote by M_i, $i = 3, 4, ...$, some generic positive constants. For $\varepsilon > 0$ and $s \ge 0$, we define

$$\varphi_\varepsilon(r) = \begin{cases} \text{sign}(r) & \text{if } \ |r| > s+\varepsilon \\[2ex] \dfrac{\text{sign}(r)(|r|-s)}{\varepsilon} & \text{if } \ s < |r| \le s+\varepsilon \\[2ex] 0 & \qquad \text{otherwise.} \end{cases}$$

Define $\widetilde{\Phi} \in (C^1(\mathbb{R}))^N$ as $\widetilde{\Phi}(s) = \displaystyle\int_0^s \Phi_n(t)\varphi_h'(t)\,dt.$

Then, formally, div $(\widetilde{\Phi}(u_n)) = \Phi_n(u_n)\cdot\varphi_h'(u_n)\nabla u_n$, and by the Divergence Theorem (see also [7]), we get

$$\int_\Omega \Phi_n(u_n).\nabla\varphi_h(u_n)\,dx = \int_\Omega \text{div }(\widetilde{\Phi}(u_n))\,dx = 0. \tag{2.16}$$

We choose $v = \varphi_\varepsilon(u_n)$ as test function in (2.12), we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla(\varphi_\varepsilon(u_n))\,dx + \int_\Omega \Phi(u_n)\cdot\nabla(\varphi_\varepsilon(u_n))\,dx\\ &\quad + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\,dx + \int_\Omega H_n(x,\nabla u_n)\varphi_\varepsilon(u_n)\,dx = \int_\Omega f_n\varphi_\varepsilon(u_n)\,dx.\end{aligned}$$

Using $g_n(x,u_n,\nabla u_n)\varphi_\varepsilon(u_n) \geq 0$, (2.7) and (2.16), we obtain

$$\frac{1}{\varepsilon}\int_{\{s<|u_n|\leq s+\varepsilon\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \leq \int_{\{s<|u_n|\}} b(x)|\nabla u_n|^{p-1}dx + \int_{\{s<|u_n|\}} |f_n|\,dx.$$

Observe that,

$$\begin{aligned}&\int_{\{s<|u_n|\}} b(x)|\nabla u_n|^{p-1}dx\\ &\quad\leq \int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma.\end{aligned}\tag{2.17}$$

Because,

$$\begin{aligned}\int_{\{s<|u_n|\}} b(x)|\nabla u_n|^{p-1}dx &= \int_s^{+\infty}\frac{-d}{d\sigma}\left(\int_{\{\sigma<|u_n|\}} b(x)|\nabla u_n|^{p-1}\,dx\right)d\sigma\\ &= \int_s^{+\infty}\lim_{\delta\to0}\frac{1}{\delta}\left(\int_{\{\sigma<|u_n|\leq\sigma+\delta\}} b(x)|\nabla u_n|^{p-1}\,dx\right)d\sigma\\ &\leq \int_s^{+\infty}\lim_{\delta\to0}\frac{1}{\delta}\left(\int_{\{\sigma<|u_n|\leq\sigma+\delta\}} b^p\,dx\right)^{\frac{1}{p}}\left(\int_{\{\sigma<|u_n|\leq\sigma+\delta\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma\\ &= \int_s^{+\infty}\left(\lim_{\delta\to0}\frac{1}{\delta}\int_{\{\sigma<|u_n|\leq\sigma+\delta\}} b^p\,dx\,dt\right)^{\frac{1}{p}}\left(\lim_{\delta\to0}\frac{1}{\delta}\int_{\{\sigma<|u_n|\leq\sigma+\delta\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma\\ &= \int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma.\end{aligned}$$

By (2.3) and (2.17), we deduce that

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\{s<|u_n|\leq s+\varepsilon\}}\alpha|\nabla u_n|^p\,dx &\leq \int_{\{s<|u_n|\}}|f_n|\,dx\\ &\quad+\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma.\end{aligned}$$

Letting ε go to zero, we obtain

$$\begin{aligned}\frac{-d}{ds}\int_{\{s<|u_n|\}}\alpha|\nabla u_n|^p\,dx &\leq \int_{\{s<|u_n|\}}|f_n|\,dx\\ &\quad+\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma,\end{aligned}\tag{2.18}$$

where $\{s < |u_n|\}$ denotes the set $\{(x) \in \Omega, s < |u_n(x)|\}$ and $\mu(s)$ stands for the distribution function of u_n, that is $\mu(s) = |\{(x) \in \Omega,\ |u_n(x)| < s\}|$ for all $s \geq 0$.

Now, we recall the following inequality (see for example [18]), we have for almost every $s > 0$

$$1 \leq \left(NC_N^{\frac{1}{N}}\right)^{-1} (\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}} \left(-\frac{d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}. \tag{2.19}$$

Using (2.19), we have

$$\begin{aligned}
&\frac{-d}{ds}\int_{\{s<|u_n|\}}\alpha|\nabla u_n|^p\,dx = \alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p}}\\
&\leq \left(\int_{\{s<|u_n|\}}|f_n|\,dx\right)\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(-\tfrac{d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}\\
&+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p}}\\
&\times\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma.
\end{aligned} \tag{2.20}$$

Which implies that,

$$\begin{aligned}
&\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}\\
&\leq \left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(\int_{\{s<|u_n|\}}|f_n|\,dx\right)+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}\\
&\times(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma.
\end{aligned} \tag{2.21}$$

Now, we consider B and ψ (see Lemma 2.2 of [1]) defined by

$$\int_{\{s<|u_n|\}}b^p(x)\,dx = \int_0^{\mu(s)}B^p(\sigma)d\sigma, \text{ and } \psi(s) = \int_{\{s<|u_n|\}}|f_n|\,dx. \tag{2.22}$$

We have

$$||B||_{L^p(\Omega)} \leq ||h||_{L^p(\Omega)} \text{ and } |\psi(s)| \leq ||f_n||_{L^1(\Omega)}. \tag{2.23}$$

From (2.21) and (2.22)

$$\begin{aligned}
&\alpha\left(\tfrac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dx\right)^{\frac{1}{p'}} \leq (NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\psi(s)\\
&+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}B(\mu(\nu))(-\mu'(\nu))^{\frac{1}{p}}\left(-\frac{d}{d\nu}\int_{\{\nu<|u_n|\}}|\nabla u_n|^p dx\right)^{\frac{1}{p'}}d\nu.
\end{aligned}$$

From Lemma 1.1, we obtain

$$\begin{aligned}
&\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}} \leq (NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\psi(s)\\
&\quad+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}\Big[(NC_N^{\frac{1}{N}})^{-1}(\mu(\sigma))^{\frac{1}{N}-1}\psi(\sigma)\Big]\\
&\quad\times B(\mu(\sigma))(-\mu'(\sigma))\,\exp\left(\int_s^{\sigma}(NC_N^{\frac{1}{N}})^{-1})B(\mu(r))(\mu(r))^{\frac{1}{N}-1}(-\mu'(r))dr\right)d\sigma.
\end{aligned}\tag{2.24}$$

Now, by a variable change and by the Hölder inequality, we estimate the argument of the exponential function on the right hand side of (2.24)

$$\begin{aligned}
\int_s^{\sigma}B(\mu(r))(\mu(r))^{\frac{1}{N}-1}(-\mu'(r))dr &= \int_s^{\sigma}B(z)z^{\frac{1}{N}-1}dz\\
&\leq \int_0^{|\Omega|}B(z)z^{\frac{1}{N}-1}dz\\
&\leq ||B||_{L^r}\left(\int_0^{|\Omega|}z^{(\frac{1}{N}-1)r'}\right)^{\frac{1}{r'}}.
\end{aligned}\tag{2.25}$$

Raising to the power p' in (2.24) and we can write

$$\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx \leq M_1. \tag{2.26}$$

where M_1 depends only on Ω, N, p, p', f, α and $||b||_{L^r(\Omega)}$, integrating between 0 and k, and then (2.14) is proved.

We now give the proof of (2.15), using $T_k(u_n)$ as test function in (2.12), gives

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\,dx+\int_\Omega\Big(g_n(x,u_n,\nabla u_n)+H_n(x,\nabla u_n)\Big)T_k(u_n)\,dx\\
&\qquad+\int_\Omega\Phi(u_n)\cdot\nabla T_k(u_n)\,dx=\int_\Omega f_nT_k(u_n)\,dx.
\end{aligned}$$

By the divergence theorem, we get

$$\int_\Omega\Phi(u_n)\cdot\nabla T_k(u_n)=0,$$

using (2.7), we deduce that,

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}}a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx+\int_\Omega\Phi(u_n)\cdot\nabla T_k(u_n)\,dx\\
&\qquad+\int_{\{|u_n|\leq k\}}g_n(x,u_n,\nabla u_n)u_n\,dx+\int_{\{|u_n|>k\}}g_n(x,u_n,\nabla u_n)T_k(u_n)\,dx\\
&\qquad\leq\int_\Omega f_nT_k(u_n)\,dx+\int_\Omega b(x)|\nabla u_n|^{p-1}|T_k(u_n)|\,dx
\end{aligned}\tag{2.27}$$

and by using in the fact that $g_n(x,u_n,\nabla u_n)u_n\geq 0$ and (2.3), we have

$$\begin{aligned}
&\alpha\int_{\{|u_n|\leq k\}}|\nabla u_n|^p\,dx+\int_{\{|u_n|>k\}}g(x,u_n,\nabla u_n)T_k(u_n)\,dx\\
&\qquad\leq k||f||_{L^1}+k\int_{\{|u_n|\leq k\}}b(x)|\nabla u_n|^{p-1}\,dx+k\int_{\{|u_n|\geq k\}}b(x)|\nabla u_n|^{p-1}\,dx,
\end{aligned}$$

which implies that,

$$\begin{aligned}&\int_{\{|u_n|>k\}} g(x,u_n,\nabla u_n)T_k(u_n)\,dx\\&\qquad\le k||f||_{L^1}k\int_{\{|u_n|\le k\}} b(x)|\nabla u_n|^{p-1}\,dx + k\int_{\{|u_n|\ge k\}} b(x)|\nabla u_n|^{p-1}\,dx.\end{aligned}$$

By the Hölder inequality and (2.14), we obtain

$$\begin{aligned}&\int_{\{|u_n|>k\}} g(x,u_n,\nabla u_n)T_k(u_n)\,dx\\&\qquad\le k||f||_{L^1(\Omega)} + k^{1+\frac{1}{p'}}M_1||b||_{L^p(\Omega)} + k\int_{\{|u_n|>k\}} b(x)|\nabla u_n|^{p-1}\,dx.\end{aligned}\tag{2.28}$$

From (2.6) and applying Young's inequality, we get for all $k>\delta$

$$\begin{aligned}\nu' k\int_{\{|u_n|>k\}}|\nabla u_n|^p\,dx &\le k||f||_{L^1(\Omega)} + k^{1+\frac{1}{p'}}M_1||b||_{L^p(\Omega)} + k\int_{\{|u_n|>k\}} b(x)|\nabla u_n|^{p-1}\,dx\\&\le k||f||_{L^1(\Omega)} + k^{1+\frac{1}{p'}}M_1||b||_{L^p(\Omega)} + M_6k||b||^p_{L^p} + \tfrac{1}{p'}\nu' k\int_{\{|u_n|>k\}}|\nabla u_n|^p\,dx.\end{aligned}\tag{2.29}$$

Hence

$$(1-\tfrac{1}{p'})\int_{\{|u_n|>k\}}|\nabla u_n|^p\,dx \le M_3||f||_{L^1(\Omega)} + k^{\frac{1}{p'}}M_5||b||_{L^p(\Omega)} + M_7||b||^p_{L^p},\tag{2.30}$$

and Lemma 2.3 is proved. □

Step 2: Almost everywhere convergence of u_n. We prove that u_n converges to some function u locally in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). We will show that u_n is a Cauchy sequence in measure in any ball B_R.

Let $k>0$ large enough, we have

$$\begin{aligned}k\ \text{meas}(\{|u_n|>k\}\cap B_R) &= \int_{\{|u_n|>k\}\cap B_R}|T_k(u_n)|dx\\&\le \int_{B_R}|T_k(u_n)|dx\\&\le C\left(\int_\Omega|\nabla T_k(u_n)|^p dx\right)^{\frac{1}{p}}\\&\le c_1.\end{aligned}\tag{2.31}$$

Which implies

$$\text{meas}(\{|u_n|>k\}\cap B_R)\le \tfrac{c_1}{k^1},\ \text{for all } k>1.\tag{2.32}$$

We have, for every $\delta>0$,

$$\begin{aligned}\text{meas}(\{|u_m-u_n|>\delta\}\cap B_R)\le\ &\text{meas}(\{|u_n|>k\}\cap B_R)+\ \text{meas}(\{|u_m|>k\}\cap B_R)\\&+\ \text{meas}(\{|T_k(u_n)-T_k(u_m)|>\delta\}).\end{aligned}\tag{2.33}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, there exists some $v_k\in W_0^{1,p}(\Omega)$, such that

$$\begin{aligned}&T_k(u_n)\rightharpoonup v_k \text{ weakly in } W_0^{1,p}(\Omega),\\&T_k(u_n)\to v_k \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega.\end{aligned}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$, then, by (2.32) and (2.33), there exists some $k(\varepsilon) > 0$ such that $\text{meas}(\{|u_n - u_m| > \delta\} \cap B_R) < \varepsilon$ for all $n, m \geq n_0(k(\varepsilon), \delta, R)$. This proves that (u_n) is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function u. Then

$$\begin{aligned}
&T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega),\\
&T_k(u_n) \to T_k(u) \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega.
\end{aligned}$$

Which implies, by using (2.1), for all $k > 0$ there exists a function $h_k \in (L^{p'}(\Omega))^N$, such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } (L^{p'}(\Omega))^N. \tag{2.34}$$

Step 3: Strong convergence of truncations. Let $k > 0$, we consider the function $\phi(s) = se^{\lambda s^2}$, with $\lambda \geq (\frac{L_1(k)}{\alpha})^2$, we have the following inequality

$$\phi'(s) - \frac{L_1(k)}{\alpha} \mid \phi(s) \mid \geq \frac{1}{2}, \tag{2.35}$$

holds for all $s \in \mathbb{R}$. Here, we define $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ where $h > 2k > 0$, and the following function

$$v_n = \phi(w_n). \tag{2.36}$$

Using v_n as test function in (2.12), we obtain

$$\begin{aligned}
\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla w_n \phi'(w_n)\, dx + \int_\Omega \Phi_n(u_n) \cdot \phi'(u_n) \nabla u_n \, dx \\
+ \int_\Omega g_n(x, u_n, \nabla u_n)\phi(w_n)\, dx + \int_\Omega H_n(x, \nabla u_n)\phi(w_n)\, dx = \int_\Omega f_n \phi(w_n)\, dx.
\end{aligned} \tag{2.37}$$

Using the fact that $\displaystyle\int_0^{u_n} \Phi_n(s) \cdot \phi'(s)\, ds \in W_0^{1,p}(\Omega)$ and Stokes formula, we get

$$\int_\Omega \Phi_n(u_n) \cdot \phi'(u_n) \nabla u_n \, dx = \int_\Omega \operatorname{div}\left[\int_0^{u_n} \Phi_n(s) \cdot \phi'(s)\, ds\right] dx = 0. \tag{2.38}$$

Note that, $\nabla w_n = 0$ on the set where $\{|u_n| > h + 4k\}$, therefore, setting $M = 4k + h$, and denoting by $\alpha_h^1(n)$, $\alpha_h^2(n)$,..., various sequences of real numbers which converge to zero when n tends to infinity for any fixed value of h, we get by (2.37) and (2.38)

$$\begin{aligned}
\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla w_n \phi'(w_n)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\phi(w_n)\, dx \\
\leq \int_\Omega f_n \phi(w_n)\, dx + \int_\Omega |H_n(x, \nabla u_n)\phi(w_n)|\, dx.
\end{aligned} \tag{2.39}$$

Since

$$\left|\int_\Omega H_n(x, \nabla u_n)\phi(w_n)\, dx\right| \leq ||\nabla u_n||_{L^p(\Omega)}^{p-1} ||b\phi(T_{2k}(u - T_h(u)))||_{L^p}, \tag{2.40}$$

(where $b\phi(w_n) \to b\phi(T_{2k}(u - T_h(u)))$ in L^p, by Lebesgue's dominated convergence theorem, because $\phi(w_n)$ is bounded).

$$\left|\int_\Omega H_n(x, \nabla u_n)\phi(w_n)\, dx\right| = M_9 ||b\phi(T_{2k}(u - T_h(u)))||_{L^p} + \alpha_h^3(n), \tag{2.41}$$

and since $g_n(x,u_n,\nabla u_n)\phi(w_n)\geq 0$ on the subset $\{x\in\Omega : |u_n(x)|>k\}$, we deduce from (2.39) that

$$\begin{aligned}\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla w_n\phi'(w_n)\,dx+\int_{\{|u_n(x)|\leq k\}}g_n(x,u_n,\nabla u_n)\phi(w_n)\,dx\\ \leq\int_\Omega f_n\phi(w_n)\,dx+M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p}+\alpha_h^3(n).\end{aligned}\tag{2.42}$$

Splitting the first integral on the left hand side of (2.42) where $|u_n|\leq k$ and $|u_n|>k$, we can write, by using (2.3)

$$\begin{aligned}&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla w_n\phi'(w_n)dx\\ &\qquad\geq\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\phi'(w_n)dx\\ &\qquad\quad -C_k\int_{\{|u_n|>k\}}|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u_n)|dx.\end{aligned}\tag{2.43}$$

where $C_k=\phi'(2k)$. Since, when n tends to infinity, we have $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ tends to 0 strongly in $(L^p(\Omega))^N$ while, $(a(x,T_M(u_n),\nabla T_M(u_n)))_n$ is bounded in $(L^{p'}(\Omega))^N$ hence the last term in the previous inequality tends to zero for every h fixed as n tends to infinity. Now, observe that

$$\begin{aligned}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)dx\\ &\ =\int_\Omega[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\cdot[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)dx\\ &\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)dx.\end{aligned}\tag{2.44}$$

By the continuity of the Nymetskii operator, we have for all $i=1,...,N$,

$$a_i(x,T_k(u_n),\nabla T_k(u))\phi'(w_n)\to a_i(x,T_k(u),\nabla T_k(u))\phi'(T_{2k}(u-T_k(u)))$$

strongly in $L^{p'}(\Omega)$, and since $\frac{\partial(T_k(u_n))}{\partial x_i}\rightharpoonup\frac{\partial(T_k(u))}{\partial x_i}$ weakly in $L^p(\Omega)$, the second term of the right hand side of (2.44) tends to zero as n tends to infinity. So that (2.43) yields

$$\begin{aligned}&\int_\Omega a(x,T_M(u_n),\nabla T_M(u))\cdot[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)dx\\ &\qquad\geq\int_\Omega[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\quad\cdot[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)dx+\alpha_h^5(n).\end{aligned}\tag{2.45}$$

For the second term of the left hand side of (2.42), we can estimate as follows

$$\begin{aligned}\int_{\{|u_n|\leq k\}}g(x,u_n,\nabla u_n)\phi(w_n)dx&\leq\int_{\{|u_n|\leq k\}}L_1(k)\Big(L_2(x)+|\nabla T_k(u_n)|^p\Big)|\phi(w_n)|dx\\ \leq L_1(k)\int_\Omega L_2(x)|\phi(w_n)|dx&+\frac{L_1(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\phi(w_n)|dx.\end{aligned}\tag{2.46}$$

Remark that, we have

$$\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\phi(w_n)|dx\\
&\quad=\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\cdot[\nabla T_k(u_n)-\nabla T_k(u)]|\phi(w_n)|dx\\
&\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)|\phi(w_n)|dx\\
&\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot[\nabla T_k(u_n)-\nabla T_k(u)]|\phi(w_n)|dx.
\end{aligned}\tag{2.47}$$

By the Lebesgue's Theorem, we have

$$\nabla T_k(u)|\phi(w_n)|\to\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\text{ strongly in }(L^p(\Omega))^N$$

Moreover, in view of (2.34) the second term of the right hand side of (2.47) tends to

$$\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx.$$

The third term of the right hand side of (2.47) tends to 0 since for all $i=1,...,N$,

$$a_i(x,T_k(u_n),\nabla T_k(u))\phi(w_n)\to a_i(x,T_k(u),\nabla T_k(u))\phi(T_{2k}(u-T_k(u)))$$

strongly in $L^{p'}(\Omega)$, while

$$\frac{\partial(T_k(u_n))}{\partial x_i}\rightharpoonup\frac{\partial(T_k(u))}{\partial x_i}\text{ weakly in }L^p(\Omega).$$

From (2.46) and (2.47), we obtain

$$\begin{aligned}
&\Big|\int_{\{|u_n|\le k\}}g(x,u_n,\nabla u_n)\phi(w_n)dx\Big|\\
&\le\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big]\cdot\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]|\phi(w_n)|dx\\
&\quad+\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx+L_1(k)\int_\Omega L_2(x)|\phi(w_n)|dx+\alpha_h^{10}(n),
\end{aligned}\tag{2.48}$$

Now, by the strongly convergence of f_n and in fact that

$$w_n\rightharpoonup T_{2k}(u-T_k(u))\text{ weakly in }W_0^{1,p}(\Omega)\text{ and weakly}_*\text{ in }L^\infty(\Omega),\tag{2.49}$$

moreover, combining (2.45) and (2.48), we conclude that

$$\begin{aligned}
&\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big]\\
&\qquad\cdot\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big](\phi'(w_n)-\tfrac{L_1(k)}{\alpha}|\phi(w_n)|)dx\\
&\quad\le L_1(k)\int_\Omega L_2(x)|\phi(T_{2k}(u-T_h(u)))|dx+\int_\Omega f\phi(T_{2k}(u-T_h(u)))|dx\\
&\quad+\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx+M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p}+\alpha_h^{11}(n),
\end{aligned}\tag{2.50}$$

which and (2.35), implies that

$$\begin{aligned}
&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big]\cdot\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dx\\
&\quad\le 2L_1(k)\int_\Omega L_2(x)|\phi(T_{2k}(u-T_h(u)))|dx + 2\int_\Omega f\phi(T_{2k}(u-T_h(u)))|dx\\
&\quad + 2\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx + 2M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p} + \alpha_h^{12}(n),
\end{aligned} \tag{2.51}$$

hence, passing to the limit over n, we obtain

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big]\cdot\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dx\\
&\quad\le 2L_1(k)\int_\Omega L_2(x)|\phi(T_{2k}(u-T_h(u)))|dx + 2\int_\Omega f\phi(T_{2k}(u-T_h(u)))|dx\\
&\qquad +2M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p} + 2\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx + \alpha_h^{13}(n).
\end{aligned} \tag{2.52}$$

It remains to show, for our purposes, that the all terms on the right hand side of (2.52) converge to zero as h goes to infinity. Therefore by (2.52), letting h go to infinity, we conclude,

$$\lim_{n\to+\infty}\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big]\cdot[\nabla T_k(u_n)-\nabla T_k(u)]dx = 0. \tag{2.53}$$

Then, Lemma 5 of [11] implies,

$$T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega). \tag{2.54}$$

Step 4: Equi-integrability of H_n and g_n. We shall now prove that $H_n(x,\nabla u_n)$ converges to $H(x,\nabla u)$ and $g_n(x,u_n,\nabla u_n)$ converges to $g(x,u,\nabla u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem. Since $H_n(x,\nabla u_n)\to H(x,\nabla u)$ a.e. Ω and $g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ a.e. Ω, thanks to (2.4) and (2.7), it suffices to prove that $H_n(x,\nabla u_n)$ and $g_n(x,u_n,\nabla u_n)$ are uniformly equi-integrable in Ω. We will now prove that $H_n(x,\nabla u_n)$ is uniformly equi-integrable, we use Hölder's inequality and (2.13), we have for any measurable subset $E\subset\Omega$:

$$\begin{aligned}
\int_E |H_n(x,\nabla u_n)|\,dx &\le \Big(\int_E b^p(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_\Omega|\nabla u_n|^p\Big)^{\frac{1}{p'}}\\
&\le C\Big(\int_E b^p(x)\,dx\Big)^{\frac{1}{p}}
\end{aligned} \tag{2.55}$$

which is small uniformly in n when the measure of E is small.

To prove the uniform equi-integrability of $g_n(x,u_n,\nabla u_n)$. For any measurable subset $E\subset\Omega$ and $m\ge 0$,

$$\begin{aligned}
&\int_E |g(x,u_n,\nabla u_n)|\,dx = \int_{E\cap\{|u_n|\le m\}}|g(x,u_n,\nabla u_n)|\,dx + \int_{E\cap\{|u_n|>m\}}|g(x,u_n,\nabla u_n)|\,dx\\
&\quad\le L_1(m)\int_{E\cap\{|u_n|\le m\}}\Big(L_2(x)+|\nabla u_n|^p\Big)\,dx + \int_{E\cap\{|u_n|>m\}}|g(x,u_n,\nabla u_n)|\,dx\\
&\quad\le L_1(m)\int_{E\cap\{|u_n|\le m\}}\Big(L_2(x)+|\nabla T_m(u_n)|^p\Big)\,dx + \int_{E\cap\{|u_n|>m\}}|g(x,u_n,\nabla u_n)|\,dx\\
&\quad= K_1+K_2.
\end{aligned} \tag{2.56}$$

For fixed m, we get

$$K_1 \quad \leq \quad L_1(m)\int_E \Big(L_2(x)+|\nabla T_m(u_n)|^p\Big)\,dx,$$

which is thus small uniformly in n for m fixed when the measure of E is small (recall that $T_m(u_n)$ tends to $T_m(u)$ strongly in $W_0^{1,p}(\Omega)$). We now discuss the behavior of the second integral of the right hand side of (2.56), let ψ_m be a function such that

$$\begin{cases} \psi_m(s)=0 & \text{if} \quad |s|\leq m-1,\\ \psi_m(s)=\mathrm{sign}(s) & \text{if} \quad |s|\geq m,\\ \psi_m'(s)=1 & \text{if} \quad m-1<|s|<m. \end{cases} \tag{2.57}$$

We choose for $m>1$, $\psi_m(u_n)$ as a test function in (2.12), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\psi_m'(u_n)\,dx+\int_\Omega \Phi_n(u_n)\cdot\nabla u_n\psi_m'(u_n)\,dx \\ +\int_\Omega g_n(x,u_n,\nabla u_n)\psi_m(u_n)\,dx+\int_\Omega H_n(x,\nabla u_n)\psi_m(u_n)\,dx=\int_\Omega f_n\psi_m(u_n)\,dx.$$

By the divergence theorem, we get

$$\int_\Omega \Phi_n(u_n)\cdot\nabla u_n\psi_m'(u_n)\,dx=0.$$

Using (2.3) and Hölder's inequality

$$\int_{\{m-1\leq|u_n|\}}|g_n(x,u_n,\nabla u_n)|\,dx\leq\int_E|H_n(x,\nabla u_n)|\,dx+\int_{\{m-1\leq|u_n|\}}|f|\,dx,$$

and by (2.13), we have

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>m-1\}}|g_n(x,u_n,\nabla u_n)|\,dx=0.$$

Thus we proved that the second term of the right hand side of (2.56) is also small, uniformly in n and in E when m is sufficiently large. Which shows that $g_n(x,u_n,\nabla u_n)$ and $H_n(x,\nabla u_n)$ are uniformly equi-integrable in Ω as required, we conclude that

$$\begin{array}{lll} H_n(x,\nabla u_n)\to H(x,\nabla u) & \text{strongly in} & L^1(\Omega),\\ g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u) & \text{strongly in} & L^1(\Omega). \end{array} \tag{2.58}$$

Step 5: Passing to the limit. We take $T_k(u_n-v)$ as test function in (2.12), with $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, we can write

$$\int_\Omega a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\cdot\nabla T_k(u_n-v)dx+\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(u_n-v)dx \\ +\int_\Omega (g(x,u_n,\nabla u_n)+H(x,\nabla u_n))T_k(u_n-v)\,dx=\int_\Omega f_nT_k(u_n-v)dx. \tag{2.59}$$

By Fatou's lemma and in fact that

$$a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\rightharpoonup a(x,T_{k+||v||_\infty}(u),\nabla T_{k+||v||_\infty}(u))$$

weakly in $(L^{p'}(\Omega))^N$. It easily see that

$$\int_\Omega a(x,T_{k+||v||_\infty}(u),\nabla T_{k+||v||_\infty}(u))\cdot\nabla T_k(u-v)dx \\ \leq \liminf_{n\to\infty}\int_\Omega a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\cdot\nabla T_k(u_n-v)dx. \tag{2.60}$$

For the second term of the right hand side of (2.59). Since $\nabla T_k(u_n - v) \rightharpoonup \nabla T_k(u - v)$ weakly in $(L^p(\Omega))^N$, for the second term of the left hand side of (2.59), we have

$$\int_\Omega \Phi_n(u_n) \cdot \nabla T_k(u_n - v)dx \to \int_\Omega \Phi(u) \cdot \nabla T_k(u - v)dx \text{ as } n \to +\infty. \tag{2.61}$$

On the other hand, we have

$$\int_\Omega f_n T_k(u_n - v)dx \to \int_\Omega f T_k(u - v)dx \text{ as } n \to +\infty. \tag{2.62}$$

Thanks to (2.58) and (2.60)-(2.62), we can pass to the limit in (2.59), and we obtain that u is a solution of the problem (1.1). This completes the proof of Theorem 2.2. □

Remark 2.4. The condition (2.4) can be replaced by the weaker one

$$|g(x, s, \xi)| \le L_2(x) + L_1(|s|)|\xi|^p, \tag{2.63}$$

where $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $L_2(x) \in L^1(\Omega)$.

References

1. A. Alvino and G. Trombetti, Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri, *Ricerche Mat.* **27** (1978) 413–428.
2. E. Beckenbak and R. Beliman, *Inequalities*, Springer-Verlag, 1965.
3. A. Benkirane and J. P. Gossez, An approximation theorem for higher order Orlicz-Sobolev spaces, *Studia Math.* **92** (1989), 231–255.
4. A. Youssfi, A. Benkirane and M. El Moumni, Bounded solutions of unilateral problems for strongly nonlinear equations in Orlicz spaces, *E. J. Qualitative Theory of Diff. Equ.* **21** (2013), 1-2-5.
5. A. Youssfi, A. Benkirane and M. El Moumni, Existence result for strongly nonlinear elliptic unilateral problems with L^1 data, *Complex Variables and Elliptic Equations* (2012) http://dx.doi.org/10.1080/17476933.2012.725166.
6. A. Bensoussan, L. Boccardo and F. Murat, On a non linear partial differential equation having natural growth terms and unbounded solution, *Ann. Inst. Poincaré* **5** (1988), 347–364.
7. L. Boccardo, J. I.Diaz, F. Murat, D. Giachetti, Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms, *J. Differential Equations* **106** (1993), no. 2, 215–237.
8. L. Boccardo and T. Gallouët, Nonliner elliptic equations with right hand side measure, *Comm. in P.D.E.* **17** (1994) , no. 3 and 4, 641–654.
9. L. Boccardo and T. Gallouët, Strongly nonlinear elliptic equation having natural growth terms and L^1 data, *Nonlinear Anal. T.M.A* **19** (1992), 573–578.
10. L. Boccardo,T. Gallouët and L. Orsina, Existence and nonexistence of solutions for some nonlinear elliptic equations, *J. Anal. Math.* **73** (1997), 203–223.
11. L. Boccardo, F. Murat and J. P. Puel, Existance of bounded solution for non linear elliptic unilateral problems, *Annali Mat. Pura Appl.* **152** (1988), 183–196.
12. H. Brezis and W. Strauss, Semilinear second-order elliptic equations in L^1, *J. Math. Soc. Japan* **25**, (1973), no. 4, 565–590.
13. T. Del Vecchio, Nonlinear elliptic equations with measure data, *Potential Analysis* **4** (1995), 185–203.
14. T. Gallouët and R. Herbin, Existence of a solution to a coupled elliptic system, *Appl. Math. Lett.* **7** (1994), 49–55.
15. T. Goudon and M. Saad, On a Fokker-Planck equation arising in population dynamics, *Rev. Mat. Complut.* **11** (1998), 353–372.
16. R. Lewandowski, The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy-viscosity, *Nonlinear Anal. T.M.A* **28** (1997), 393–417.
17. J.-L. Lions, *Mathematical topics in fluid mechanics, incompressible models*, Oxford Lecture Series in Math and its Applications **3**, Clarendon Press, 1996.

18. G. H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1964.
19. J.-L. Lions, *Quelques méthodes de résolution des problème aux limites non linéaires*, Dunod, Paris, 1969.
20. V. M. Monetti and L. Randazzo, Existence results for nonlinear elliptic equations with p-growth in the gradient, *Ricerche di Mat.* **XLIX** (2000), fasc. 1, 163–181.
21. V. Rădulescu, *Analyse de quelques problèmes aux limites elliptiques non linéaires*, Habilitation à Diriger des Recherches, Université Pierre et Marie Curie, Paris, 2003.
22. V. Rădulescu and M. Willem, Elliptic systems involving finite Radon measures, *Differential and Integral Equations* **16** (2003), 221–229.
23. A. Porretta, Existence for elliptic equations in L^1 having lower order terms with natural growth, *Portugual. Math.* **57** (2000), 179–190.
24. G. Talenti, Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces, *Ann. Mat. Pura Appl.* **120** (1979), 159–184.

(Mostafa El Moumni) University Sidi Mohamed Ben Abdellah, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P 1796 Atlas Fez, Morocco
E-mail address: mostafaelmoumni@gmail.com