Entropy solution for strongly nonlinear elliptic problems with lower order terms and $L^1$-data

Mostafa El Moumni

Abstract. We give an existence result for strongly nonlinear elliptic equations of the type

$$-\text{div} \left(a(x,u,\nabla u) + \Phi(u)\right) + g(x,u,\nabla u) + H(x,\nabla u) = f \quad \text{in } \Omega,$$

where the right hand side $f$ belongs to $L^1(\Omega)$, $-\text{div}(a(x,u,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in $\nabla u$ and $\Phi \in C^0(\mathbb{R},\mathbb{R}^N)$. The critical growth condition on $g$ is with respect to $\nabla u$ and no growth condition with respect to $u$, while the function $H(x,\nabla u)$ grows as $|\nabla u|^{p-1}$.

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1. Introduction

In the present paper, we show the existence of an entropy solution for strongly nonlinear elliptic problem of the type

$$-\text{div} \left(a(x,u,\nabla u) + \Phi(u)\right) + g(x,u,\nabla u) + H(x,\nabla u) = f \quad \text{in } \Omega, \quad (1.1)$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$. The operator $-\text{div}(a(x,u,\nabla u))$ is a Leray-Lions operator acting from $W^{1,p}_0(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, which is coercive and grows like $|\nabla u|^{p-1}$ with respect to $\nabla u$ and $\Phi \in C^0(\mathbb{R},\mathbb{R}^N)$. Furthermore, the functions $g$ and $H$ are two the Carathéodory functions with suitable assumptions (see Assumption H(2)). The function $\Phi$ is just assumed to be continuous on $\mathbb{R}$. Main difficulties in this work arise from the fact that we consider data $f$ which only belong to $L^1(\Omega)$.

Many physical models lead to elliptic and parabolic problems. For instance, in [14] the authors study the modeling of an electronically device. The derived elliptic system coupled the temperature (denoted $u$) and the electronically potential (denoted $\varphi$). The temperature equation is considered as an elliptic equation where the second member $f = |\nabla \varphi|^2$ belongs to $L^1(\Omega)$. In [15] a Fokker-Planck equation arising in populations dynamics is studied. Models of turbulent flows in oceanography and climatology also lead to such kind of problems (see [16] and the references therein).

In [17] the author studies the Navier-Stokes equations completed by an equation for the temperature ($u = T$). Note that for compressible flows the divergence of the velocity does not vanish, and the temperature equation can be considered with linear terms having the form $b(x)\nabla u$. These linear terms introduce new difficulties in the sense that the compactness results, do not apply directly which needs further technical investigations.

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We are interested in existence results for entropy solutions to (1.1). For instance, in the variational case (i.e. when $f \in W^{-1,p'}(\Omega)$), existence result can be found in [6] while if $f \in L^1(\Omega)$ initiated basic works were given in [12, 9, 23], also an existence result for (1.1) was proved in [8] (see the references therein). Related topics can be found in [21, 22].

When $H$ is not necessarily the null function and $\Phi \equiv 0$, existence result for problem (1.1) was proved first in [13] in the case where $g$ does not depend on the gradient and then in [20] using, in both works, the rearrangement techniques. For different approach used in the setting of Orlicz Sobolev space the reader can refer to [3, 4]. See also [5] for related topics.

The main features of (1.1) are both the fact that the operator has two lower order terms, which produce a lack of coercivity and the right-hand side which is a measure. The operator has no lower order terms (i.e. $H \equiv g \equiv 0$), in this case the difficulties in studying problem (1.1) are due only to the right-hand side belongs to $L^1(\Omega)$ and the functions $a(x,u,\nabla u)$ and $\Phi(u)$ do not belong to $(L^1_{loc}(\Omega))^N$ in general. In the present paper we consider operators where both the two lower order terms $\Phi(u)$ and $H(x,\nabla u)$ appear without any coerciveness assumption on the operator. Simple examples (the Laplace operator in a ball, i.e. $p = 2$, $\Phi \equiv H \equiv g \equiv 0$, and second member the Dirac mass in the center) show that, in general, the solution of (1.1) does not belong to the space $W^{1,1}_{loc}(\Omega)$. Thus it is necessary to change the classical framework of Sobolev spaces in order to prove existence results. In the present paper we consider operators where both the lower order terms $H(x,\nabla u)$ appear without any coerciveness assumption on the operator.

Our aim in this paper is to investigate the existence of entropy solutions to strongly nonlinear elliptic equations (1.1), in the case where the right-hand side belongs to $L^1(\Omega)$, $\Phi \not\equiv 0$. The function $g(x,s,\xi)$ is assumed to have exactly the natural growth (i.e. of order $p$), but no growth assumption is imposed with respect to $s$ to the function $g$ which only satisfies the sign condition and the coercivity condition. The function $H(x,\xi)$, which induces a convection term, is assumed only to grow at most as $|\xi|^{p-1}$.

Now we state a slight modification of Gronwall’s Lemma (see [2]).

**Lemma 1.1.** Given the function $\lambda, \gamma, \varphi, \rho$ defined on $[a, +\infty[$, suppose that $a \geq 0$, $\lambda \geq 0$, $\gamma \geq 0$ and that $\lambda \gamma, \lambda \varphi$ and $\lambda \rho$ belong to $L^1(a, +\infty)$. If for a.e. $t \geq 0$, we have

$$\varphi(t) \leq \rho(t) + \gamma(t) \int_{-\infty}^{+\infty} \lambda(\tau) \varphi(\tau) d\tau.$$ 

Then, for a.e. $t \geq 0$,

$$\varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \rho(\tau) \lambda(\tau) \left( \int_{\tau}^{+\infty} \lambda(r) \gamma(r) dr \right) d\tau.$$

We recall that, for $k > 1$ and $s$ in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} 
  s & \text{if } |s| \leq k, \\
  k \frac{s}{|s|} & \text{if } |s| > k.
\end{cases}$$

2. Main results

Let us now give the precise hypotheses on the problem (1.1), we assume that the following assumptions:
Assumption H(1). \( \Omega \) is a bounded open set of \( \mathbb{R}^N \) (\( N \geq 1 \)), let \( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) be a Carathéodory function, such that
\[
|a(x, s, \xi)| \leq \beta |k(x)| + |s|^{p-1} + |\xi|^{p-1},
\]
for a.e. \((x) \in \Omega\), all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\), some positive function \(k(x) \in L^p'(\Omega)\) and \(\beta > 0\).
\[
[a(x, s, \xi) - a(x, s, \eta)] \cdot (\xi - \eta) > 0 \quad \text{for all } (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ with } \xi \neq \eta,
\]
\[
a(x, s, \xi) \cdot \xi \geq \alpha |\xi|^p,
\]
where \(\alpha\) is a strictly positive constant.

Assumption H(2). Furthermore, let \(g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}\) and \(H(x, \xi) : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}\) be two Carathéodory functions which satisfy, for almost every \(x \in \Omega\) and for all \(s \in \mathbb{R}, \xi \in \mathbb{R}^N\), the following conditions
\[
|g(x, s, \xi)| \leq L_1(|s|)(L_2(x) + |\xi|^p),
\]
\[
g(x, s, \xi)s \geq 0,
\]
where \(L_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is a continuous increasing function, while \(L_2(x)\) is positive and belongs to \(L^1(\Omega)\).
\[
\exists \quad \delta > 0, \nu' > 0 \quad : \quad |s| \geq \delta, \quad |g(x, s, \xi)| \geq \nu' |\xi|^p,
\]
\[
|H(x, \xi)| \leq b(x)|\xi|^{p-1},
\]
where \(b(x)\) is positive and belongs to \(L^r(\Omega)\) with \(r > \max(N, p)\).
\[
\Phi \in C^0(\mathbb{R}, \mathbb{R}^N),
\]
we point out that no growth hypothesis is assumed on the function \(\Phi\). This implies that for a function \(u \in W^{1,p}_0(\Omega)\), the term \(\text{div } \Phi(u)\) may be meaningless, even as a distribution.

Assumption H(3). As far as the right-hand side of (1.1) is concerned, we assume that
\[
f \in L^1(\Omega).
\]
We shall use the following definitions of entropy solutions solutions for problem (1.1) in the following sense:

Definition 2.1. An entropy solution of (1.1) is a function \(u : \Omega \rightarrow \mathbb{R}\) such that
\[
T_k(u) \in W^{1,p}_0(\Omega), \quad \text{and } \forall \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)
\]
\[
\int_{\Omega} a(x, u, \nabla u) \cdot \nabla T_k(u - \varphi) \, dx + \int_{\Omega} \Phi(u) \cdot \nabla T_k(u - \varphi) \, dx
\]
\[
+ \int_{\Omega} g(x, u, \nabla u) T_k(u - \varphi) \, dx + \int_{\Omega} H(x, \nabla u) T_k(u - \varphi) \, dx = \int_{\Omega} f T_k(u - \varphi) \, dx.
\]

Existence result. Our main results are collected in the following theorems:

Theorem 2.2. Assume that (2.1)–(2.9) hold true. Then the problem (1.1) has at least one solution \(u\) in the sense of definition 2.1.

Proof. The proof of Theorem 2.2 is done in five steps.
Step 1: Approximate problem and a priori estimates. For $n > 0$, let us define the following approximation of $\Phi$, $g$, $H$, and $f$. First, let $\Phi_n$ be a Lipschitz continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$, such that $\Phi_n$ uniformly converges to $\Phi$ on any compact subset of $\mathbb{R}^N$ as $n$ tends to $+\infty$. Set

$$
g_{n}(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{n}{2} |g(x, s, \xi)|} \quad \text{and} \quad H_{n}(x, \xi) = \frac{H(x, \xi)}{1 + \frac{n}{2} |H(x, \xi)|}. \tag{2.11}
$$

Note that $g_{n}(x, s, \xi)$ and $H_{n}(x, \xi)$ are satisfying the following conditions

$$
|g_{n}(x, s, \xi)| \leq n \quad \text{and} \quad |H_{n}(x, \xi)| \leq n.
$$

Let $f_n$ is a regular functions such that $f_n$ strongly converges to $f$ in $L^1(\Omega)$ and $||f_n||_{L^1} \leq c_1$ for some constant $c_1$.

Let us now consider the approximate problem

$$
-\text{div} \left( a(x, u_n, \nabla u_n) + \Phi_n(u_n) \right) + g_n(x, u_n, \nabla u_n) + H_n(x, \nabla u_n) = f_n \quad \text{in } \Omega. \tag{2.12}
$$

From the Leray-Lions existence theorem (cf. Theorem 2.1 and Remark 2.1 in chapter 2 of [19]), there exists at least one weak solution $u_n \in W^{1, p}_0(\Omega)$ of the approximate problem (2.12).

Now, we prove the solution $u_n$ of problem (2.12) is bounded in $W^{1, p}_0(\Omega)$, we prove the following

**Lemma 2.3.** Let $u_n \in W^{1, p}_0(\Omega)$ be a weak solution of (2.12). Then, the following estimate holds,

$$
||u_n||_{W^{1, p}_0(\Omega)} \leq D, \tag{2.13}
$$

where $D$ depends only on $\Omega$, $N$, $p$, $p'$, $f$ and $||b||_{L^r(\Omega)}$.

**Proof.** To get (2.13), we divide the integral $\int_{\Omega} |\nabla u_n|^p \, dx$ in two parts and we prove the following estimates: for all $k \geq 0$

$$
\int_{\{|u_n| \leq k\}} |\nabla u_n|^p \, dx \leq M_1 k, \tag{2.14}
$$

and

$$
\int_{\{|u_n| > k\}} |\nabla u_n|^p \, dx \leq M_2, \tag{2.15}
$$

where $M_1$ and $M_2$ are positive constants. In what follows we will denote by $M_i$, $i = 3, 4, \ldots$, some generic positive constants. For $\varepsilon > 0$ and $s \geq 0$, we define

$$
\varphi_{\varepsilon}(r) = \begin{cases} 
\text{sign}(r) & \text{if } |r| > s + \varepsilon \\
\frac{\text{sign}(r)(|r| - s)}{\varepsilon} & \text{if } s < |r| \leq s + \varepsilon \\
0 & \text{otherwise}
\end{cases}
$$

Define $\tilde{\Phi} \in (C^1(\mathbb{R}))^N$ as $\tilde{\Phi}(s) = \int_0^s \Phi_n(t) \varphi'_{\varepsilon}(t) \, dt$.

Then, formally, $\text{div} \left( \tilde{\Phi}(u_n) \right) = \Phi_n(u_n) \cdot \varphi'_{\varepsilon}(u_n) \nabla u_n$, and by the Divergence Theorem (see also [7]), we get

$$
\int_{\Omega} \Phi_n(u_n) \nabla \varphi_{\varepsilon}(u_n) \, dx = \int_{\Omega} \text{div} \left( \tilde{\Phi}(u_n) \right) \, dx = 0. \tag{2.16}
$$
We choose \( v = \varphi_\varepsilon(u_n) \) as test function in (2.12), we have
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla (\varphi_\varepsilon(u_n)) \, dx + \int_{\Omega} \Phi(u_n) \cdot \nabla (\varphi_\varepsilon(u_n)) \, dx \\
+ \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_\varepsilon(u_n) \, dx + \int_{\Omega} H_n(x, \nabla u_n) \varphi_\varepsilon(u_n) \, dx = \int_{\Omega} f_n \varphi_\varepsilon(u_n) \, dx.
\]
Using \( g_n(x, u_n, \nabla u_n) \varphi_\varepsilon(u_n) \geq 0 \), (2.7) and (2.16), we obtain
\[
\frac{1}{\varepsilon} \int_{\{s < |u_n| \leq s + \varepsilon\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx \leq \int_{\{s < |u_n|\}} b(x)|\nabla u_n|^{p-1} \, dx + \int_{\{s < |u_n|\}} |f_n| \, dx.
\]
Observe that,
\[
\int_{\{s < |u_n|\}} b(x)|\nabla u_n|^{p-1} \, dx \\
\leq \int_{s}^{+\infty} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} b^p \, dx \right) \frac{1}{p} \left( \frac{d}{d\sigma} \int_{\{\sigma < |u_n|\}} |\nabla u_n|^p \, dx \right) \frac{1}{p} \, d\sigma. \tag{2.17}
\]
Because,
\[
\int_{\{s < |u_n|\}} b(x)|\nabla u_n|^{p-1} \, dx = \int_{s}^{+\infty} \frac{-d}{d\sigma} \left( \int_{\{\sigma < |u_n|\}} b(x)|\nabla u_n|^{p-1} \, dx \right) \, d\sigma \\
= \int_{s}^{+\infty} \lim_{\delta \to 0} \frac{1}{\delta} \left( \int_{\{\sigma < |u_n| \leq \sigma + \delta\}} b(x)|\nabla u_n|^{p-1} \, dx \right) \, d\sigma \\
\leq \int_{s}^{+\infty} \lim_{\delta \to 0} \frac{1}{\delta} \left( \int_{\{\sigma < |u_n| \leq \sigma + \delta\}} b^p \, dx \right)^{\frac{1}{p}} \left( \int_{\{\sigma < |u_n| \leq \sigma + \delta\}} |\nabla u_n|^p \, dx \right)^{\frac{1}{p}} \, d\sigma \\
= \int_{s}^{+\infty} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} b^p \, dx \right)^{\frac{1}{p}} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} |\nabla u_n|^p \, dx \right)^{\frac{1}{p}} \, d\sigma.
\]
By (2.3) and (2.17), we deduce that
\[
\frac{1}{\varepsilon} \int_{\{s < |u_n| \leq s + \varepsilon\}} a|x|\nabla u_n|^p \, dx \leq \int_{\{s < |u_n|\}} |f_n| \, dx \\
+ \int_{s}^{+\infty} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} b^p \, dx \right)^{\frac{1}{p}} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} |\nabla u_n|^p \, dx \right)^{\frac{1}{p}} \, d\sigma.
\]
Letting \( \varepsilon \) go to zero, we obtain
\[
\frac{-d}{ds} \int_{\{s < |u_n|\}} a|x|\nabla u_n|^p \, dx \leq \int_{\{s < |u_n|\}} |f_n| \, dx \\
+ \int_{s}^{+\infty} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} b^p \, dx \right)^{\frac{1}{p}} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} |\nabla u_n|^p \, dx \right)^{\frac{1}{p}} \, d\sigma, \tag{2.18}
\]
where \( \{s < |u_n|\} \) denotes the set \( \{(x) \in \Omega, s < |u_n(x)|\} \) and \( \mu(s) \) stands for the distribution function of \( u_n \), that is \( \mu(s) = |\{(x) \in \Omega, |u_n(x)| < s\}| \) for all \( s \geq 0 \).
From (2.21) and (2.22) we have

\[
1 \leq (NC_N^{\frac{1}{p}})^{-1} (\mu(s))^{\frac{1}{p} - 1} (-\mu'(s))^\frac{1}{p} \left( -\frac{d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}}. \tag{2.19}
\]

Using (2.19), we have

\[
\frac{-d}{ds} \int_{\{s < |u_n|\}} \alpha |\nabla u_n|^p dx = \alpha \left( \frac{-d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p} - 1} \left( \frac{-d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}} + (NC_N^{\frac{1}{p}})^{-1} \left( \frac{-d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}}
\]

\[
\times \int_s^{+\infty} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} b^p d\sigma \right)^{\frac{1}{p} - 1} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}} d\sigma. \tag{2.20}
\]

Which implies that,

\[
\alpha \left( \frac{-d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p} - 1} \left( \frac{-d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}}
\]

\[
\leq (NC_N^{\frac{1}{p}})^{-1} (\mu(s))^{\frac{1}{p} - 1} (-\mu'(s))^\frac{1}{p} \left( \int_{\{s < |u_n|\}} |f_n| dx \right) + (NC_N^{\frac{1}{p}})^{-1} \left( \frac{-d}{d\sigma} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}}
\]

\[
\times (-\mu'(s))^\frac{1}{p} \int_s^{+\infty} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} b^p d\sigma \right)^{\frac{1}{p} - 1} \left( \frac{-d}{d\sigma} \int_{\{\sigma < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}} d\sigma. \tag{2.21}
\]

Now, we consider \( B \) and \( \psi \) (see Lemma 2.2 of [1]) defined by

\[
\int_{\{s < |u_n|\}} b^p(x) dx = \int_0^{\mu(s)} B^p(\sigma) d\sigma \text{, and } \psi(s) = \int_{\{s < |u_n|\}} |f_n| dx. \tag{2.22}
\]

We have

\[
||B||_{L^p(\Omega)} \leq ||h||_{L^p(\Omega)} \text{ and } |\psi(s)| \leq ||f_n||_{L^1(\Omega)}. \tag{2.23}
\]

From (2.21) and (2.22)

\[
\alpha \left( \frac{-d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p} - 1} \left( \frac{-d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}}
\]

\[
\leq (NC_N^{\frac{1}{p}})^{-1} (\mu(s))^{\frac{1}{p} - 1} (-\mu'(s))^\frac{1}{p} \psi(s)
\]

\[
+ (NC_N^{\frac{1}{p}})^{-1} (\mu(s))^{\frac{1}{p} - 1} (-\mu'(s))^\frac{1}{p} \int_s^{+\infty} B(\mu(\nu))(-\mu'(\nu))^\frac{1}{p} \left( \frac{-d}{d\nu} \int_{\{\nu < |u_n|\}} |\nabla u_n|^p dx \right)^{\frac{1}{p}} d\nu.
\]
From Lemma 1.1, we obtain
\[
\alpha \left( \frac{d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p \, dx \right)^{\frac{1}{p}} \leq (NC_N^{\frac{1}{p}})^{-1} (\mu(s))^{\frac{1}{p} - 1}(-\mu'(s))^{\frac{1}{p}} \psi(s)
\]
\[
+ (NC_N^{\frac{1}{p}})^{-1} (\mu(s))^{\frac{1}{p} - 1}(-\mu'(s))^{\frac{1}{p}} \int_{s}^{+\infty} \left[(NC_N^{\frac{1}{p}})^{-1} (\mu(\sigma))^{\frac{1}{p} - 1} \psi(\sigma) \right] \times B(\mu(\sigma))(-\mu'(\sigma)) \exp \left( \int_{s}^{\sigma} (NC_N^{\frac{1}{p}})^{-1} B(\mu(r))(\mu(r))^{\frac{1}{p} - 1}(-\mu'(r)) \, dr \right) \, d\sigma.
\]
(2.24)

Now, by a variable change and by the H"older inequality, we estimate the argument of the exponential function on the right hand side of (2.24)
\[
\int_{s}^{\sigma} B(\mu(r))(\mu(r))^{\frac{1}{p} - 1}(-\mu'(r)) \, dr = \int_{s}^{\sigma} B(z)z^{\frac{1}{p} - 1} \, dz
\]
\[
\leq \int_{0}^{\sigma} B(z)z^{\frac{1}{p} - 1} \, dz
\]
\[
\leq ||B||_{L^r} \left( \int_{0}^{\sigma} z^{\left(\frac{1}{p} - 1\right)r'} \right)^{\frac{1}{r'}}.
\]
(2.25)

Raising to the power $p'$ in (2.24) and we can write
\[
\frac{-d}{ds} \int_{\{s < |u_n|\}} |\nabla u_n|^p \, dx \leq M_1,
\]
(2.26)

where $M_1$ depends only on $\Omega, N, p, p', f, \alpha$ and $||b||_{L^r(\Omega)}$, integrating between 0 and $k$, and then (2.14) is proved.

We now give the proof of (2.15), using $T_k(u_n)$ as test function in (2.12), gives
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n) \, dx + \int_{\Omega} \left( g_n(x, u_n, \nabla u_n) + H_n(x, \nabla u_n) \right) T_k(u_n) \, dx
\]
\[
+ \int_{\Omega} \Phi(u_n) \cdot \nabla T_k(u_n) \, dx = \int_{\Omega} f_n T_k(u_n) \, dx.
\]

By the divergence theorem, we get
\[
\int_{\Omega} \Phi(u_n) \cdot \nabla T_k(u_n) = 0,
\]
using (2.7), we deduce that,
\[
\int_{\{u_n \leq k\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx + \int_{\Omega} \Phi(u_n) \cdot \nabla T_k(u_n) \, dx
\]
\[
+ \int_{\{u_n \leq k\}} g_n(x, u_n, \nabla u_n) u_n \, dx + \int_{\{u_n > k\}} g_n(x, u_n, \nabla u_n) T_k(u_n) \, dx
\]
\[
\leq \int_{\Omega} f_n T_k(u_n) \, dx + \int_{\Omega} b(x)|\nabla u_n|^{p-1} T_k(u_n) \, dx
\]
(2.27)

and by using in the fact that $g_n(x, u_n, \nabla u_n) u_n \geq 0$ and (2.3), we have
\[
\alpha \int_{\{u_n \leq k\}} |\nabla u_n|^p \, dx + \int_{\{u_n > k\}} g(x, u_n, \nabla u_n) T_k(u_n) \, dx
\]
\[
\leq k ||f||_{L^1} + k \int_{\{u_n \leq k\}} b(x)|\nabla u_n|^{p-1} \, dx + k \int_{\{u_n > k\}} b(x)|\nabla u_n|^{p-1} \, dx,
\]
which implies that,
\[
\int_{\{|u_n|>k\}} g(x, u_n, \nabla u_n)T_k(u_n) \, dx
\]
\[
\leq k\|f\|_{L^1(\Omega)} k \int_{\{|u_n|\leq k\}} b(x)|\nabla u_n|^{p-1} \, dx + k \int_{\{|u_n|\geq k\}} b(x)|\nabla u_n|^{p-1} \, dx.
\]
By the Hölder inequality and (2.14), we obtain
\[
\int_{\{|u_n|>k\}} g(x, u_n, \nabla u_n)T_k(u_n) \, dx
\]
\[
\leq k\|f\|_{L^1(\Omega)} + k^{1+\frac{1}{p^*}} M_1\|b\|_{L^p(\Omega)} + k \int_{\{|u_n|>k\}} b(x)|\nabla u_n|^{p-1} \, dx.
\]
From (2.6) and applying Young’s inequality, we get for all \(k > \delta\)
\[
u' k \int_{\{|u_n|>k\}} |\nabla u_n|^p \, dx \leq k\|f\|_{L^1(\Omega)} + k^{1+\frac{1}{p^*}} M_1\|b\|_{L^p(\Omega)} + k \int_{\{|u_n|>k\}} b(x)|\nabla u_n|^{p-1} \, dx
\]
\[
\leq k\|f\|_{L^1(\Omega)} + k^{1+\frac{1}{p^*}} M_1\|b\|_{L^p(\Omega)} + M_b k\|b\|_{L^p}^p + \frac{1}{\nu'} k \nu' k \int_{\{|u_n|>k\}} |\nabla u_n|^p \, dx.
\]
Hence
\[
(1 - \frac{1}{p^*}) \int_{\{|u_n|>k\}} |\nabla u_n|^p \, dx \leq M_3\|f\|_{L^1(\Omega)} + k^{1+\frac{1}{p^*}} M_5\|b\|_{L^p(\Omega)} + M_7\|b\|_{L^p}^p,
\]
and Lemma 2.3 is proved. \(\square\)

**Step 2: Almost everywhere convergence of \(u_n\).** We prove that \(u_n\) converges to some function \(u\) locally in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). We will show that \(u_n\) is a Cauchy sequence in measure in any ball \(B_R\).

Let \(k > 0\) large enough, we have
\[
k \text{meas}\{\{|u_n|>k\} \cap B_R\} = \int_{\{|u_n|>k\} \cap B_R} |T_k(u_n)| \, dx
\]
\[
\leq \int_{B_R} |T_k(u_n)| \, dx
\]
\[
\leq C \left(\int_{\Omega} |\nabla T_k(u_n)|^p \, dx\right)^\frac{1}{p}
\]
\[
\leq \frac{c_1}{k^\frac{1}{p^*}}, \text{ for all } k > 1.
\]
Which implies
\[
\text{meas}\{\{|u_n|>k\} \cap B_R\} \leq \frac{c_1}{k^\frac{1}{p^*}}, \text{ for all } k > 1.
\]
We have, for every \(\delta > 0\),
\[
\text{meas}\{\{|u_m-u_n|>\delta\} \cap B_R\} \leq \text{meas}\{\{|u_n|>k\} \cap B_R\} + \text{meas}\{\{|u_m|>k\} \cap B_R\} + \text{meas}\{\{|T_k(u_n)-T_k(u_m)|>\delta\}\}
\]
\[
(2.33)
\]
Since \(T_k(u_n)\) is bounded in \(W^{1,p}_0(\Omega)\), there exists some \(v_k \in W^{1,p}_0(\Omega)\), such that
\[T_k(u_n) \rightharpoonup v_k \text{ weakly in } W^{1,p}_0(\Omega),
\[T_k(u_n) \rightarrow v_k \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega.
\]
Consequently, we can assume that \(T_k(u_n)\) is a Cauchy sequence in measure in \(\Omega\).
Let $\epsilon > 0$, then, by (2.32) and (2.33), there exists some $k(\epsilon) > 0$ such that $\text{meas}\{ |u_n - u_m| > \delta \} \cap B_R < \epsilon$ for all $n, m \geq n_0(\epsilon, \delta, R)$. This proves that $(u_n)$ is a Cauchy sequence in measure in $B_R$, thus converges almost everywhere to some measurable function $u$. Then

$$T_k(u_n) \to T_k(u) \text{ weakly in } W_0^{1,p}(\Omega),$$

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega.$$ Which implies, by using (2.1), for all $k > 0$ there exists a function $h_k \in (L^p(\Omega))^N$, such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \to h_k \text{ weakly in } (L^p(\Omega))^N. \quad (2.34)$$

**Step 3: Strong convergence of truncations.** Let $k > 0$, we consider the function $\phi(s) = s\epsilon^{\gamma s^2}$, with $\lambda \geq (\frac{L_k}{\alpha})^2$, we have the following inequality

$$\phi'(s) - \frac{L_k}{\alpha} |\phi(s)| \geq \frac{1}{2}, \quad (2.35)$$

holds for all $s \in \mathbb{R}$. Here, we define $v_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ where $h > 2k > 0$, and the following function

$$v_n = \phi(w_n). \quad (2.36)$$

Using $v_n$ as test function in (2.12), we obtain

$$\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla v_n \phi'(w_n) \, dx + \int_{\Omega} \Phi_n(u_n) \cdot \phi'(u_n) \nabla u_n \, dx$$

$$+ \int_{\Omega} g_n(x, u_n, \nabla u_n) \phi(w_n) \, dx + \int_{\Omega} H_n(x, \nabla u_n) \phi(w_n) \, dx = \int_{\Omega} f_n \phi(w_n) \, dx. \quad (2.37)$$

Using the fact that $\int_{0}^{u_n} \Phi_n(s) \cdot \phi'(s) \, ds \in W_0^{1,p}(\Omega)$ and Stokes formula, we get

$$\int_{\Omega} \Phi_n(u_n) \cdot \phi'(u_n) \nabla u_n \, dx = \int_{\Omega} \text{div} \left[ \int_{0}^{u_n} \Phi_n(s) \cdot \phi'(s) \, ds \right] \, dx = 0. \quad (2.38)$$

Note that, $\nabla w_n = 0$ on the set where $\{ |u_n| > h + 4k \}$, therefore, setting $M = 4k + h$, and denoting by $\alpha_1, \alpha_2, \ldots$, various sequences of real numbers which converge to zero when $n$ tends to infinity for any fixed value of $h$, we get by (2.37) and (2.38)

$$\int_{\Omega} a(x, M(u_n), \nabla M(u_n)) \cdot \nabla w_n \phi'(w_n) \, dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) \phi(w_n) \, dx$$

$$\leq \int_{\Omega} f_n \phi(w_n) \, dx + \int_{\Omega} |H_n(x, \nabla u_n)\phi(w_n)| \, dx. \quad (2.39)$$

Since

$$\int_{\Omega} H_n(x, \nabla u_n) \phi(w_n) \, dx \leq \||\nabla u_n||_{L^p(\Omega)}^{p-1}||b\phi(T_{2k}(u - T_h(u)))||_{L^p}, \quad (2.40)$$

(where $b\phi(w_n) \to b\phi(T_{2k}(u - T_h(u)))$ in $L^p$, by Lebesgue’s dominated convergence theorem, because $\phi(w_n)$ is bounded),

$$\int_{\Omega} H_n(x, \nabla u_n) \phi(w_n) \, dx = M_0 ||b\phi(T_{2k}(u - T_h(u)))||_{L^p} + \alpha^2_3(n), \quad (2.41)$$

for all $k > 0$ and $n_0$ sufficiently large.
and since \( g_n(x, u_n, \nabla u_n) \phi(w_n) \geq 0 \) on the subset \( \{x \in \Omega : |u_n(x)| > k\} \), we deduce from (2.39) that

\[
\int_{\Omega} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla w_n \phi'(w_n) \, dx + \int_{\{u_n(x) \leq k\}} g_n(x, u_n, \nabla u_n) \phi(w_n) \, dx
\leq \int_{\Omega} f_n \phi(w_n) \, dx + M_3 \|b\phi(T_{2k}(u - T_k(u)))\|_{L^p} + \alpha_k^h(n).
\] (2.42)

Splitting the first integral on the left hand side of (2.42) where \( |u_n| \leq k \) and \( |u_n| > k \), we can write, by using (2.3)

\[
\int_{\Omega} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla w_n \phi'(w_n) \, dx
\geq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \phi'(w_n) \, dx
- C_k \int_{\{u_n \leq k\}} |a(x, T_M(u_n), \nabla T_M(u_n))| |\nabla T_k(u_n)| \, dx.
\] (2.43)

where \( C_k = \phi'(2k) \). Since, when \( n \) tends to infinity, we have \( \nabla T_k(u) \chi_{\{|u_n| > k\}} \) tends to 0 strongly in \( (L^p(\Omega))^N \) while, \( (a(x, T_M(u_n), \nabla T_M(u_n)))_n \) is bounded in \( (L^p(\Omega))^N \) hence the last term in the previous inequality tends to zero for every \( h \) fixed as \( n \) tends to infinity. Now, observe that

\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \phi'(w_n) \, dx
= \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \phi'(w_n) \, dx
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \phi'(w_n) \, dx.
\] (2.44)

By the continuity of the Nymetskii operator, we have for all \( i = 1, ..., N \),

\[
a_i(x, T_k(u_n), \nabla T_k(u)) \phi'(w_n) \to a_i(x, T_k(u), \nabla T_k(u)) \phi'(T_{2k}(u - T_k(u)))
\]

strongly in \( L^p(\Omega) \), and since \( \frac{\partial T_k(u_n)}{\partial x_i} \to \frac{\partial T_k(u)}{\partial x_i} \) weakly in \( L^p(\Omega) \), the second term of the right hand side of (2.44) tends to zero as \( n \) tends to infinity. So that (2.43) yields

\[
\int_{\Omega} a(x, T_M(u_n), \nabla T_M(u)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \phi'(w_n) \, dx
\geq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \phi'(w_n) \, dx + \alpha_k^h(n).
\] (2.45)

For the second term of the left hand side of (2.42), we can estimate as follows

\[
\int_{\{u_n \leq k\}} g(x, u_n, \nabla u_n) \phi(w_n) \, dx \leq \int_{\{u_n \leq k\}} L_1(k) \left( L_2(x) + |\nabla T_k(u_n)|^p \right) \phi(w_n) \, dx
\leq L_1(k) \int_{\Omega} L_2(x) \phi(w_n) \, dx + \frac{L_1(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) \phi(w_n) \, dx.
\] (2.46)
Remark that, we have
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n)|\phi(w_n)|dx
\]
= \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \cdot [\nabla T_k(u_n) - \nabla T_k(u)]|\phi(w_n)|dx
\]
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)) \cdot \nabla T_k(u)|\phi(w_n)|dx
\]
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)) \cdot \nabla T_k(u)|\phi(w_n)|dx.
\]
(2.47)

By the Lebesgue’s Theorem, we have
\[
\nabla T_k(u)|\phi(w_n)| \rightarrow \nabla T_k(u)|\phi(T_{2k}(u - T_k(u)))| \text{ strongly in } (L^p(\Omega))^N
\]
Moreover, in view of (2.34) the second term of the right hand side of (2.47) tends to
\[
\int_{\Omega} h_k \nabla T_k(u)|\phi(T_{2k}(u - T_k(u)))| dx.
\]
The third term of the right hand side of (2.47) tends to 0 since for all \(i = 1, \ldots, N\),
\[
a_i(x, T_k(u_n), \nabla T_k(u))|\phi(w_n)| \rightarrow a_i(x, T_k(u), \nabla T_k(u))|\phi(T_{2k}(u - T_k(u)))
\]
strongly in \(L^p(\Omega)\), while
\[
\frac{\partial(T_k(u_n))}{\partial x_i} \rightarrow \frac{\partial(T_k(u))}{\partial x_i} \text{ weakly in } L^p(\Omega).
\]
From (2.46) and (2.47), we obtain
\[
\left| \int_{\{u_n \leq k\}} g(x, u_n, \nabla u_n) \phi(w_n)dx \right|
\]
\[
\leq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \cdot [\nabla T_k(u_n) - \nabla T_k(u)]|\phi(w_n)|dx
\]
+ \int_{\Omega} h_k \nabla T_k(u)|\phi(T_{2k}(u - T_k(u)))| dx + L_1(k) \int_{\Omega} L_2(x)|\phi(w_n)|dx + \alpha_1^{10}(n),
\]
(2.48)

Now, by the strongly convergence of \(f_n\) and in fact that
\[
w_n \rightarrow T_{2k}(u - T_k(u)) \text{ weakly in } W_0^{1,p}(\Omega) \text{ and weakly* in } L^\infty(\Omega),
\]
(2.49)
moreover, combining (2.45) and (2.48), we conclude that
\[
\int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))]
\]
\[
\left[\nabla T_k(u_n) - \nabla T_k(u) \right]|\phi'(w_n) - \frac{L_1(k)}{\alpha}\phi(w_n)|dx
\]
\[
\leq L_1(k) \int_{\Omega} L_2(x)|\phi(T_{2k}(u - T_k(u)))|dx + \int_{\Omega} f\phi(T_{2k}(u - T_k(u)))|dx
\]
+ \int_{\Omega} h_k \nabla T_k(u)|\phi(T_{2k}(u - T_k(u)))| dx + M_0||\phi(T_{2k}(u - T_k(u)))||_{L^p} + \alpha_1^{11}(n).
\]
(2.50)

which and (2.35), implies that
\[
\int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right] \cdot \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \\
\leq 2L_1(k) \int_{\Omega} L_2(x) |\phi(T_{2k}(u - T_k(u)))| dx + 2 \int_{\Omega} f \phi(T_{2k}(u - T_k(u))) dx \\
+ 2 \int_{\Omega} h_k. \nabla T_k(u) |\phi(T_{2k}(u - T_k(u)))| dx + 2M_0 |\phi(T_{2k}(u - T_k(u)))|_{L^p} + \alpha_h^{12}(n).
\] 

(2.51)

hence, passing to the limit over \( n \), we obtain

\[
\limsup_{n \to +\infty} \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right] \cdot \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \\
\leq 2L_1(k) \int_{\Omega} L_2(x) |\phi(T_{2k}(u - T_k(u)))| dx + 2 \int_{\Omega} f \phi(T_{2k}(u - T_k(u))) dx \\
+ 2M_0 |\phi(T_{2k}(u - T_k(u)))|_{L^p} + 2 \int_{\Omega} h_k. \nabla T_k(u) |\phi(T_{2k}(u - T_k(u)))| dx + \alpha_h^{13}(n).
\] 

(2.52)

It remains to show, for our purposes, that the all terms on the right hand side of (2.52) converge to zero as \( h \) goes to infinity. Therefore by (2.52), letting \( h \) go to infinity, we conclude,

\[
\lim_{n \to +\infty} \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right] \cdot \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx = 0.
\] 

(2.53)

Then, Lemma 5 of [11] implies,

\[
T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega).
\] 

(2.54)

**Step 4: Equi-integrability of \( H_n \) and \( g_n \).** We shall now prove that \( H_n(x, \nabla u_n) \) converges to \( H(x, \nabla u) \) and \( g_n(x, u_n, \nabla u_n) \) converges to \( g(x, u, \nabla u) \) strongly in \( L^1(\Omega) \) by using Vitali’s theorem. Since \( H_n(x, \nabla u_n) \to H(x, \nabla u) \) a.e. \( \Omega \) and \( g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \) a.e. \( \Omega \), thanks to (2.4) and (2.7), it suffices to prove that \( H_n(x, \nabla u_n) \) and \( g_n(x, u_n, \nabla u_n) \) are uniformly equi-integrable in \( \Omega \). We will now prove that \( H_n(x, \nabla u_n) \) is uniformly equi-integrable, we use Hölder’s inequality and (2.13), we have for any measurable subset \( E \subset \Omega \):

\[
\int_{E} |H_n(x, \nabla u_n)| dx \\
\leq \left( \int_{E} b^p(x) dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\nabla u_n|^{p'} \right)^{\frac{1}{p'}} \\
\leq C \left( \int_{E} b^p(x) dx \right)^{\frac{1}{p}}
\] 

(2.55)

which is small uniformly in \( n \) when the measure of \( E \) is small.

To prove the uniform equi-integrability of \( g_n(x, u_n, \nabla u_n) \). For any measurable subset \( E \subset \Omega \) and \( m \geq 0 \),

\[
\int_{E} |g(x, u_n, \nabla u_n)| dx \\
= \int_{E \cap \{|u_n| \leq m\}} |g(x, u_n, \nabla u_n)| dx + \int_{E \cap \{|u_n| > m\}} |g(x, u_n, \nabla u_n)| dx \\
\leq L_1(m) \int_{E \cap \{|u_n| \leq m\}} |L_2(x) + |\nabla u_n|^{p'}| dx + \int_{E \cap \{|u_n| > m\}} |g(x, u_n, \nabla u_n)| dx \\
\leq L_1(m) \int_{E \cap \{|u_n| \leq m\}} \left(L_2(x) + |\nabla T_m(u_n)|^{p'}\right) dx + \int_{E \cap \{|u_n| > m\}} |g(x, u_n, \nabla u_n)| dx \\
= K_1 + K_2.
\] 

(2.56)
For fixed $m$, we get
\[ K_1 \leq L_1(m) \int_E \left( L_2(x) + |\nabla T_m(u_n)|^p \right) dx, \]
which is thus small uniformly in $n$ for $m$ fixed when the measure of $E$ is small (recall that $T_m(u_n)$ tends to $T_m(u)$ strongly in $W_0^{1,p}(\Omega)$). We now discuss the behavior of the second integral of the right hand side of (2.56), let $\psi_m$ be a function such that
\[ \begin{align*}
\psi_m(s) &= 0 \quad \text{if } |s| \leq m - 1, \\
\psi_m(s) &= \text{sign}(s) \quad \text{if } |s| \geq m, \\
\psi'_m(s) &= 1 \quad \text{if } m - 1 < |s| < m.
\end{align*} \tag{2.57} \]
We choose for $m > 1$, $\psi_m(u_n)$ as a test function in (2.12), we obtain
\[ \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla u_n \psi'_m(u_n) \, dx + \int_{\Omega} \Phi_n(u_n) \cdot \nabla u_n \psi'_m(u_n) \, dx \\
+ \int_{\Omega} g_n(x, u_n, \nabla u_n) \psi_m(u_n) \, dx + \int_{\Omega} H_n(x, \nabla u_n) \psi_m(u_n) \, dx = \int_{\Omega} f_n \psi_m(u_n) \, dx. \]
By the divergence theorem, we get
\[ \int_{\Omega} \Phi_n(u_n) \cdot \nabla u_n \psi'_m(u_n) \, dx = 0. \]
Using (2.3) and H"{o}lder’s inequality
\[ \int_{\{m-1 \leq |u_n|\}} |g_n(x, u_n, \nabla u_n)| \, dx \leq \int_{E} |H_n(x, \nabla u_n)| \, dx + \int_{\{m-1 \leq |u_n|\}} |f| \, dx, \]
and by (2.13), we have
\[ \lim_{m \to \infty} \sup_{n \in \mathbb{N}} \int_{\{|u_n| > m-1\}} |g_n(x, u_n, \nabla u_n)| \, dx = 0. \]
Thus we proved that the second term of the right hand side of (2.56) is also small, uniformly in $n$ and in $E$ when $m$ is sufficiently large. Which shows that $g_n(x, u_n, \nabla u_n)$ and $H_n(x, \nabla u_n)$ are uniformly equi-integrable in $\Omega$ as required, we conclude that
\[ H_n(x, \nabla u_n) \to H(x, \nabla u) \quad \text{strongly in } L^1(\Omega), \]
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \tag{2.58} \]

**Step 5: Passing to the limit.** We take $T_k(u_n - v)$ as test function in (2.12), with $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we can write
\[ \int_{\Omega} a(x, T_{k+|v|_\infty} u_n, \nabla T_{k+|v|_\infty} u_n) \cdot \nabla T_k(u_n - v) \, dx + \int_{\Omega} \Phi_n(u_n) \cdot \nabla T_k(u_n - v) \, dx \\
+ \int_{\Omega} (g(x, u_n, \nabla u_n) + H(x, \nabla u_n)) T_k(u_n - v) \, dx = \int_{\Omega} f_n T_k(u_n - v) \, dx. \tag{2.59} \]
By Fatou’s lemma and in fact that
\[ a(x, T_{k+|v|_\infty} u_n, \nabla T_{k+|v|_\infty} u_n) \rightharpoonup a(x, T_{k+|v|_\infty} u, \nabla T_{k+|v|_\infty} u) \]
weakly in $(L^p(\Omega))^N$. It easily see that
\[ \int_{\Omega} a(x, T_{k+|v|_\infty} u, \nabla T_{k+|v|_\infty} u) \cdot \nabla T_k(u - v) \, dx \\
\leq \liminf_{n \to \infty} \int_{\Omega} a(x, T_{k+|v|_\infty} u_n, \nabla T_{k+|v|_\infty} u_n) \cdot \nabla T_k(u_n - v) \, dx. \tag{2.60} \]
For the second term of the right hand side of (2.59). Since $\nabla T_k(u_n - v) \to \nabla T_k(u - v)$ weakly in $(L^p(\Omega))^N$, for the second term of the left hand side of (2.59), we have

$$\int_{\Omega} \Phi(u_n) \cdot \nabla T_k(u_n - v)dx \to \int_{\Omega} \Phi(u) \cdot \nabla T_k(u - v)dx \text{ as } n \to +\infty. \tag{2.61}$$

On the other hand, we have

$$\int_{\Omega} f_n T_k(u_n - v)dx \to \int_{\Omega} f T_k(u - v)dx \text{ as } n \to +\infty. \tag{2.62}$$

Thanks to (2.58) and (2.60)-(2.62), we can pass to the limit in (2.59), and we obtain that $u$ is a solution of the problem (1.1). This completes the proof of Theorem 2.2.

Remark 2.4. The condition (2.4) can be replaced by the weaker one

$$|g(x,s,\xi)| \leq L_2(x) + L_1(|s|)|\xi|^p, \tag{2.63}$$

where $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $L_2(x) \in L^1(\Omega)$.

References


(Mostafa El Moumni) University Sidi Mohamed Ben Abdellah, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P 1796 Atlas Fez, Morocco
E-mail address: mostafaelmoumni@gmail.com